数理解析研究所講究録
1162巻 2000年 99-106

CRITICAL SOBOLEV INEQUALITY AND ITS APPLICATION TO NONLINEAR EVOLUTION EQUATIONS IN THE FLUID MECHANICS

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1. INTRODUCTION

Our main concern, in this note, is to discuss on the uniqueness problem for the Navier-Stokes equation and the Euler equations.

We firstly consider the Navier-Stokes equation under auxiliary regularity assumption to the solution.

$$(1.1)\qquad \begin{cases} \partial_t u + u\cdot\nabla u = -\nabla p + \Delta u + f, & t>0, x\in\mathbb{R}^n, \\ \operatorname{div} u = 0, \quad t>0, x\in\mathbb{R}^n, \\ u(0,x) = u_0(x). \end{cases}$$

By virtue of the energy inequality associated with (1.1), i.e.,

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u(\tau)\|_2^2 d\tau \le \|u_0\|_2^2, \quad a.e. \quad t>0,$$

it is well known that there exists a global energy class weak solution (so called Lerey-Hopf's weak solution) $u\in L^\infty([0,T];L^2_\sigma(\mathbb{R}^n))\cap L^2(0,T;\dot{H}^1(\mathbb{R}^n))$. The natural regularity for Leray-Hopf weak solutions is then

$$(1.2)\qquad u\in L^\sigma([0,T];L^q), \quad \frac{2}{\sigma}+\frac{n}{q}=\frac{n}{2}, \quad 2\le q\le \frac{2n}{n-2}.$$

The uniqueness problem is then considered under the auxiliary assumption on the one of the weak solution. The condition suggested by Serrin is

$$(1.3)\qquad u\in L^\theta([0,T];L^p), \quad \frac{2}{\theta}+\frac{n}{p}=1, \quad n<p.$$

See Ohyama [22], Serrin [28], Giga [15],

An interesting problem is to consider the corresponding condition for the vorticity $\omega = \operatorname{rot} u$. By the Sobolev embedding theorem, the corresponding condition to $|\nabla|^s u$ is

$$(1.4)\qquad |\nabla|^r u\in L^\theta([0,T];L^p), \quad \frac{2}{\theta}+\frac{n}{p}=1+r.$$

Those conditions, (1.3) and (1.4) are closely related to the estimate for the bi-linear form induced from the nonlinear term $(u\cdot\nabla v, w)$. Recent research for estimating this term develops both the regularity theory and decay problems for the Navier-Stokes system. Among others, Chanillo considered the bi-linear estimate in [8] by a real analytical argument. This result essentially extracted a better regularity from the nonlinear coupling, more precisely the estimate saves a logarithmic singularity for the solutions and in fact, it was applied to the regularity problem to the harmonic map on the sphere (see e.g., [12], [13]). By an elegant proof, Coifman-Lions-Mayer-Semmes [9] showed the $\mathcal{H}^1$ regularity of the nonlinear coupling $u \cdot \nabla u$ for the Leray-Hopf solution. This showed that the nonlinear term has a better regularity because of its special algebraic structure by divergence free - rotation free coupling. In fact, for the 2-dimensional case, $u \cdot \nabla u \in \mathcal{H}^1$ and since Leray-Hopf solution belong to H^1 and hence in BMO, the coupling $(u \cdot \nabla u, u)$ makes its sense. Then it is developed to the regularity problem in a different setting in the Besov spaces ([14], [6]) and to the decay problem in the Hardy space corresponding to the L^p where $p \le 1$ by Miyakawa [21].

Our attention here is devoted to the uniqueness condition in terms of the vorticity. In views of the above conditions, for example, $\nabla u \in L^1([0,T]; L^\infty)$ is considered as the limiting case of the uniqueness condition. On the other hand, from the observation of the break down condition to the Euler equation (1.5),

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + f, & t > 0, x \in \mathbb{R}^n, \\ \quad \text{div } u = 0, \quad t > 0, x \in \mathbb{R}^n, \\ \quad u(0,x) = u_0(x). \end{cases} \tag{1.5}$$

it is meaningful to control the situation in terms of the vorticity of fluid, rot $u(t)$. In the celebrated result by Beale-Kato-Majda [2], the solution of the 3D Euler equation is shown to be regular under rot $u(t) \in L^1([0,T]; L^\infty)$. This result is extended into the slightly larger class of condition by Kozono-Taniuichi [19]. The corresponding uniqueness result, however, seems to have a difficulty, since $\|\nabla u\|_\infty$ can not be controlled only by $\|\omega\|_\infty$. We introduce here a possible substitution $\|\omega\|_{BMO}$ to $\|\nabla u\|_\infty$ and generalize the situation in terms of the Besov spaces.

