数理解析研究所講究録
1159 巻 2000 年 81-86

Conjectures about the differential operators in an algorithm for computing the residues.

Shinichi TAJIMA (田島 慎一) Niigata Univ.
Yayoi NAKAMURA (中村 弥生) Ochanomizu Univ.

Let $X=\mathbf{C}^{2}$ and fix a coordinate system $z=(x,y)$ of X. We denote by $\mathcal{O}_{X}$ the sheaf of holomorphic functions on X. Let $f_{1}, f_{2}\in \mathcal{O}_{X}$ and (f_{1}, f_{2}) be a regular sequence. Denote by I the sheaf of ideal of $\mathcal{O}_{X}$ generated by f_{1}, f_{2}. Put $A=\{z\in X|f_{1}=f_{2}=0\}$. Assume that at least one zero has multiplicity greater than 1. We denote by m the algebraic local cohomology class associated to the meromorphic function $1/f_{1}f_{2}$.

In [4], we gave an algorithm to compute the cohomology class m and the residues. This algorithm has been constructed by the aid of the theory of $\mathcal{D}_{X}$-module and is based on the properties of the annihilators of m.

In this note, we examine the more detailed properties of annihilators which are useful for our algorithm. We use the computer algebra system Kan ([5]) and Risa/Asir ([2]).

1 The operators used in our algorithm

Let Ω_{X} be the sheaf of holomorphic differential form on X. We assume that the set of common zeros A consists of finitely many points $A_{1}, \ldots, A_{\nu}$. There is a pairing

$$Res_{A_{\ell}}:\Omega_{X}/I\Omega_{X}\otimes \mathcal{E}xt_{\mathcal{O}_{X}}^{2}(\mathcal{O}_{X}, /I, \mathcal{O}_{X})\rightarrow \mathbf{C}.$$

For m, this pairing yields a unique linear mapping $\Omega_{X}/I\Omega_{X}\ni\phi(z)dz\mapsto \mathrm{Res}_{A_{\ell}}\langle\phi(z)dz, m\rangle\in \mathbf{C}$ defined by the residue of the differential form $\phi(z)dz/f_{1}f_{2}$ at A_{ℓ}.

Put $V_{K}=\{\phi(z)dz\in\Omega_{X}/I\Omega_{X}\mid Res_{A_{j}}\langle\phi(z)dz, m\rangle=0, j=1, \ldots, \nu\}$. Let μ_{j} be the multiplicity of A_{j}, $j=1, \ldots, \nu$ and $\mu=\mu_{1}+\cdots+\mu_{\nu}$. Then, V_{K} can be regarded as $\mu-\nu$ dimensional vector space. Denote by $\mathcal{A}nn$ the ideal generated by differential operators which annihilate m. Then we have the following theorem.

Theorem 1

$$V_{K}=\{(R^{*}\psi(z))dz\mid R\in \mathcal{A}nn, \psi(z)dz\in\Omega_{X}/I\Omega_{X}\}.$$

Now we give conjectures about the properties of operators $P_{1}, \ldots, P_{k}\in \mathcal{A}nn$ which we use in our algorithm for computing the residues.

Conjecture (A) There exist $P_{1}, \ldots, P_{k}\in \mathcal{A}nn$ whose adjoint operators act on the vector space $\mathbf{C}[x, y]/I$ and $Im(P_{1}^{*}, \ldots, P_{k}^{*})$ span V_{K}, where $Im(P_{1}^{*}, \ldots, P_{k}^{*})$ stands for the set of images of the adjoint operators P_{j}^{*}, $j=1, \ldots, k$ associated to $\mathbf{C}[x, y]/I$.

If there exist operators $P_{1}, \ldots, P_{k}\in \mathcal{A}nn$ which satisfy the property in the conjecture (A), we have following conjectures about construction of them.

Conjecture $(C1)$ P_{j}'s are first-order differential operators.

Put $P_{j}=c_{j1}\partial_{x}+c_{j2}\partial_{y}+c_{j0}$ where $c_{j0}, c_{j1}, c_{j2}\in \mathbf{C}[x, y]$ and $\partial_{x}:=\partial/\partial x$, $\partial_{y}:=\partial/\partial y$.

Conjecture $(C2)$ $\langle c_{11}, c_{12}, \ldots, c_{k1}, c_{k2}, f_{1}, f_{2}\rangle=\sqrt{\langle f_{1},f_{2}\rangle}$ as the ideal of $\mathbf{C}[x, y]$.

