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ANALYTIC AND GEVREY REGULARITY FOR SOME MODEL EQUATIONS

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ABSTRACT. We state some Gevrey hypoellipticity results for some model equations representing certain classes of sums of squares of vector fields operators.

1. INTRODUCTION AND STATEMENTS

The purpose of this talk is to present some results concerning the analytic or Gevrey regularity of solutions of "sums of squares of vector fields" type equations with smooth—i.e. analytic—data.

More precisely we are concerned with the regularity of the solutions of second order differential equations

$$P(x, D)u(x) = f(x)$$

in an open subset Ω of $\mathbb{R}^n$, where

$$P(x, D) = \sum_{j=1}^{r} (X_j(x, D))^2, \qquad x \in \Omega, \tag{1.1}$$

X_j denoting a homogeneous vector field with analytic coefficients.

It is well known since the fundamental paper of Hörmander [10] that the operator in (1.1) is C^∞–hypoelliptic if the vector fields X_j, $j = 1, \dots, r$, and their brackets up to a finite length N generate the n-dimensional Lie algebra which we identify with $\mathbb{R}^n$ itself. When this occurs we say that P satisfies Hörmander's condition of order N.

We shall always assume that Hörmander's condition up to a finite order N is verified.

A very natural question then can be asked: assume that the vector fields X_j, $j = 1, \dots, r$, have real analytic coefficients. Is then P analytic hypoelliptic?

It is known since the famous example of Baouendi-Goulaouic [1] that this is not true (see also Métivier [13]), even though six years later

Tartakoff [21] and Treves [25] independently showed that if the characteristic set is a symplectic manifold and the localized operator is "transversally non-degenerate", then C^∞–hypoellipticity entails analytic hypoellipticity.

In particular in the above mentioned paper Treves formulated the following

Conjecture 1 (Treves' first conjecture). *If* Char P, *assumed to be an analytic manifold, contains a smooth curve whose tangent vector at some point is orthogonal, with respect to the symplectic form, to the tangent space to* Char P *at that point, then P is not analytic hypoelliptic.*

This conjecture is still standing unproved and a number of authors have worked on it.

It is easy to see though that the above conjecture cannot account for the following operator produced by Oleinik and Radkevič in [14], [15]:

$$P(x, D_x, D_t, D_s) = D_x^2 + x^{2(p-1)} D_t^2 + x^{2(q-1)} D_s^2, \tag{1.2}$$

where $(x, t, s) \in \mathbb{R}^3$, p, q are non negative integers and $q \geq p$.

Let us denote by G^s the class of Gevrey functions of type s and by $G^{(s_1, \dots, s_n)}$ the class of Gevrey functions of partial type s_j, $j = 1, \dots, n$, where s, s_j are real numbers ≥ 1. They can be defined as follows:

$$G^s = \{u \mid u \in C^\infty(\mathbb{R}^n), \ \ |\partial^\alpha u(x)| \leq C^{1+|\alpha|} \alpha!^s, \text{ locally in } x\}, \tag{1.3}$$

where C is a positive constant depending only on u;

$$G^{(s_1, \dots, s_n)} = \{u \mid u \in C^\infty(\mathbb{R}^n), \ \ |\partial^\alpha u(x)| \leq C^{1+|\alpha|} \alpha_1!^{s_1} \dots \alpha_n!^{s_n}, \text{ locally in } x\}, \tag{1.4}$$

where $C > 0$ depends on u only.

We remark that if $s_j = 1$ for some $j \in \{1, \dots, n\}$ we get a function partially analytic with respect to the variable x_j.

Coming back to the operator in (1.2) we have the following

Theorem 1 ([15], [7], [3]). *The operator P in (1.2) is $G^{q/p}$ hypoelliptic and not better. More precisely we have that if u solves the equation $Pu = f$ and f is analytic, then $u \in G^{(s_1, s_2, s_3)}$ where*

$$s_1 \geq 1 + \frac{1}{p} - \frac{1}{q}, \quad s_2 \geq 1, \quad s_3 \geq \frac{q}{p}.$$

Moreover each of these threshold values is optimal.

It is then evident that, since the characteristic set of P is a symplectic manifold, Conjecture 1 does not yield any result in this case.

To account for such cases F. Treves proposed a second conjecture; it essentially deals with operators of the form "sum of squares" of vector fields and makes no assumption on the regularity of the characteristic set. For the precise statement we refer to Treves' original paper [27]. Here we formulate a much less general form of this conjecture, which is suitable for our needs in the present discussion.