2. Uniqueness for the Navier-Stokes equation

Before presenting our result, we recall some notations and definition of the Besov spaces (c.f., [31]). Let ϕ_j $j = 0, \pm 1, \pm 2, \pm 3, \cdots$ be the Littlewood-Payley dyadic decomposition satisfying $\hat{\phi}_j(\xi) = \phi(2^{-j}\xi)$ and $\sum_{j=-\infty}^{\infty} \phi_j(\xi) = 1$ except $\xi = 0$. We put a smooth cut off to fill the origin $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\hat{\psi}(\xi) \in C_0^\infty(B_1)$ such that $\hat{\psi} + \sum_{j=0}^{\infty} \hat{\phi}_j(\xi) = 1$.

Definition The homogeneous Besov space $\dot{B}^s_{p,\rho} = \{f \in \mathcal{S}; \|f\|_{\dot{B}^s_{p,\rho}} < \infty\}$ is introduced by the norm

$$\|f\|_{\dot{B}^s_{p,\rho}} = (\sum_{j=-\infty}^{\infty} \|2^{js}\phi_j * f\|_p^\rho)^{1/\rho'}$$

for $s \in \mathbb{R}$, $1 \le p, \rho \le \infty$ and the inhomogeneous Besov space similarly defined by

$$\|f\|_{B^s_{p,\rho}} = (\|\psi * f\|_p^\rho + \sum_{j=0}^{\infty} \|2^{js}\phi_j * f\|_p^\rho)^{1/\rho'}.$$

Theorem 2.1 (Uniqueness for Navier-Stokes [24]). *Let u and $\tilde{u}$ be the Leray-Hopf weak solutions for the Navier-Stokes system with the same initial data u_0 with the same initial data u_0. Suppose that the vorticity ω for one of the solution satisfies* rot $u = \omega \in L(\log L)^{1/\rho'}([0,T]; \dot{B}^0_{\infty,\rho})$, $1 \le \rho \le \infty$ *and the other solution $\tilde{u}$ satisfies the energy inequality*

$$\|\tilde{u}(t)\|_2^2 + 2\int_0^t \|\nabla\tilde{u}(\tau)\|_2^2 d\tau \le \|u_0\|_2.$$

Then $u = \tilde{u}$.

Recalling the embedding relation $BMO(\mathbb{R}^n) \subset \dot{B}^0_{\infty,\infty}(\mathbb{R}^n)$, we have;

Corollary 2.2 (limiting vorticity condition). *Let u and $\tilde{u}$ be the Leray-Hopf weak solutions for the Navier-Stokes system with the same initial data u_0. Suppose that the vorticity of the one of the solution u satisfies* rot $u = \omega \in L(\log L)^{1/\rho'}([0,T]; \dot{B}^s_{p,\rho})$ *with $s = n/p$ and $1 \le p, \rho \le \infty$. and the other solution $\tilde{u}$ satisfies the energy inequality*

$$\|\tilde{u}(t)\|_2^2 + 2\int_0^t \|\nabla\tilde{u}(\tau)\|_2^2 d\tau \le \|u_0\|_2.$$

Then $u = \tilde{u}$. Especially, if rot $u \in L(\log L)([0,T]; BMO)$ *then $u = \tilde{u}$.*

In fact, for the partial regularity problem, Beale-Kato-Majda [2] showed that the solution of Euler equation is regular if rot $u \in L^1([0,T]; L^\infty)$. In this case, the vorticity rot $u = \omega$ can dominate $\|\nabla u\|_\infty$ via the Bio-Savart law with aid of extra regularity assumption. (see also Ponce [27] and Kozono-Taniuchi [19] and Vishik [32]). In our case, however the regularity can be covered by the viscosity of the equation.