Conjecture $(C3)$ $\langle F_{1}, F_{2}, P_{1}, \ldots, P_{k}\rangle=\mathcal{A}nn$, where $F_{j}=f_{j}$, $j=1,2$ stands for differential operators of order 0.

Conjecture $(C4)$ As for the number of first order differential operators, we have $1\leq k\leq 2$.

2 Illustration of conjectures

We use the following procedure to investigate the annihilators P_j, $j = 1, \ldots, k$.

(i) Construct annihilators of order zero and of order one.

(ii) Take the gröbner bases GB of operators in (i).

(iii) Find first order operators which generate GB together with 0th order operators. (we shall see the particular case in 2.2.2)

(iv) Verify the condition (1).

These computation can be carried by computer algebra system Kan and Risa/Asir.

2.1 The case $A = \{(0,0)\}$.

2.1.1 Example : $f_1 = x^5$, $f_2 = y^2 + x^4 + x^3$

In this case, f_1 and f_2 have common zero only at the origin with multiplicity 10.

(i) Computing syzygies on the ring of polynomials, we obtain
$F_1 = x^5$,
$F_2 = y^2 + x^4 + x^3$,
as annihilators of m of order zero and

- $-2yx\partial_x + (4x^4 + 3x^3)\partial_y - 10y$,
- $2yx\partial_x + (x^3 + 4y^2)\partial_y + 18y$,
- $(2x^2 + 2x)\partial_x + (4yx + 3y)\partial_y + 18x + 16$,
- $2yx\partial_x + (-4x^4 - 3x^3)\partial_y + 10y$,
- $(-2y^2x + 2y^2)\partial_x + (4yx^4 - yx^3 - 3yx^2)\partial_y - 10x^2 - 10y^2x$,
- $-2yx^2\partial_x + (4x^5 + 3x^4)\partial_y - 10yx$,
- $(-2x^2 + 6x)\partial_x + 9y\partial_y - 10x + 48$,

as annihilators of m of order one (see Section 3).

(ii) The gröbner basis GB of the ideal generated by these operators with respect to the lexicographic order $y \succ x$ is given by following 8 operators;
$F_1 = x^5$,
$F_2 = y^2 + x^4 + x^3$,
$P_1 = (-2x^2 + 6x)\partial_x + 9y\partial_y - 10x + 48$,
$P_2 = x^3\partial_x + 5x^2$,
$P_3 = 2yx\partial_x + (-4x^4 - 3x^3)\partial_y + 10y$,
$P_4 = 3x^2\partial_x^2 + (-4x^2 + 24x)\partial_x - 20x + 30$,
$P_5 = 9x\partial_x^2 + (-16x^2 + 12x + 54)\partial_x - 9x^2\partial_y^2 - 80x + 60$,
$P_6 = -x\partial_x^4 - 8\partial_x^3 - 4x\partial_y^2\partial_x + (4x - 8)\partial_y^2$.

(iii) We find that the operators F_1, F_2 and P_1 generate GB.

(iv) The ideal generated by f_1, f_2 and the coefficients of ∂_x and ∂_y in P_1 is equal to the radical of the ideal I, i.e. , $\langle f_1, f_2, -2x^2 + 6x, 9y\rangle = \langle x, y\rangle = \sqrt{\langle f_1, f_2\rangle}$.

In fact, we can see that the operator P_1 satisfies the property in the conjecture (A) and the other first order operators not as follows. Under the isomorphism $\Omega_X/I\Omega_X \cong \mathbf{C}[x,y]/I$, these operators P_j, $j = 1, 2, 3$ act on the 10 dimensional vector space $\mathbf{C}[x,y]/I$. Using the gröbner basis with respect to the lexicographic order $y \succ x$, the monomial basis MB of $\mathbf{C}[x,y]/I$ is $MB = \{1, y, x, yx, x^2, yx^2, x^3, yx^3, x^4, yx^4\}$. Then $Im(P_1^*)$ is given by

$P_1^*1 = -6x + 33 \mod I,$
$P_1^*y = -6yx + 24y \mod I,$
$P_1^*x = -4x^2 + 27x \mod I,$
$P_1^*yx = -4yx^2 + 18yx \mod I,$
$P_1^*x^2 = -2x^3 + 21x^2 \mod I,$
$P_1^*yx^2 = -2yx^3 + 12yx^2 \mod I,$
$P_1^*x^3 = 15x^3 \mod I,$
$P_1^*yx^3 = 6yx^3 \mod I,$
$P_1^*x^4 = 9x^4 \mod I,$
$P_1^*yx^4 = 0 \mod I.$

From this computation, it follows that $\dim Im(P_1^*) = 9$. The other side, $\dim Im(P_j^*) < 9$, $j = 2, 3$. Thus, we verify that the operator P_1 enjoys (A).

