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Alpha-spiral mappings of a Banach space into the complex plane

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Abstract

Let E be a complex Banach space and let B be the unit ball in E, i.e. $B=\{x\in E:\|x\|<1\}$. In this paper we define the class of α-spiral mappings of the unit ball B into the complex plane **C**.

1 Introduction

Let E^{*} be the dual space of E. For any $A\in E^{*}$ we consider $\chi(A)=\{x\in E:A(x)\neq 0\}$ and $\gamma(A)=E\setminus\chi(A)$. If $A\neq 0$ then $\chi(A)$ is dense in E and $\chi(A)\cap\hat{B}$ is dence in $\hat{B}$, where $\hat{B}=\{x\in E:\|x\|=1\}$.

Let $H(B)$ be the family of all functions $f:B\rightarrow\mathbf{C}, f(0)=0$, which are holomorphic in B, i.e. have the Fréchet derivative $f'(x)$ in each point $x\in B$. If $f\in H(B)$, then, in some neighbourhoods V of the origin, $f(x)=\sum_{m=1}^{\infty}P_{m,f}(x)$, where the series is uniformly convergent on V and $P_{m,f}:E\rightarrow\mathbf{C}$ are continuous and homogeneous polynomials of degree m.

Let $\alpha\in\mathbf{R}$ with $|\alpha|<\frac{\pi}{2}$ and let $z_0\in\mathbf{C}\setminus\{0\}$. The condition

$$z(t)=z_0e^{-(\cos\alpha+i\sin\alpha)t},\qquad t\in\mathbf{R}$$

defines an *α-spiral curve* in the complex plane.

Let D be a domain in **C**, such that $0\in D$. If for any $z_0\in D\setminus\{0\}$ the arc of α-spiral curve between the points z_0 and the origin is contained in D, then D is an *α-spiral domain with respect to the origin.*

Let $U = \{z \in \mathbf{C} : |z| < 1\}$. We denote by $SP(\alpha)$ the family of all univalent functions $f : U \to \mathbf{C}$,

$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are α-spiral in U, i.e. $f(U)$ is an α-spiral domain with respect to the origin.

Theorem 1 ([3]) *Let f be an holomorphic function from U into $\mathbf{C}$ such that $f(0) = 0$, $f'(0) = 1, f(z) \neq 0$, for all $z \in U \setminus \{0\}$ and let $\alpha \in \mathbf{R}$ with $|\alpha| < \frac{\pi}{2}$. Then $f \in SP(\alpha)$ if and only if*

$$\operatorname{Re}\left[e^{i\alpha}\frac{zf'(z)}{f(z)}\right] > 0 \qquad \textit{for all } z \in U.$$

2 The class of alpha-spiral mappings on a Banach space

Let $A \in E^*, A \neq 0$ and $\alpha \in R, |\alpha| < \frac{\pi}{2}$. We denote by $SP_A(\alpha)$ the family of all functions $f \in H(B)$ which have the form

$$f(x) = A(x) + \sum_{n=2}^{\infty} P_{n,f}(x) \tag{1}$$

such that, for any $a \in \chi(A) \cap \hat{B}, f$ is univalent on $B_a = \{za : z \in U\}$ and $f(B_a)$ is an α-spiral domain with respect to the origin.

For any function f of the form (1) and $a \in \chi(A) \cap \hat{B}$ we consider the function $f_a : U \to \mathbf{C}$.

$$f_a(z) = \frac{f(za)}{A(a)}, \qquad z \in U.$$

Obviously

$$f_a(z) = z + \sum_{n=2}^{\infty} \frac{P_{n,f}(a)}{A(a)} z^n, \qquad z \in U. \tag{2}$$

Moreover, it is easy to check that

$$f_a^{(n)}(z) = \frac{f^{(n)}(za)(a, ..., a)}{A(a)}, \qquad n \in \mathbf{N}, z \in U.$$

By using the properties of α-spiral functions in the unit disk, we obtain some estimations of $|P_{n,f}(a)|$ and $\|P_{n,f}\|$ in the class $SP_A(\alpha)$.

Theorem 2 *If $f \in SP_A(\alpha)$ and $a \in \hat{B}$, then*

$$|P_{n,f}(a)| \le \frac{|A(a)|}{(n-1)!} \prod_{k=1}^{n-1} \left[(k-1)^2 + 4k\cos^2\alpha\right]^{\frac{1}{2}}, \qquad n \ge 2 \tag{3}$$

This inequality is sharp and the equality holds for the function

$$f(x) = \frac{A(x)}{(1 - H(x))^{2s}}, \qquad x \in B$$

where $s = e^{-i\alpha}\cos\alpha, H \in E^*, H(a) = 1$ and $\|H\| = 1$.