3. The Euler equation

The global existence in time solution for the Euler equation (1.5) in $\mathbb{R}^2$ is known by the result of Yudovich [33] for $\omega \in L^1 \cap L^\infty$, Diperna-Majda[10] for $\omega_0 \in L^1 \cap L^p$ and Chae [7] for $\omega_0 \in L^1 \cap L\log L$. In particular, by the a priori estimate for rot $u(t)$ in $L^2 \cap L^\infty$, it is known that the solution in the class $\omega(t) \in L^1 \cap L^\infty$ is unique ([33]). This uniqueness result was extended into the case when the unbounded vorticity case in [34] in the case of bounded domain. We present here a slightly different uniqueness result than the result in [34].

Theorem 3.1 (Uniqueness for Euler [24]). *Let u and $\tilde{u}$ be a weak solutions for the Euler equation in $L^\infty(0,T;H^1_\sigma)$. Suppose that one of the solutions satisfies* rot $u \in L(\log L)^{1/2}(0,T;\dot{B}^0_{\infty,\infty})$, *then $u = \tilde{u}$.*

The condition of the above theorem is realizable if we assume that $\nabla\omega_0 \in L^2(\mathbb{R}^2)$. Note that this condition does not necessarily implies $\omega_0 \in L^\infty$ which is the uniqueness condition obtained by Yudovich [33]. The result has a slight difference on the regularity assumption on the generalized solution of the Euler equations. Since Yudovich employ the variant of the argument of the Perron-Nagumo criterion to the uniqueness on the ordinary differential inequality, it is required the continuity for the solution in L^2. Allowing this extra regularity, it is also possible to see the weaker regularity assumption like in [34] in terms of a generalization of the Besov spaces (c.f. [25]).

4. Critical Sobolev inequality

It is well known that the critical case of the Sobolev imbedding theorem is involving the certain kind of special feature when the relation between the integrability exponent p and regularity s satisfies $s = n/p$. For example, when $n = 2$, $W^{1,2}(\mathbb{R}^2)$ can not be embedded into L^∞.

Here we give some generalization of the logarithmic Sobolev inequality originally due to Brezis-Gallouet [4], Brezis-Wainger[5] and Beale-Kato-Majda [2] (see for some generalization [11], [30], [26], [19], [17]).

Theorem 4.1 ([17]). *For any $p, \rho, q, \nu \in [1,\infty]$, $r_1, r_2, \sigma_1, \sigma_2 \in [1,\infty)$, $\nu \le \min(\rho, \sigma_1, \sigma_2)$, $1/q = 1/p - s/n$, $1/r_1 - s_1/n < 1/q < 1/r_2 - s_2/n$, there exists a constant C which is only depending on n, p and q such that for $f \in \dot{B}^{s_1}_{r_1,\sigma_1} \cap \dot{B}^{s_2}_{r_2,\sigma_2}$, we have For , we have*

$$\|f\|_{\dot{B}^0_{q,\nu}} \le C\|f\|_{\dot{B}^s_{p,\rho}}\left(1 + \left(\frac{1}{\kappa}\log^+ \frac{\|f\|_{\dot{B}^{s_1}_{r_1,\sigma_1}} + \|f\|_{\dot{B}^{s_2}_{r_2,\sigma_2}}}{\|f\|_{\dot{B}^s_{p,\rho}}}\right)^{1/\rho'-1/\nu'}\right), \tag{4.1}$$

where $\kappa = \min(n(1/q - 1/r_1) + s_1, n(1/r_2 - 1/q) - s_2)$.

The above inequality is a sort of the interpolation inequality for functions in the Besov space. In fact the embedding

$$\dot{B}^0_{q,\nu} \subset \dot{B}^{s_1}_{r_1,\sigma_1} \cap \dot{B}^{s_2}_{r_2,\sigma_2}$$

is well known. The advantage of the above inequality is at the logarithmic order from the higher order norms. If $\nu < \rho$, then the inequality always holds without the extra logarithmic term. To compensate the deficiency for the second summability exponent ν to ρ, we need a higher regularity of order given by the logarithim. The extra regularity