The functions f_1 and f_2 are semiquasihomogeneous polynomials of degree 10 and 6 with weights $wt(x) = 2$, $wt(y) = 3$. Put $wt(\partial_x) = -2$ and $wt(\partial_y) = -3$. Then the operator P_1 is the semiquasihomogeneous polynomial in $\mathbf{C}[x, y, \partial_x, \partial_y]$ with the quasihomogeneous part $3(\underline{2}x\partial_x + \underline{3}y\partial_y + \underline{10} + \underline{6})$. The underlined parts indicate that the quasihomogeneous part of the operator is determined by the degree of f_1 and f_2 as semiquasihomogeneous polynomials.

2.1.2 Example : $f_1 = x^7$, $f_2 = y^2 + x(x^4 + 2x^3y - 3x^5y - x^6)$

In this case, f_1 and f_2 have common zero only at the origin with multiplicity 14.

(i) Computing syzygies on the ring of polynomials, we obtain
$F_1 = x^7,$
$F_2 = y^2 + x(x^4 + 2x^3y - 3x^5y - x^6),$
as annihilators of m of order zero and

- $(-2x^3 - 2yx^2)\partial_x + (-5yx^2 - 8y^2x))\partial_y - 24x^2 - 30yx,$
- $(37x^3 + 36yx^2))\partial_x + (94yx^2 + 144y^2x))\partial_y + 447x^2 + 540yx,$
- $((16y + 37)x^3 - 20yx^2 + 16x))\partial_x + (-24x^4 - 24yx^3 + (64y^2 + 94y)x^2 - 80y^2x + 40y))\partial_y$
  $-48x^3 + (240y + 447)x^2 - 300yx + 192,$
- $((4y^2 - 2)x^2 - 2yx))\partial_x + ((16y^3 - 5y)x - 8y^2))\partial_y + (60y^2 - 24)x - 30y,$
- $yx^3\partial_x + 4y^2x^2\partial_y + 15yx^2,$
- $((-16y - 37)x^3 + (10y^2 + 55y)x^2 - 26x))\partial_x + (39x^4 + 39yx^3 + (-79y^2 - 94y)x^2$
  $+(40y^3 + 220y^2)x - 65y))\partial_y + 78x^3 + (-270y - 447)x^2 + (150y^2 + 825y)x - 312,$
- $((16y^2 + 57y)x^3 - 10x^2 + 6yx))\partial_x + (-24yx^4 - 24y^2x^3 + (64y^3 + 174y^2)x^2 - 25yx))\partial_y$
  $-48yx^3 + 747yx^2 + (480y^3 - 120)x + 42y,$
- $((4y^2 - 2)x^2 - 2yx))\partial_x + ((16y^3 - 5y)x - 8y^2))\partial_y + (60y^2 - 24)x - 30y,$
- $((-16y^2 - 57y)x^3 + (32y^3 + 114y^2 + 10)x^2 - 26yx + 12y^2))\partial_x$
  $+(24yx^4 - 24y^2x^3 + (-112y^3 - 174y^2)x^2 + (128y^4 + 348y^3 + 25y)x - 50y^2))\partial_y$
  $-84x^4 - 120yx^3 - 747yx^2 + (-192y^3 + 1410y^2 + 120)x - 84y^3 - 282y,$
- $((-16y^2 - 57y)x^3 + 10x^2 - 6yx))\partial_x + (24yx^4 + 24y^2x^3 + (-64y^3 - 174y^2)x^2 + 25yx))\partial_y$
  $+48yx^3 - 747yx^2 + (-480y^3 + 120)x - 42y,$
- $((-48y^4 - 171y^3)x^3 + (-16y^3 - 27y^2)x^2 + (-18y^3 + 10y)x - 6y^2))\partial_x$
  $+(72y^3x^4 + (72y^4 + 24y^2)x^3 + (-192y^5 - 522y^4 + 24y^3)x^2 + (-64y^4 - 99y^3)x + 25y^2))\partial_y$
  $+42x^4 + 84yx^3 + (-144y^3 - 345y^2)x - 84y^3 + 120y$

as annihilators of m of order one.

