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SOME SUFFICIENT CONDITIONS FOR STRONGLY STARLIKENESS

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Abstract. *We consider strongly starlikeness of order* α *of functions* $f(z)=z+a_{n+1}z^{n+1}+\ldots$ *which are analytic in the unit disc and satisfy the condition of the form*

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}-1\right|<\lambda,\ 0<\mu<1,\ 0<\lambda<1.$$

1. INTRODUCTION AND PRELIMINARIES

Let H denote the class of functions analytic in the unit disc $U=\{z:|z|<1\}$ and let $A\subset H$ be the class of normalized analytic functions f in U such that $f(0)=f'(0)-1=0$. For $n\geq 1$ we put

$$A_n=\{f: f(z)=z+a_{n+1}z^{n+1}+\ldots \quad \textit{is analytic in } U\}$$

and $A_1=A$.

A function $f\in A$ is said to be *strongly starlike or order* α , $0<\alpha\leq 1$, if and only if

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha},$$

where $\prec$ denotes the usual *subordination.* We denote this class by $S(\alpha)$. If $\alpha=1$, then $S(1)\equiv S^*$ is the well-known class of *starlike functions* in U (see, for example, [1]).

In this paper we find a condition so that $f\in A_n$ satisfying

(1) $$f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu} \prec 1+\lambda z, \quad 0<\mu<1, 0<\lambda<1,$$

is in $S(\alpha)$. Also, we consider an integral transformation.

We note that the author in [4] determined the values for λ in (1) which implies starlikeness in U. Recently, Ponnusamy and Singh [5] found the condition which implies the strongly starlikeness of order α, but for $\mu<0$ in (1). By using the similar method as in [5] we consider strongly starlikeness in the case (1).

First, we cite the following

LEMMA A. *Let* $Q\in H$ *satisfy the subordination condition*

$$Q(z)\prec 1+\lambda_1 z, \quad Q(0)=1,$$

where $0<\lambda_1\leq 1$. *For* $0<\alpha\leq 1$, *let* $p\in H$, $p(0)=1$ *and* p *satisfy the condition*

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$$Q(z)p^{\alpha}(z) \prec 1+\lambda z, \quad 0<\lambda\le 1.$$

Then for

(2) $$\sin^{-1}\lambda+\sin^{-1}\lambda_1\le\frac{\alpha\pi}{2}$$

we have $\operatorname{Re}\{p(z)\}>0$ *in* U.

This is the special case of the more general lemma given in [5].

2. RESULTS AND CONSEQUENCES

For our results we also need the following two lemmas.

LEMMA 1. *Let* $p\in H$, $p(z)=1+p_n z^n+\ldots$, $n\ge 1$, *satisfy the condition*

$$p(z)-\frac{1}{\mu}zp'(z) \prec 1+\lambda z, \quad 0<\mu<1, \quad 0<\lambda\le 1.$$

Then

$$p(z)\prec 1+\lambda_1 z,$$

where

$$\lambda_1=\frac{\lambda\mu}{n-\mu}.$$

The proof of this lemma for n=1 is given by the author in [4]. For any $n\in N$ we also can apply Jack's lemma [3].

LEMMA 2. *If* $0<\mu<1$, $0<\lambda\le 1$ *and* $Q\in H$ *satisfying*

$$Q(z)\prec 1+\frac{\lambda\mu}{n-\mu}z, \quad Q(0)=1, \quad n\in N,$$

and if $p\in H$, $p(0)=1$ *and satisfies*

$$Q(z)p^{\alpha}(z)\prec 1+\lambda z,$$

where

(3) $$0<\lambda\le\frac{(n-\mu)\sin(\pi\alpha/2)}{\left|\mu+(n-\mu)e^{i\pi\alpha/2}\right|},$$

then $\operatorname{Re}\{p(z)\}>0$ *in* U.

Proof. If in Lemma A we put $\lambda_1=\dfrac{\lambda\mu}{n-\mu}$, then the condition (2) is equivalent to

$$\sin^{-1}\lambda+\sin^{-1}\frac{\lambda\mu}{n-\mu}\le\frac{\alpha\pi}{2}.$$

This inequality is equivalent to

$$\sin^{-1}\left(\lambda\sqrt{1-\frac{\lambda^2\mu^2}{(n-\mu)^2}}+\frac{\lambda\mu}{n-\mu}\sqrt{1-\lambda^2}\right)\le\frac{\alpha\pi}{2},$$

or to the inequality

$$\lambda\left[\sqrt{(n-\mu)^2-\lambda^2\mu^2}+\mu\sqrt{1-\lambda^2}\right]\le(n-\mu)\sin(\alpha\pi/2).$$

From there, after some transformations, we get the following equivalent inequality

$$\left(\left[\mu^2+(n-\mu)^2\right]^2-4\mu^2(n-\mu)^2\cos^2(\alpha\pi/2)\right)\lambda^4-2(n-\mu)^2[(\mu^2+(n-\mu)^2]\sin^2(\alpha\pi/2)\lambda^2+$$
$$+(1-\mu)^4\sin^4(\alpha\pi/2)\ge 0$$

which is equivalent to the condition (3).

By Lemma A we have that $\mathrm{Re}\{p(z)\}>0$ in U.

