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FRACTIONAL CALCULUS OPERATOR AND ITS APPLICATIONS IN THE UNIVALENT FUNCTIONS

Jae Ho Choi and Yong Chan Kim
Department of Mathematics
Yeungnam University, 214-1
Daedong, Gyongsan 712-749, Korea

AND

Shigeyoshi Owa
Department of Mathematics
Kinki University, Higashi-Osaka
Osaka 577, Japan

ABSTRACT. In this paper we introduce the class $\mathcal{A}(\alpha,\beta,\gamma)$ consisting of analytic functions which is defined by using the fractional calculus operator $\mathcal{J}_z^\lambda$ in the unit disk $\mathcal{U}$. We shall determine the relationships of this class and well known classes $\mathcal{S}^*(\gamma)$ and $\mathcal{K}(\gamma)$ and investigate coefficient estimates and growth theorems for functions belonging to the class $\mathcal{A}(\alpha,\beta,\gamma)$. Integral operator F_c is also considered for the class $\mathcal{A}(\alpha,\beta,\gamma)$.

1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disk $\mathcal{U} = \{z : |z| < 1\}$. Also let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in the unit disk $\mathcal{U}$.

A function $f(z)$ belonging to the class $\mathcal{S}$ is said to be starlike of order γ $(0 \leq \gamma < 1)$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \gamma \qquad (z \in \mathcal{U}; 0 \leq \gamma < 1). \tag{1.2}$$

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We denote by $\mathcal{S}^*(\gamma)$ the subclass of $\mathcal{S}$ consisting of functions which are starlike of order γ in $\mathcal{U}$.

Further, a function $f(z)$ belonging to the class $\mathcal{S}$ is said to be convex of order γ $(0 \leq \gamma < 1)$ if and only if

$$\text{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \gamma \qquad (z \in \mathcal{U}; 0 \leq \gamma < 1). \tag{1.3}$$

We denote by $\mathcal{K}(\gamma)$ the subclass of $\mathcal{S}$ consisting of functions which are convex of order γ in $\mathcal{U}$.

We note that

$$f(z) \in \mathcal{K}(\gamma) \Leftrightarrow zf'(z) \in \mathcal{S}^*(\gamma), \tag{1.4}$$

and that $\mathcal{S}^*(\gamma) \subset \mathcal{S}^*(0) \equiv \mathcal{S}^*$ and $\mathcal{K}(\gamma) \subset \mathcal{K}(0) \equiv \mathcal{K}$ $(0 \leq \gamma < 1)$, where $\mathcal{S}^*$ and $\mathcal{K}$ denote the subclasses of $\mathcal{A}$ consisting of functions which are starlike and convex in $\mathcal{U}$, respectively.

Many essentially equivalent definitions of fractional calculus have been given in the literature (*cf.*, *e.g.*, [6] and [7, p.45]). We state the following definitions due to Owa and Srivastava [5] which have been used rather frequently in the theory of analytic functions (see also [3]).

Definition 1. The fractional integral of order λ is defined, for a function $f(z)$, by

$$\mathcal{D}_z^{-\lambda} f(z) := \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta \qquad (\lambda > 0), \tag{1.5}$$

and the fractional derivative of order λ is defined, for a function $f(z)$, by

$$\mathcal{D}_z^{\lambda} f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \qquad (0 \leq \lambda < 1), \tag{1.6}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ involved in (1.5) (and that of $(z-\zeta)^{-\lambda}$ involved in (1.6)) is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 2. Under the hypotheses of Definition 1, the fractional derivative of order $n+\lambda$ is defined by

$$\mathcal{D}_z^{n+\lambda} f(z) := \frac{d^n}{dz^n} \mathcal{D}_z^{\lambda} f(z) \qquad (0 \leq \lambda < 1; n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}). \tag{1.7}$$

With the aid of the above definitions, Owa and Srivastava [5] defined the fractional operator $\mathcal{J}_z^{\lambda}$ by

$$\mathcal{J}_z^{\lambda} f(z) = \Gamma(2-\lambda) z^{\lambda} \mathcal{D}_z^{\lambda} f(z) \qquad (\lambda \neq 2, 3, 4, \cdots) \tag{1.8}$$

for functions (1.1) belonging to the class $\mathcal{A}$.