(ii) The gröbner basis GB of the ideal generated by these operators with respect to the lexicographic order $y \succ x$ is given by following 10 operators
$F_1 = x^7,$
$F_2 = y^2 + x(x^4 + 2x^3y - 3x^5y - x^6),$
$P_1 = (21x^3 + 16x)\partial_x + (-24x^4 + 40y)\partial_y + 147x^2 + 192,$
$P_2 = x^4\partial_x + 7x^3,$
$P_3 = -x^3\partial_x + 4x^6\partial_y - 7x^2,$
$P_4 = -2yx\partial_x + 5x^5\partial_y + 36x^6 - 16x^4 - 14y,$
$P_5 = (-5x^3\partial_y - 24x^2)\partial_x + (96x^5 - 35x^2)\partial_y - 168x,$
$P_6 = 4x^2\partial_x^2 + (9x^3 + 40x)\partial_x + 16x^4\partial_y + 63x^2 + 56,$
$P_7 = 3x\partial_x^2 + 24\partial_x - 5x^4\partial_y^2 + (-27x^5 + 36x^3)\partial_y,$
$P_8 = 5x\partial_x^3 + 45\partial_x^2 - 288x^2\partial_x + 25x^3\partial_y^2 + (1152x^5 + 90x^4 - 240x^2)\partial_y - 2016x.$

(iii) We find that the operators F_1, F_2 and P_1 generate GB.

(iv) Then, the ideal generated by f_1, f_2 and the coefficients of ∂_x and ∂_y in P_1 is equal to the radical of the ideal I, i.e. , $\langle f_1, f_2, 21x^3+16x, -24x^4+40y\rangle = \langle x, y\rangle = \sqrt{\langle f_1, f_2\rangle}$.

In fact, we can verify that the operator P_1 satisfies the property in the conjecture (A) and the other first order operators not as follows. Under the isomorphism $\Omega_X/I\Omega_X \cong \mathbf{C}[x,y]/I$, these operators P_j, $j=1,2,3,4$ act on the 14 dimensional vector space $\mathbf{C}[x,y]/I$. Using the gröbner basis with respect to the lexicographic order $y \succ x$, we have $MB = \{1, y, x, yx, x^2, yx^2, x^3, yx^3, x^4, yx^4, x^5, yx^5, x^6, yx^6\}$. Then $Im(P_1^*)$ is given by

$$\begin{aligned}
P_1^*1 &= 84x^2+136 && \text{mod } I,\\
P_1^*y &= 24x^4+84yx^2+96y && \text{mod } I,\\
P_1^*x &= 63x^3+120x && \text{mod } I,\\
P_1^*yx &= 24x^5+63yx^3+80yx && \text{mod } I,\\
P_1^*x^2 &= 42x^4+104x^2 && \text{mod } I,\\
P_1^*yx^2 &= 24x^6+42yx^4+64yx^2 && \text{mod } I,\\
P_1^*x^3 &= 21x^5+88x^3 && \text{mod } I,\\
P_1^*yx^3 &= 21yx^5+48yx^3 && \text{mod } I,\\
P_1^*x^4 &= 72x^4 && \text{mod } I,\\
P_1^*yx^4 &= 32yx^4 && \text{mod } I,\\
P_1^*x^5 &= 56x^5 && \text{mod } I,\\
P_1^*yx^5 &= 16yx^5 && \text{mod } I,\\
P_1^*x^6 &= 40x^6 && \text{mod } I,\\
P_1^*yx^6 &= 0 && \text{mod } I.
\end{aligned}$$

From this computation, it follows that $\dim Im(P_1^*) = 13$. The other side, $\dim Im(P_j^*) < 13$, $j = 2,3,4$.