Proof. Suppose that $f \in SP_A(\alpha)$ and $n \ge 2$. If $a \in \chi(A) \cap \hat{B}$, then $f_a \in SP(\alpha)$ and hence we get (3). If $a \in \gamma(A) \cap \hat{B}$, then evidently $a = \lim_{m\to\infty} a_m$, where $a_m \in \chi(A), m \in \mathbf{N}$. There exists $r_m \in /\!R_+$ such that $a_m/r_m \in \hat{B}$. Clearly $(r_m)_{m\ge 0}$ is bounded for 0 is an interior point of B.
Since $a_m/r_m \in \chi(A) \cap \hat{B}, m \in /\!\!N$, by the first part of the proof we have

$$\left|P_{n,f}\left(\frac{a_m}{r_m}\right)\right| \le \left|A\left(\frac{a_m}{r_m}\right)\right| \frac{1}{(n-1)!}\prod_{k=1}^{n-1}\left[(k-1)^2+4k\cos^2\alpha\right]^{\frac{1}{2}}, \qquad m \in \mathbf{N}.$$

Hence

$$|P_{n,f}(a_m)| \le r_m^{n-1}\frac{|A(a_m)|}{(n-1)!}\prod_{k=1}^{n-1}\left[(k-1)^2+4k\cos^2\alpha\right]^{\frac{1}{2}}, \qquad m \in \mathbf{N}.$$

By taking the limit with $m \to \infty$, we obtain $P_{n,f}(a) = 0$.

Corollary 1
Any $f \in SP_A(\alpha)$ vanishes on $\gamma(A) \cap B$.

Proof. Let $f \in SP_A(\alpha)$. Since $P_{n,f}(a) = 0$ for all $a \in \gamma(A) \cap \hat{B}$, f vanishes on B_a. Let $x \in \gamma(A) \cap B, x \neq 0$. Then $a = \frac{x}{\|x\|} \in \gamma(A) \cap \hat{B}$ and $f(za) = 0$ for all $z \in U$. Putting $z = \|x\|$, we get $f(x) = 0$.

Corollary 2 *If $f \in SP_A(\alpha)$ and $n \ge 2$, then*

$$\|P_{n,f}\| \le \frac{\|A\|}{(n-1)!}\prod_{k=1}^{n-1}\left[(k-1)^2+4k\cos^2\alpha\right]^{\frac{1}{2}} \tag{4}$$

The inequality is sharp, being attained by

$$f(x) = \frac{A(x)}{(1-H(x))^{2s}}, \qquad x \in B.$$

We shall give some necessary and sufficient conditions for holomorphic functions to belong to the class $SP_A(\alpha)$.

Theorem 3 *If* $f \in SP_A(\alpha)$ *,then*

$$\operatorname{Re}\left[e^{i\alpha}\frac{f'(x)(x)}{f(x)}\right] > 0, \qquad \textit{for any}\, x \in \chi(A) \cap B \tag{5}$$

Proof. Let $x \in \chi(A) \cap B, x \neq 0$. Then $a = \frac{x}{\|x\|} \in \chi(A) \cap \hat{B}$ and hence the function f_a belongs to the class $SP(\alpha)$ .We have

$$\operatorname{Re}\left[e^{i\alpha}\frac{zf_a'(z)}{f_a(z)}\right] > 0, \qquad z \in U.$$

From the equality

$$\frac{f'(za)(za)}{f(za)} = \frac{zf_a'(z)}{f_a(z)}, \qquad z \in U,$$

we obtain

$$\operatorname{Re}\left[e^{i\alpha}\frac{f'(za)(za)}{f(za)}\right] > 0, \qquad z \in U.$$

Putting $z = \|x\|$, we get (5).

Theorem 4 *Let* $f \in H(B), f'(0) = A$ *and* $f(x) \neq 0$*, for all* $x \in B \setminus \{0\}$. *If*

$$\operatorname{Re}\left[e^{i\alpha}\frac{f'(x)(x)}{f(x)}\right] > 0, \qquad x \in B$$

then $f \in SP_A(\alpha)$.

Proof. Let $a \in \chi(A) \cap \hat{B}$. Since $f_a(0) = 0, f_a'(0) = 1, f_a(z) \neq 0$, for all $z \in U \setminus \{0\}$ and

$$\operatorname{Re}\left[e^{i\alpha}\frac{zf_a'(z)}{f_a(z)}\right] = \operatorname{Re}\left[e^{i\alpha}\frac{f'(za)(za)}{f(za)}\right] > 0, \qquad z \in U,$$

we obtain that f_a is an α-spiral function in U. Then f is univalent in B_a and $f(B_a)$ is an α-spiral domain with respect to the origin. Hence $f \in SP_A(\alpha)$.

Remark

The above results can be generalized by replacing the unit ball B with a bounded and open set $D \subset E, D \neq \Phi$ such that $zD \subset D$, for $z \in \bar{U} = \{z \in \mathbf{C}, |z| \leq 1\}$. In this case, for $\alpha = 0$ some of the results due to E.Janiec [4] are obtained.

References

[1] Dobrovolska, K., Dziubinski, I., *On starlike and convex functions of many variables*, Demonstratio Math. 11(1978), 545-556.

[2] Dziubinski, I., Sitarski, R., *On classes of holomorph functions of many variables starlike and convex on some hypersurfaces*, Demonstratio Math. 13(1980), 619-632.

[3] Goodman, A. W., Univalent functions, Mariner Publishing Company, Inc., 1983

[4] Janiec, E., *On starlike and convex maps of a Banach space into the complex plane*, Demonstratio Math., XXVII, 2(1994), 511-516.

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