Definition 1. *Let $I = (i_1, i_2, \ldots, i_k)$ be a multiindex and $i_j \in \{1, \ldots, r\}$, $j = 1, \ldots, k$. We set*

$$X_I = \{X_{i_1}, \{X_{i_2}, \{X_{i_3}, \ldots, \{X_{i_{k-1}}, X_{i_k}\} \ldots\}\}\}, \tag{1.5}$$

where $\{X_i, X_j\}$ denotes the Poisson bracket of the (symbols of the) vector fields X_i and X_j, so that X_I is again a vector field with analytic coefficients.

$$|I| = k. \tag{1.6}$$

Then

Conjecture 2 (Treves' second conjecture). *Let P be as in (1.1) and let us assume that all the sets involved are analytic manifolds, at least near a fixed base point. Define:*

$$\begin{aligned} &\Sigma_1 = \operatorname{Char} P \\ &\Sigma_2 = \Sigma_1 \cap \bigcap_{|I|=2} X_I^{-1}(0) \\ &\cdots \\ &\Sigma_j = \Sigma_{j-1} \cap \bigcap_{|I|=j} X_I^{-1}(0) \\ &\cdots \end{aligned} \tag{1.7}$$

The above sets are called Poisson strata. We point out explicitly that, since we are assuming that P satisfies Hörmander condition, the above sequence of Poisson strata comes to an end, i.e. there exists an integer N such that $\Sigma_N = \varnothing$. Evidently we have that

$$\Sigma_1 \supseteq \Sigma_2 \supseteq \cdots \supseteq \Sigma_{N-1} \supseteq \Sigma_N = \varnothing.$$

Then the operator P in (1.1) is analytic hypoelliptic if and only if every Poisson stratum is a symplectic manifold.

It is quite straightforward to verify that in the case of the Oleinik-Radkevič operator in (1.2) we have:

$$\begin{aligned} \Sigma_1 &= \{(x,t,s;\xi,\tau,\sigma) \mid x=\xi=0\} \\ \Sigma_2 &= \cdots = \Sigma_{p-1} = \Sigma_1 \\ \Sigma_p &= \{(x,t,s;\xi,\tau,\sigma) \mid x=\xi=0,\ \tau=0\} \\ \Sigma_{p+1} &= \cdots = \Sigma_{q-1} = \Sigma_p \\ \Sigma_q &= \varnothing, \end{aligned} \tag{1.8}$$

near the point $(0, e_n)$. In this case we see that the strata $\Sigma_1, \ldots, \Sigma_{p-1}$ are symplectic.

Here we address the following question: consider an operator which is a sum of 3 squares of vector fields in 3 variables and assume that the associated Poisson stratification has the same symplectic character (and the same Hörmander numbers) as that of the operator (1.2). By this we mean that the lengths of the two stratifications are the same-and that each stratum of one is symplectically diffeomorphic to the corresponding stratum of the other. In particular this implies that the relative codimensions are the same.

The question is: does such an operator then exhibit the same hypoellipticity behaviour as that in (1.2)?

In this talk we consider only model operators and and refer to a paper in preparation [4] for more general results, as well as for the proofs.

Actually we have the

Theorem 2. *Let $q \geq p \geq 1$.*

(i) Consider the operator

$$P_1(x, D_x, D_t, D_s) = D_x^2 + x^{2(p-1)}(D_t + x^{q-p}D_s)^2 + x^{2(q-1)}D_s^2. \tag{1.9}$$

Then P_1 is $G^{q/p}$-hypoelliptic.

(ii) Consider the operator

$$P_2(x, D_x, D_t, D_s) = D_x^2 + x^{2(p-1)}(D_t + x^{q-p}D_s)^2 + x^{2(q-1)}D_t^2. \tag{1.10}$$

Then:

(a) If $q \geq 2p$, P_2 is $G^{q/p}$-hypoelliptic.
(b) If $p \leq q < 2p$, P_2 is $G^{3-2(p/q)}$-hypoelliptic.

Some comments to Theorem 2 are in order.