The functions f_1 and f_2 are semiquasihomogeneous polynomials of degree 14 and 10 with weights $wt(x)=2$, $wt(y)=5$. Put $wt(\partial_x)=-2$ and $wt(\partial_y)=-5$. Then the operator P_1 is the semiquasihomogeneous polynomial in $\mathbf{C}[x,y,\partial_x,\partial_y]$ with quasihomogeneous part $8(\underline{2}x\partial_x+\underline{5}y\partial_y+\underline{14}+\underline{10})$. The underlined parts indicate that the quasihomogeneous part of the operator is determined by the degree of f_1 and f_2 as semiquasihomogeneous polynomials.

2.2 In the case that A consists of several points

2.2.1 Example: $f_1 = (x^2+y^2)^2 + 3x^2y - y^3$, $f_2 = x^2+y^2-1$

In this case, $A = \{(0,1), (\sqrt{3}/2, -1/2), (-\sqrt{3}/2, -1/2)\}$ with multiplicities 2 at each points.
(i) Computing syzygies on the ring of polynomials, we obtain

$F_1 = 16x^6 - 24x^4 + 9x^2$,
$F_2 = 4x^4 - 5x^2 - y + 1$

as annihilators of m of order zero and

- $(x^2+y^2-1)\partial_y + 2y$,
- $(x^2+y^2-1)\partial_x + 2x$,
- $(2y^2+y)x\partial_x + (-2y-1)x^2\partial_y + 6y^2 + 3y - 3$,
- $(2y^3-y^2-y)\partial_x + (-2y^2+y+1)x\partial_y + (-6y+3)x$,
- $(2yx^2+y^2-y)\partial_x + (-2x^3+(-y+1)x)\partial_y + (6y-3)x$,
- $(2y^2+y)x\partial_x + (-2y-1)x^2\partial_y + 6y^2 + 3y - 3$,
- $(-2y^2-y)x\partial_x + (2y+1)x^2\partial_y - 6y^2 - 3y + 3$,
- $(2y+1)x\partial_x + (-2x^2-4y^2+y+3)\partial_y - 6y + 5$,

as annihilators of m of order one.

(ii) The gröbner basis GB of these operators with respect to the lexicographic order $y \succ x$ is given by following 6 operators;

$F_1 = 16x^6 - 24x^4 + 9x^2$,
$F_2 = 4x^4 - 5x^2 - y + 1$
$P_1 = (4x^3-3x)\partial_x + (8x^4-6x^2)\partial_y - 16x^4 + 36x^2 - 6$,
$P_2 = (-16x^5+24x^3-9x)\partial_x - 96x^4 + 96x^2 - 18$,
$P_3 = (8x^4-6x^2)\partial_x^2 + ((12x^3-9x)\partial_y + 64x^3 - 12x)\partial_x + (48x^2-18)\partial_y + 96x^2 + 12$,
$P_4 = (4x^3-3x)\partial_x^3 + (48x^2-12)\partial_x^2 + ((-12x^3+9x)\partial_y^2 + (24x^3-30x)\partial_y + 144x)\partial_x$
$+(-48x^2+18)\partial_y^2 + (96x^2-60)\partial_y + 96$.

(iii) We find that the operators F_1, F_2 and P_1 generate GB.

(iv) The ideal generated by f_1, f_2 and the coefficients of ∂_x and ∂_y in P_1 is equal to the radical of the ideal I, i.e. , $\langle f_1, f_2, 4x^3-3x, 8x^4-6x^2\rangle = \langle 4x^3-3x, 2x^2+y-1\rangle = \sqrt{\langle f_1, f_2\rangle}$.

In fact, we can verify that the operator P_1 satisfies the property in the conjecture (A) and the other first order operators are not as follows. Under the isomorphism $\Omega_X/I\Omega_X \cong \mathbf{C}[x,y]/I$, the operators P_j, $j=1,2$ act on the 6 dimensional vector space $\mathbf{C}[x,y]/I$. Using the gröbner basis with respect to the lexicographic order $y \succ x$, we have $MB = \{1, x, x^2, x^3, x^4, x^5\}$. Then $Im(P_1^*)$ is given by

$$\begin{array}{ll} P_1^*1 = -16x^4+24x^2-3 & \bmod\ I, \\ P_1^*x = -16x^5+20x^3 & \bmod\ I, \\ P_1^*x^2 = -8x^4+12x^2 & \bmod\ I, \\ P_1^*x^3 = -12x^5+15x^3 & \bmod\ I, \\ P_1^*x^4 = -6x^4+9x^2 & \bmod\ I. \\ P_1^*x^5 = -9x^5+45/4x^3 & \bmod\ I. \end{array}$$

From this computation, it follows that $\dim Im(P_1^*) = 3(=6-3)$. The other side, $\dim Im(P_2^*) = 1 < 3$.

