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New coefficient inequalities for starlike and convex functions

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Abstract. The object of the present paper is to derive new coefficient inequalities for univalent and starlike, and univalent and convex functions defined in the open unit disk U. Our results are the improvements of the previous theorems given by J. Clunie and F. R. Keogh ([1]) and by H. Silverman ([2]).

1 Introduction

Let A denote the class of functions $f(z)$ of the form

$$f(z)=\sum_{n=1}^{\infty}a_{n}z^{n} \qquad (a_{1}=1)$$

which are analytic in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$. A function $f(z)\in A$ is said to be univalent and starlike in U if it satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0$$

for all $z\in U$. Also a function $f(z)\in A$ is said to be univalent and convex in U if it satisfies

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0$$

for all $z\in U$.

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Clunie and Keogh ([1]) (also Silverman ([2])) have proved the following result: If $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} n\,|a_n| \leq 1,$$

then $f(z)$ is univalent and starlike in U. If $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} n^2\,|a_n| \leq 1,$$

then $f(z)$ is univalent and convex in U.
In the present paper, we consider new coefficient inequalities for functions $f(z)$ to be univalent and starlike, and univalent and convex in U.

2 Coefficient inequalities

Our main result for the coefficient inequality of $f(z)$ to be univalent and starlike in U is contained in

Theorem 1. *Let $f(z)$ be in the class A and*

$$\max_{n\geq 1} n\,|a_n| \;=\; p\,|a_p|\,.$$

If $f(z)$ satisfies

$$\sum_{n=1,n\neq p}^{\infty} (|n-p|+p)\,|a_n| \;\leq\; p\,|a_p|,$$

then $f(z)$ is univalent and starlike in U.

Proof. Applying the maximum principle of analytic functions, the following inequality folds true on $|z|=1$

$$|zf'(z)-pf(z)| \;-\; |pf(z)| \;=\; \left|\sum_{n=1}^{\infty}(n-p)a_n z^n\right| - p\left|\sum_{n=1}^{\infty} a_n z^n\right|$$

$$\leq\; \sum_{n=1}^{\infty} |n-p|\,|a_n|\,|z|^n \;-\; p\left(|a_p|\,|z|^p - \sum_{n=1,n\neq p}^{\infty} |a_n|\,|z|^n\right)$$

$$=\; \sum_{n=1,n\neq p}^{\infty} (|n-p|+p)\,|a_n| - p\,|a_p| \;\leq 0.$$

Therefore, it follows that

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p$$

for all $z \in U$. This shows that $f(z)$ is univalent and starlike in U. $\square$

Remark 1. If

$$\max_{n\geq 1} n\,|a_n| = |a_1| = 1,$$

then Theorem 1 becomes the result by Clunie and Keogh ([1]) (also by Silverman ([2]).

Corollary 1. *If a function $f(z) \in A$ satisfies*

$$\max_{n\geq 1} n\,|a_n| = 2\,|a_2|$$

and

$$\sum_{n=3}^{\infty} n\,|a_n| \leq 2\,|a_2| - 3,$$

then $f(z)$ is univalent and starlike in U.

By means of the definitions between starlike functions and convex functions, it follows that $f(z) \in A$ is univalent and convex in U if and only if $zf'(z)$ is univalent starlike in U. Therefore Theorem 1 gives us

Theorem 2. *Let $f(z)$ be in the class A and*

$$\max_{n\geq 1} n^2\,|a_n| = p^2\,|a_p|.$$

If $f(z)$ satisfies

$$\sum_{n=1,n\neq p}^{\infty} n(|n-p|+p)\,|a_n| \leq p^2\,|a_p|,$$

then $f(z)$ is univalent and convex in U.

Remark 2. If

$$\max_{n\geq 1} n^2\,|a_n| = |a_1| = 1,$$

then Theorem 2 becomes the result by Silverman ([2]).

Corollary 2. *If a function $f(z) \in A$ satisfies*

$$\max_{n\geq 1} n^2\,|a_n| = 4\,|a_2|$$

and

$$\sum_{n=3}^{\infty} n\,|a_n| \leq 4\,|a_2| - 3,$$

then $f(z)$ is univalent and convex in U.

References

[1] J.Clunie and F.R.Keogh, *On starlike and convex schlicht functions*, J. London Math. Soc. **35**(1960),229-233.

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