1– It is easy to check that the Poisson stratification associated to the model operators P_1 and P_2 is the same as that of the Oleinik-Radkevič operator in (1.2), namely (1.8).
2– In the case of a generic sum of three squares of analytic ector fields with a Poisson stratification symplectically diffeomorphic to (1.8) it is possible to deduce a standard form for the vector fields. By inspection of the construction the standard forms can be classified in a symplectically invariant way into two broad classes of which P_1 and P_2 are model representatives.
3– The index $\frac{q}{p}$ is obviously optimal in this generality, since it is so in the case of the operator (1.2). In the range $p \leq q < 2p$ we have $3 - 2\frac{p}{q} < \frac{q}{p}$, hence the threshold obtained in (ii)(b) is worse than that in (ii)(a). We are not able to prove that (ii)(b) is an optimal result.
4– The (motivation of the) proof of (ii)(b) is deeply microlocal. When $q < 2p$ we obtain an apparently less sharp result because of the existence of null bicharacteristics of the vector field D_x issuing from points in the intersection of the characteristic varieties of the other vector fields.
5– When $p = q$ we obtain analytic hypoellipticity.

As a final remark we want to point out that if the number of symplectic strata of the Poisson stratification "increases", then we can hope to obtain a better Gevrey hypoellipticity threshold. This is the case for the following

Theorem 3 ([5]). *Let p, q, ℓ, $k \in \mathbb{N}$, $q \geq p \geq 1$ and $k \leq \ell(q-1)$. Set*

(1.11)
$$P_3(x, t; D_x, D_t, D_s) = D_x^2 + x^{2(q-1)}(D_t + (x^{2k} + t^{2\ell})D_s)^2 + x^{2(p-1)}D_t^2.$$

Then P_3 is G^s-hypoelliptic for every $s \geq s_0$ with

$$s_0 = \frac{(q-1)(q+2k)}{(p-1)(q+2k)+q-p}.$$

We remark that the Poisson stratification associated to the operator in (1.11) is

$$\begin{aligned} \Sigma_1 &= \{(x,t,s;\xi,\tau,\sigma) \mid x = \xi = 0\} \\ \Sigma_2 &= \cdots = \Sigma_{p-1} = \Sigma_1 \\ \Sigma_p &= \{(x,t,s;\xi,\tau,\sigma) \mid x = \xi = 0 \tau = 0\} \\ \Sigma_{p+1} &= \cdots = \Sigma_{q-1} = \Sigma_p \\ \Sigma_q &= \{(x,t,s;\xi,\tau,\sigma) \mid x = \xi = 0 t = \tau = 0\} \\ \Sigma_{q+1} &= \cdots = \Sigma_{q+2k-1} = \Sigma_q \\ \Sigma_{q+2k} &= \varnothing. \end{aligned} \tag{1.12}$$

Moreover we have

$$s_0 \leq \frac{q+2k}{q},$$

and $s_0 = 1$ if $p = q$.

2. Proof of (i) in Theorem 2

Just to give the flavor of the technique we employ we want to prove here part (i) of Theorem 2. Let

$$X_1 = D_x, \quad X_2 = x^{p-1}(D_t + x^{q-p}D_s), \quad X_3 = x^{q-1}D_s, \tag{2.1}$$

be the three vector fields the sum of whose squares equals P_1. From now on we shall write P instead of P_1.

Denote by φ an Ehrenpreis type cut off function; this means that for any pair of open sets ω, $\Omega \subset \mathbb{R}^3$, $\omega \subset\subset \Omega$, there is a positive constant C_0 such that $\varphi \equiv 1$ on ω and

$$|D^\alpha \varphi(x)| \leq C_0^{1+|\alpha|} N^{|\alpha|},$$

for $|\alpha| \leq qN$. Here N denotes an arbitrarily large positive integer. Of course, whatever the choice of N is, the so defined function φ depends on N, but we omit to write this dependence to keep the notation simple.

¿From now on N will be as large as required; we stress that when $|\alpha|$ is close to N the bound for φ is essentially a bound for analytic functions.

The other ingredient we need for our proof is an a priori estimate of Rothschild-Stein type:

$$\sum_{j=1}^{3} \|X_j u\|^2 + \|u\|^2_{\frac{1}{q}} \leq C\left(|\langle Pu, u\rangle| + \|u\|^2\right), \tag{2.2}$$

where $\|\cdot\|_s$ denotes the norm in the Sobolev space of order s and $\|\cdot\|_0 = \|\cdot\|$ is the L^2 norm.