$f \in \dot{B}^{s_1}_{r_1,\sigma_1}$ is devoted for the regularity of f around the low frequency and $f \in \dot{B}^{s_1}_{r_1,\sigma_1}$ for high frequency. The proof follows belows shows

$$\|f\|_{\dot{B}^0_{q,\nu}} \le C\|f\|_{\dot{B}^s_{p,\rho}}\left(1+\Big(\frac{1}{\kappa}\log^+\frac{\|\psi * f\|_{\dot{B}^{s_1}_{r_1,\sigma_1}}+\|(1-\psi)*f\|_{\dot{B}^{s_2}_{r_2,\sigma_2}}}{\|f\|_{\dot{B}^s_{p,\rho}}}\Big)^{1/\rho'-1/\nu'}\right). \tag{4.2}$$

This is a generalization to the known logarithmic inequalities in [4], [5], [2], [30] and [18] mentioned above (see more detailed discussions [17]).

Remark We may also have by the different choice of N that

$$\|f\|_{\dot{B}^0_{q,\nu}} \le C\left\{1+\|f\|_{\dot{B}^s_{p,\rho}}\Big(\frac{1}{\kappa}\log^+(\|f\|_{\dot{B}^{s_1}_{r_1,\sigma_1}}+\|f\|_{\dot{B}^{s_2}_{r_1,\sigma_2}})\Big)^{1/\rho'-1/\nu'}\right\} \tag{4.3}$$

under the same conditions.

5. Proof of the Uniqueness

Proof of Theorem 2.1 . Set $w = u - \tilde{u}$. We note that $w \in L^\infty([0,T]; L^2_\sigma \cap \dot{H}^1_\sigma) \cap L^2([0,T]; \dot{H}^1_\sigma)$. Since w satisfies

$$\begin{cases} \partial_t w + \Delta w - w\cdot\nabla w + w\cdot\nabla u + u\cdot\nabla w + \nabla(p-q) = 0, & t>0, x\in\mathbb{R}^n,\\ \text{div } w = 0, \quad t>0, x\in\mathbb{R}^n, & \\ w(t,0) = 0, & \end{cases} \tag{5.1}$$

in the sense of distribution, we have the following weak form

$$\frac{d}{dt}\|w(t)\|_2^2 + 2\|\nabla w\|_2^2 = (w\cdot\nabla u, w). \tag{5.2}$$

This process is in fact justified by the following argument.

Under the assumption $\omega = \text{rot } u \in L\log L([0,T]; \dot{B}^s_{p,\rho})$, it is possible to show that u belongs to $C^1((0,T]; H^s)$ for any $s>0$, i,e, u is smooth except $t=0$, and hence satisfies the energy equality:

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u(\tau)\|_2^2 d\tau = \|u_0\|_2^2, \tag{5.3}$$

(see Kozono-Taniuchi [18] and Kozono-Ogawa-Taniuchi [17]).

We note that the energy equality guarantees strong continuity of $u(t)$ for t in L^2 on $[0,T]$. On the other hand, by assumption, $\tilde{u}$ satisfies the energy inequality:

$$\|\tilde{u}(t)\|_2^2 + 2\int_0^t \|\nabla \tilde{u}(\tau)\|_2^2 d\tau \le \|u_0\|_2^2. \tag{5.4}$$

Hence we have the energy inequality for the difference $w(t)$;

$$\|w(t)\|_2^2 + 2\int_0^t \|\nabla w(\tau)\|_2^2 d\tau \le 2\int_0^t |(w\cdot\nabla u, w)| d\tau \tag{5.5}$$

Now we decompose the smoother solution u into the three parts in the phase variables such as

$$
\begin{aligned}
u(x) =&\psi_{-N} * u(x) + \sum_{|j|\le N} \phi_j * u(x) + \sum_{j>N} \phi_j * u(x) \\
=&u_l(x) + u_m(x) + u_h(x)
\end{aligned} \tag{5.6}
$$

Then by the Hausdorff-Young inequality, the low frequency part is estimated as

$$
\begin{aligned}
|(w\cdot\nabla u_l, w)| =&|(w\cdot\nabla w, u_l)| \\
\le&\|\psi_{-N} * \nabla(w\otimes w)\|_2\|u\|_2 \\
\le&C2^{-N(n+2)/2}\|w\|_2^2\|u\|_2
\end{aligned} \tag{5.7}
$$