Put $I_1 = \langle (4x^2-3)^2, 4x^2-4y-5\rangle$ and $I_2 = \langle x^2, y-1\rangle$. Then $I = I_1 \cap I_2$. Let m_1 be the cohomology class with support at $V(I_1)$ and m_2 the cohomology class with support at $V(I_2)$ which satisfy $m = m_1 + m_2$. From the ideals $\langle (4x^2-3)^2, 4x^2-4y-5, P_1\rangle$ and $\langle x^2, y-1, P_1\rangle$, we obtain $R_1 = (12xy+6x)\partial_x + (18y+9)\partial_y + 12y + 42$ as an annihilator of first order of m_1 and $R_2 = x\partial_x + 2$ as an annihilator of first order of m_2. These operators satisfy the localization of the property in the conjecture (A) to $\mathcal{O}_X/I_j$, $j=1,2$.

2.2.2 Example : $f_1 = x^6 + (y^2-3)x^4 + (y^4+y^2+3)x^2 + y^6 - y^4 + y^2 - 1$, $f_2 = x^6 + (3y^2-3)x^4 + (3y^4+3y^2+3)x^2 + y^6 - 3y^4 + 3y^2 - 1$

In this case, A consists of $\{(x,y) | x^8 - x^6 + 3x^4 - x^2 + 1 = x^6 + 2x^2 - y^2 = 0\}$ with multiplicity 1, $(0,1)$ with multiplicity 2, $(0,-1)$ with multiplicity 2, $(1,0)$ with multiplicity 6, and $(-1,0)$ with multiplicity 6.

(i) Computing syzygies on the ring of polynomials, we obtain

$F_1 = -6x^{14} + 25x^{12} - 56x^{10} + 85x^8 - 82x^6 + 47x^4 - 16x^2 - 3y^2 + 3$,

$F_2 = x^{16} - 4x^{14} + 9x^{12} - 14x^{10} + 14x^8 - 9x^6 + 4x^4 - x^2$

as annihilators of m of order zero and 26 operators of order one.

(ii) The gröbner basis GB of these operators with respect to the lexicographic order $y \succ x$ is given by following 5 operators;

$$\begin{array}{rl} F_1 = & -6x^{14} + 25x^{12} - 56x^{10} + 85x^8 - 82x^6 + 47x^4 - 16x^2 - 3y^2 + 3, \\ F_2 = & x^{16} - 4x^{14} + 9x^{12} - 14x^{10} + 14x^8 - 9x^6 + 4x^4 - x^2, \\ P_1 = & (-13x^{11} + 26x^9 - 52x^7 + 52x^5 - 26x^3 + 13x)\partial_x - 132x^{10} + 176x^8 - 344x^6 \\ & +152x^4 + (44y^2 - 96)x^2 + 16y^2 + 10, \\ P_2 = & (yx^{10} - yx^8 + 3yx^6 - yx^4 + yx^2)\partial_y + 2x^{10} - 2x^8 + 6x^6 - 2x^4 + 2x^2, \\ P_3 = & ((yx^9 - yx^7 + 3yx^5 - yx^3 + yx)\partial_y + 2x^9 - 2x^7 + 6x^5 - 2x^3 + 2x)\partial_x \\ & +(10yx^8 - 8yx^6 + 18yx^4 - 4yx^2 + 2y)\partial_y + 20x^8 - 16x^6 + 36x^4 - 8x^2 + 4. \end{array}$$

(iii) In this case, we need four operators F_1, F_2, P_1 and P_2 to generate GB.

(iv) Then the ideal generated by f_1, f_2 and the coefficients of ∂_x and ∂_y in P_1 and P_2 is equal to the radical of the ideal I, i.e. , $\langle F_1, F_2, -13x^{11} + 26x^9 - 52x^7 + 52x^5 - 26x^3 + 13x, yx^{10} - yx^8 + 3yx^6 - yx^4 + yx^2\rangle = \langle -x^{11} + 2x^9 - 4x^7 + 4x^5 - 2x^3 + x, -yx^9 + yx^7 - 3yx^5 + yx^3 - yx, -2x^{10} + 3x^8 - 6x^6 + 5x^4 - x^2 - y^2 + 1\rangle = \sqrt{\langle f_1, f_2\rangle}$.