Let φ be a cut off function of the type described above and let us replace u by $\varphi D_s^r u$ in (2.2). Evidently the Gevrey (analytic) regularity for u can be deduced from from a suitable estimate of $\varphi D_s^r u$, where r is a large positive integer and $N \sim r$:

(2.3)

$$\sum_{j=1}^{3} \|X_j \varphi D_s^r u\|^2 + \|\varphi D_s^r u\|^2_{\frac{1}{q}} \leq C\left(|\langle P\varphi D_s^r u, \varphi D_s^r u\rangle| + \|\varphi D_s^r u\|^2\right).$$

Let us consider the term containing P in the right hand side. Commuting P with φD_s^r we must estimate expressions of the type

$$\langle \left[X_j^2, \varphi D_s^r\right] u, \varphi D_s^r u\rangle,$$

with $j = 1, 2, 3$. Let us start with $j = 3$. We may write

$$\begin{aligned} &|\langle \left[X_3^2, \varphi D_s^r\right] u, \varphi D_s^r u\rangle| \\ &\quad \leq 2|\langle X_3 \varphi_s' D_s^{r-1} u, X_3 \varphi D_s^r u\rangle| + |\langle \frac{1}{N} X_3 \varphi_{ss}'' D_s^{r-1} u, N X_3 \varphi D_s^{r-1} u\rangle|, \end{aligned}$$

where N is a large integer comparable in size with r and we are neglecting terms in which one of the r s-derivatives has been transferred onto φ, thus yielding a shift with a net gain whose (pure) iteration would lead to analyticity.

The above quantity can be estimated by:

$$\begin{aligned} &|\langle \left[X_3^2, \varphi D_s^r\right] u, \varphi D_s^r u\rangle| \\ &\qquad \leq \frac{1}{2}\|X_3 \varphi D_s^r u\|^2 + C\Big[\|X_3 \varphi' D_s^{r-1} u\|^2 \\ &\qquad\qquad + \|\frac{1}{N} X_3 \varphi'' D_s^{r-1} u\|^2 + \|N X_3 \varphi D_s^{r-1} u\|^2\Big]. \end{aligned} \tag{2.4}$$

Let us take a look at the term with $j = 2$. We have:

$$\langle [X_2^2, \varphi D_s^r]\, u, \varphi D_s^r u\rangle$$
$$= 2\langle x^{p-1}\left(\varphi_t' + x^{q-p}\varphi_s'\right) D_s^r u, X_2 \varphi D_s^r u\rangle$$
$$+ \langle \frac{1}{N} x^{p-1}\left(D_t + x^{q-p} D_s\right)^2 D_s^r u, N x^{p-1}\varphi D_s^r u\rangle.$$

Before proceeding further we need two remarks: (a) Since $p \leq q$ we cannot in general recover an X vector field using one s-derivative. Hence, to place a vector field before the main term, we must use the a priori estimate (2.3), thus using (i.e., gaining) less than one s-derivative. (b) A term of the form $x^{p-1}\left(\varphi_t' + x^{q-p}\varphi_s'\right)$ can be estimated by $|x|^{p-1}|\varphi'|$ near the origin.

We can then conclude:

$$(2.5) \quad |\langle [X_2^2, \varphi D_s^r]\, u, \varphi D_s^r u\rangle|$$
$$\leq \frac{1}{2}\|X_2 \varphi D_s^r u\|^2 + C\left[\|x^{p-1}\varphi' D_s^{r-\frac{1}{q}} u\|_{\frac{1}{q}}^2\right.$$
$$\left. + \|\frac{1}{N} x^{p-1}\varphi'' D_s^{r-\frac{1}{q}} u\|_{\frac{1}{q}}^2 + \|N x^{p-1}\varphi D_s^{r-\frac{1}{q}} u\|_{\frac{1}{q}}^2\right],$$

where φ', φ'' stand for first and second derivatives of φ (in s or t).

The term $j = 1$ is negligible at this stage, since we may take $D_x \varphi = 0$ near $x = 0$ and if $x \neq 0$ our operator is evidently analytic hypoelliptic (actually it is elliptic). In spite of this fact though, terms involving brackets with X_1 do play an important role in the following because of the presence of the powers of x scattered around by the other fields.

To stress this fact it is more convenient to replace $\varphi D_s^r u$ with an expression of the form $x^a \varphi^{(m)} D_s^{r-\frac{b}{q}} u$, where a, m and b are integers.

Using (2.4) and (2.5) we then obtain:

$$
\begin{aligned}
(2.6)\quad & \sum_{j=1}^{3} \|X_j x^a \varphi^{(m)} D_s^r u\|^2 + \|x^a \varphi^{(m)} D_s^r u\|_{\frac{1}{q}}^2 \\
& \leq C \left[|\langle P x^a \varphi^{(m)} D_s^r u, x^a \varphi^{(m)} D_s^r u \rangle| + \|x^a \varphi^{(m)} D_s^r u\|^2 \right] \\
& \leq \frac{1}{2} \sum_{j=1}^{3} \|X_j x^a \varphi^{(m)} D_s^r u\|^2 + C \Big[\|x^a \varphi^{(m)} D_s^r P u\|^2 \\
& + \sum_{j=1}^{3} \|X_j x^a \varphi^{(m+1)} D_s^{r-1} u\|^2 + \sum_{j=1}^{3} \|\frac{1}{N} X_j x^a \varphi^{(m+2)} D_s^{r-1} u\|^2 \\
& + \sum_{j=1}^{3} \|N X_j x^a \varphi^{(m)} D_s^{r-1} u\|^2 \\
& + \|x^{a+p-1} \varphi^{(m+1)} D_s^{r-\frac{1}{q}} u\|_{\frac{1}{q}}^2 + \|\frac{1}{N} x^{a+p-1} \varphi^{(m+2)} D_s^{r-\frac{1}{q}} u\|_{\frac{1}{q}}^2 \\
& + \|N x^{a+p-1} \varphi^{(m)} D_s^{r-\frac{1}{q}} u\|_{\frac{1}{q}}^2 \Big].
\end{aligned}
$$

Actually the exponent on x never needs to increase beyond $q-2$; if $a+p-1 \geq q-1$, instead of using the subelliptic part of (2.5) we convert $x^{a+p-1}D_s$ into $x^{a+p-1-(q-1)}X_3$, and, since $p < q$, the exponent on x has actually decreased.

We stress the fact that this trick only works in the case of the operator in (1.9). It is evident that, in order to do the same for the model operator in (1.10), we must "bound" D_s by D_t (or rather some power of D_s by D_t) and this in turn means that we need to microlocalize the estimation procedure. Furthermore, in case (1.10), even if we could apply such a procedure, there should be some (in general not conic) region of the cotangent bundle for which no treatment of this type would be possible, so that we must then follow another approach.

Following [3], it is possible to iterate inequality (2.6), with the exponent on x never exceeding $q-2$.

Denoting by ρ the number of x-derivatives landing onto the various powers of x (we recall that the behavior of ϕ with respect to x plays no role here), by c the number of X_2 vector fields landing onto φ (they each carry the factor x^{p-1}), by e the number of times that a power of x exceeds $q-1$, thus allowing us to decrease by e the power of D_s and by

f the number of X_3 fields landing onto the cut off function and yielding good analytic-growth terms, we obtain:

$$(2.7)\quad \sum_{j=1}^{3} \|X_j x^a \varphi^{(m)} D_s^r u\|^2 + \|x^a \varphi^{(m)} D_s^r u\|^2_{\frac{1}{q}} \leq \|x^a \varphi^{(m)} D_s^r P u\|^2$$

$$+ \sup_{\Delta \geq 0} C^\Delta \Bigg[\sum_{j=1}^{3} \|N^{-\ell} X_j x^{a-\rho+c(p-1)-e(q-1)} \varphi^{(m+c+f+\ell)} D_s^{r-e-f-\frac{c+\rho}{q}} u\|^2$$

$$+ \|N^{-\ell} x^{a-\rho+c(p-1)-e(q-1)} \varphi^{(m+c+f+\ell)} D_s^{r-e-f-\frac{c+\rho-1}{q}} u\|^2_{\frac{1}{q}} \Bigg],$$

where

$$\Delta = e + f + c + \rho$$

is the quantity by which D_s^r is decreased in the process, C is a fixed positive constant and the following constraints hold:

$$(2.8)\qquad \begin{cases} 0 \leq a - \rho + c(p-1) - e(q-1) < q - 1, \\ |\ell| \leq c + f. \end{cases}$$

Pursuing this task until the s-derivatives are used up and choosing $a = m = 0$ as a starting point, we obtain (suppressing the term with Pu),

$$\sum_{j=1}^{3} \|X_j \varphi D_s^r u\|^2 + \|\varphi D_s^r u\|^2_{\frac{1}{q}}$$

$$\leq \sup_{r-1 \leq \Delta \leq r} C^\Delta \left(N^{-\ell} |\varphi^{(c+f+\ell+1)}| \, \|u\| \right)^2 \leq C C_1^r N^{2(c+f)}.$$

Keeping into account the relations (2.8), the definition of Δ and the fact that the worst estimate occurs if f is minimum and c is maximum, we get that $-\rho + c(p-1) - e(q-1) \sim 0$ and

$$e\frac{q-1}{q} + \frac{c}{q} + \frac{\rho}{q} \sim r,$$

from which we deduce that

$$c \sim \frac{q}{p} r,$$

so that

$$c + f \lesssim \frac{q}{p} r.$$

Choosing $N \sim r$ we reach the desired conclusion.

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