The second term giving a core part of the solutions, can be bound by the logarithmic Sobolev inequality that for small $\varepsilon>0$ and $s=n/p$ with $s+=s+\varepsilon$ and $s-=s-\varepsilon$,

$$
\begin{aligned}
&|(w\cdot\nabla u_m,w)| \le \|w\|_2^2\|\nabla u_m\|_{\dot B^0_{\infty,1}} \\
&\le C\|w\|_2^2\|\nabla u\|_{\dot B^s_{p,\rho}}\Big\{1+\Big(\frac{1}{\varepsilon}\log^+\frac{2^{\varepsilon N}\|\nabla u_m^+\|_{\dot B^s_{p,\rho}}+2^{\varepsilon N}\|\nabla u_m^-\|_{\dot B^s_{p,\rho}}}{\|\nabla u\|_{\dot B^s_{p,\rho}}}\Big)^{1/\rho'}\Big\} \\
&\le CN^{1/\rho'}\|w\|_2^2\|\mathrm{rot}\ u\|_{\dot B^s_{p,\rho}}
\end{aligned} \tag{5.8}
$$

where we decompose $u_m = u_m^+ + u_m^- = \sum_{j\ge 0}\phi_j * u_m + \sum_{j<0}\phi_j * u_m$. While the last term is simply estimated by the Hausdorff-Young inequality that

$$
\begin{aligned}
|(w\cdot\nabla u_h, w)| =&|(w\cdot\nabla w, u_h)| \\
\le&\|w\|_2\|\nabla w\|_2\|(-\Delta)^{-1}\mathrm{rot}\ \mathcal{F}^{-1}(1-\hat\psi_N) * \sum_{j>N}\phi_j * \mathrm{rot}\ u)\|_\infty \\
\le&\|w\|_2\|\nabla w\|_2\|(-\Delta)^{-1}\nabla\mathcal{F}^{-1}(1-\hat\psi_N)\|_{\dot B^{-s}_{p',\rho'}}\|\mathrm{rot}\ u\|_{\dot B^s_{p,\rho}} \\
\le&C2^{-N}\|w\|_2\|\nabla w\|_2\|\mathrm{rot}\ u\|_{\dot B^s_{p,\rho}}
\end{aligned} \tag{5.9}
$$

Gathering the estimates (5.7)-(5.9) with (5.6) and choosing N properly large satisfying $2^{-N/2}\|u\|_2\le 1$,$2^{-N}\|\mathrm{rot}\ u\|_{\dot B^s_{p,\rho}}\simeq 1$, we see that

$$
|(w\cdot\nabla u, w)| \le C\|w\|_2^2(1+\|\mathrm{rot}\ u\|_{\dot B^s_{p,\rho}}(1+(\log^+\|\mathrm{rot}\ u\|_{\dot B^s_{p,\rho}})^{1/\rho'})+\|\nabla w\|_2^2 \tag{5.10}
$$

Hence we obtain from (5.5) and (5.10) that

(5.11)

$$
\begin{aligned}
\|w(t)\|_2^2+2\int_0^t\|\nabla w\|_2^2 \le&C\int_0^t\{\|w(\tau)\|_2^2(1+\|\mathrm{rot}\ u(\tau)\|_{\dot B^s_{p,\rho}}(1+(\log^+\|\mathrm{rot}\ u(\tau)\|_{\dot B^s_{p,\rho}})^{1/\rho'}) \\
&+\|\nabla w(\tau)\|_2^2\}d\tau
\end{aligned}
$$

and

$$\|w(t)\|_2^2 \leq C \int_0^t \{\|w(\tau)\|_2^2(1 + \|\text{rot } u(\tau)\|_{\dot{B}^s_{p,\rho}}(1 + (\log^+ \|\text{rot } u(\tau)\|_{\dot{B}^s_{p,\rho}})^{1/\rho'})\} d\tau \tag{5.12}$$

Now the Gronwall argument gives

$$\|w(t)\|_2^2 \leq C\|w(0)\|_2^2 \exp\Big(\int_0^t \{(\|\text{rot } u(\tau)\|_{\dot{B}^s_{p,\rho}}(\log^+ \|\text{rot } u(\tau)\|_{\dot{B}^s_{p,\rho}})^{1/\rho'})\} d\tau\Big) \tag{5.13}$$

The right hand side is 0 under the condition rot $u \in L(\log L)^{1/\rho'}([0,T]; \dot{B}^s_{p,\rho})$.

□

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