THEOREM 1. *Let* $f\in A_n$, $0<\mu<1$ *and* f *satisfy the subordination*

$$f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}\prec 1+\lambda z,$$

where

$$0<\lambda\le\frac{n-\mu}{\sqrt{\mu^2+(n-\mu)^2}}.$$

Then $f\in S^*$.

Proof. If we put $Q(z)=\left(\frac{z}{f(z)}\right)^{\mu}$ $(=1+q_nz^n+\ldots)$, then after some calculation, we get

$$Q(z)-\frac{1}{\mu}zQ'(z)=f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}\prec 1+\lambda z.$$

From Lemma 1 we have

$$Q(z)\prec 1+\lambda_1 z,\ \lambda_1=\frac{\lambda\mu}{n-\mu}.$$

The rest part of the proof is the same as in the case n=1 (Theorem 1 in [4]) and we omit the details.

THEOREM 2. *Let* $0<\mu<1$ *and* $0<\alpha\le 1$. *If* $f\in A_n$ *satisfies*

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}-1\right|<\frac{(n-\mu)\sin(\pi\alpha/2)}{\left|\mu+(n-\mu)e^{i\pi\alpha/2}\right|},\ z\in U, \tag{4}$$

then $f\in S(\alpha)$.

Proof. If we put $\lambda=\dfrac{(n-\mu)\sin(\pi\alpha/2)}{\left|\mu+(n-\mu)e^{i\pi\alpha/2}\right|}$, then, since $0<\alpha\le 1$, we have

$0<\lambda\leq\dfrac{n-\mu}{\sqrt{\mu^2+(n-\mu)^2}}$, and by Theorem 1, $f\in S^*$. Later, the function

$Q(z)=\left(\dfrac{z}{f(z)}\right)^{\mu}=1+q_nz^n+\ldots$ is analytic in U and satisfies $Q(z)\prec 1+\lambda_1 z$, $\lambda_1=\dfrac{\lambda\mu}{n-\mu}$.

Now, if we define

$$p(z)=\left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\alpha}},$$

then p is analytic in *U*, $p(0)=1$ and the condition (4) is equivalent to

$$Q(z)p^{\alpha}(z)\prec 1+\lambda z.$$

Finally, from Lemma 2 we obtain

$$\left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\alpha}}\prec\frac{1+z}{1-z}\quad\left(\Leftrightarrow\frac{zf'(z)}{f(z)}\prec\left(\frac{1+z}{1-z}\right)^{\alpha}\right),$$

i.e. $f\in S(\alpha)$. □

We note that for α=1 we have the statement of Theorem 1.

For $n=1, \mu=1/2, \alpha=2/3$ we get the following

COROLLARY 1. *Let f∈A and let*

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^{\frac{3}{2}}-1\right|<\frac{1}{2},\ z\in U.$$

Then

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right|<\frac{\pi}{3},\ z\in U,$$

i.e. f∈S(2/3).

THEOREM 3. *Let* $0<\mu<1$, $\mathrm{Re}\{c\}>-\mu$, *and* $0<\alpha\leq 1$. *If* $f\in A_n$ *satisfies*

(5) $$\left|f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}-1\right|<\left|\frac{n+c-\mu}{c-\mu}\right|\frac{(n-\mu)\sin(\pi\alpha/2)}{\left|\mu+(n-\mu)e^{i\pi\alpha/2}\right|},\ z\in U,$$

then the function

(6) $$F(z)=z\left[\frac{c-\mu}{z^{c-\mu}}\int_0^z\left(\frac{t}{f(t)}\right)^{\mu}t^{c-\mu-1}dt\right]^{-\frac{1}{\mu}}$$

belongs to S(α).

Proof. If we put that λ is equal to the right hand side of the inequality (5) and

$$Q(z)=F'(z)\left(\frac{z}{F(z)}\right)^{1+\mu} \quad \left(=1+q_n z^n+\dots\right)$$

then from (5) and (6) we obtain

$$Q(z)+\frac{1}{c-\mu}zQ'(z)=f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu} \prec 1+\lambda z \ .$$

Hence, by using the result of Hallenbeck and Ruscheweyh [2, Th.1] we have that

$$Q(z) \prec 1+\lambda_1 z \ , \ \lambda_1=\frac{|(c-\mu)|\lambda}{|n+c-\mu|}=\frac{(n-\mu)\sin(\pi\alpha/2)}{\left|\mu+(n-\mu)e^{i\pi\alpha/2}\right|} \ ,$$

and the desired result easily follows from Theorem 2.

REMARK 1 . For $\alpha=1$ and $n=1$ we have the corresponding result given earlier in [4].

For $c=\mu+1$, we have

COROLLARY 2. *Let* $0<\mu<1$ *and* $0<\alpha\le 1$. *If* $f\in A_n$ *satisfies the condition*

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}-1\right|<\frac{n(n-\mu)\sin(\pi\alpha/2)}{\left|\mu+(n-\mu)e^{i\pi\alpha/2}\right|} \quad , z\in U \ ,$$

then the function

$$F(z)=z\left[\frac{1}{z}\int_0^z\left(\frac{t}{f(t)}\right)^{\mu}dt\right]^{-\frac{1}{\mu}}$$

belongs to $S(\alpha)$.

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