In fact, we can verify that the operators P_1 and P_2 satisfy the property in the conjecture (A). Under the isomorphism $\Omega_X/I\Omega_X \cong \mathbf{C}[x,y]/I$, the operators P_1 and P_2 act on the 32 dimensional vector space $\mathbf{C}[x,y]/I$. And it follows that the vector space $Im(P_1^*, P_2^*)$ is 12 dimension.

Put $I_1 = \langle x^4 + (y^2+1)x^2 - y^2 + 1, 2x^4 - x^2 + y^4 + 2, x^6 + 2x^2 - y^2\rangle$, $I_2 = \langle x^2, y-1\rangle$, $I_3 = \langle x^2, y+1\rangle$, $I_4 = \langle (x-1)^3, y^2\rangle$, $I_5 = \langle (x+1)^3, y^2\rangle$. Then $I = I_1 \cap I_2 \cap I_3 \cap I_4 \cap I_5$. Let m_j be the cohomology class with support at $V(I_j)$, $j = 1,2,3,4,5$, which satisfy $m = m_1 + m_2 + m_3 + m_4 + m_5$. From the ideals generated by P_1, P_2, and I_j, we obtain the annihilators of each m_j. For m_2 and m_3, we have $x\partial_x + 2$. Concerning to m_4, we have $\langle (x-1)^3, y^2, (12x-12)\partial_x - x^2 + 44x - 7, y\partial_y + 2\rangle$ as annihilators of m_4. Note that I_4 is generated by $(x+1)^3$ and y^2, both are univariate polynomials. For such a case, we need two first order differential operators. In the same way, we have $\langle (x+1)^3, y^2, (12x+12)\partial_x - x^2 - 44x - 7, y\partial_y + 2\rangle$ as annihilators of m_5. Note that since the ideal I_1 is simple, m_1 does not require any first order differential operators.

3 Construction of annihilators of first order

We can find annihilators of first order by the computations of syzygies. Put $P = a\partial_x + b\partial_y + c$ where $a, b, c \in \mathbf{C}[x, y]$. If there exist u_{11}, u_{12}, u_{21} and u_{22} which satisfy $-af_{1x} - bf_{1y} = u_{11}f_1 + u_{12}f_2$ and $-af_{2x} - bf_{2y} = u_{21}f_1 + u_{22}f_2$, P annihilates the cohomology class associated to the meromorphic function $1/f_1f_2$ with $c = -u_{11} - u_{22}$. In other words, $(a, b, u_{11}, u_{12}, u_{21}, u_{22})$ is a syzygy of $\begin{pmatrix} -f_{1x} \\ -f_{2x} \end{pmatrix}$, $\begin{pmatrix} -f_{1y} \\ -f_{2y} \end{pmatrix}$, $\begin{pmatrix} f_1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} f_2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ f_1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ f_2 \end{pmatrix}$. Thus, we can obtain the first order differential operators annihilating the cohomology class m with respect to the given meromorphic function by using Kan. This obserbation is due to T. Oaku ([3]) and the algorithm has been implemented by him.

If these conjectures are right, we can compute the algebraic local cohomology group as left $\mathcal{D}_X$-module without any information on the b-function. Then, we will able to obtain more effecient algorithm for computing the residues.

References

[1] M. Kashiwara, *On the holonomic systems of linear differntial eqauations, II.* Inventiones mathematicae **49** (1978), 121–135.

[2] M. Noro and T. Takeshima, *Risa/Asir–a computer algebra system*,Proceedings of International Symposium on Symbolic and Algebraic Computation (ed. Paul S. Wang), 387–396, ACM, New York, 1992. (ftp:endeavor.fujitsu.co.jp /pub/isis/asir).

[3] T. Oaku, *Algorithms for the b-functions, restrictions, and algebraic local cohomology groups of D-modules*, Adv. in Appl. Math. **19** (1997), 61–105.

[4] S. Tajima and Y. Nakamura, *Computing residues of a several variables rational function* , Sûrikaiseki Kenkyûsho kôyûroku, Kyoto Univ. **1085** (1999), 71–81.

[5] N. Takayama, *Kan: A system for computation in algebraic analysis* (1991–), (http://www. math. s. kobe-u. ac. jp).