We introduce the class $\mathcal{A}(\alpha,\beta,\gamma)$ of analytic functions $f(z)$ belonging to $\mathcal{A}$ satisfying the condition

$$\text{(1.9)} \qquad \operatorname{Re}\left(\frac{\mathcal{J}_z^{\alpha} f(z)}{\mathcal{J}_z^{\beta} f(z)}\right) > \gamma \qquad (z \in \mathcal{U}).$$

for $\alpha < 2$, $\beta < 2$ and $\gamma < 1$.

We note that $\mathcal{A}(1,0,\gamma) = \mathcal{S}^*(\gamma)$ and $\mathcal{A}(\alpha+1,0,\gamma) = \mathcal{S}^*(\gamma,\alpha)$ which was studied by Owa and Shen [4]. Also, for $\lambda < 1$ and $-\lambda/(1-\lambda) \leq \gamma < 1$, $\mathcal{A}(\lambda+1,\lambda,\gamma) = \mathcal{V}(2, 2-\lambda, (1-\lambda)\gamma+\lambda)$, which was studied by Kim and Srivastava [3].

In this paper, we find coefficient estimates and growth theorems for analytic functions belonging to the class $\mathcal{A}(\alpha,\beta,\gamma)$ associated with the fractional calculus operator. We also point out the relationships between the class $\mathcal{A}(\alpha,\beta,\gamma)$ and $\mathcal{S}^*(\gamma)$ (or $\mathcal{K}(\gamma)$).

2. Preliminary Results

In order to establish our results, we need the following lemmas.

Lemma 1. *Let the function $f(z)$ is defined by (1.1) and let $\lambda < 1$. Then*

$$\text{(2.1)} \qquad z(\mathcal{J}_z^{\lambda} f(z))' = (1-\lambda)\mathcal{J}_z^{\lambda+1} f(z) + \lambda \mathcal{J}_z^{\lambda} f(z) \qquad (z \in \mathcal{U}).$$

Proof. Using the definition of fractional calculus, we have

$$\text{(2.2)} \qquad \mathcal{J}_z^{\lambda} f(z) = z + \sum_{n=2}^{\infty} \phi(n,\lambda) a_n z^n,$$

where

$$\text{(2.3)} \qquad \phi(n,\lambda) = \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} \qquad (n \geq 2).$$

By applying (2.2), we obtain

$$\begin{aligned} z(\mathcal{J}_z^{\lambda} f(z))' &= z + \sum_{n=2}^{\infty} n\phi(n,\lambda) a_n z^n \\ &= (1-\lambda)\{z + \sum_{n=2}^{\infty} \phi(n,\lambda+1) a_n z^n\} + \lambda\{z + \sum_{n=2}^{\infty} \phi(n,\lambda) a_n z^n\} \end{aligned}$$

which completes the proof of Lemma 1.

Lemma 2. (Jack [2]) *Let $\omega(z)$ be analytic in $\mathcal{U}$ with $\omega(0) = 0$. Then if $|\omega(z)|$ attains its maximum value on the circle $|z| = r$ at a point z_0, we can write*

$$z_0\omega'(z_0) = k\omega(z_0),$$

where k is real and $k \geq 1$.

Lemma 3. (Srivastava and Owa [8]) *If the function $f(z)$ defined by* (1.1) *satisfies* $\operatorname{Re}(f(z)/z) > \delta$ $(0 \leq \delta < 1)$, *then*

$$\sum_{n=2}^{\infty} |a_n| \leq 1 - \delta. \tag{2.4}$$

The result (2.4) *is sharp.*

Lemma 4. (Twomey [10]) *Let the function $f(z)$ defined by* (1.1) *be in the class $\mathcal{S}^*$. Then*

$$\left|\frac{zf'(z)}{f(z)}\right| \leq 1 + \frac{|z| \ln\left(\frac{(1+|z|)^2|f(z)|}{|z|}\right)}{(1-|z|)\ln\left(\frac{1+|z|}{1-|z|}\right)} \qquad (z \in \mathcal{U}). \tag{2.5}$$

Equality in (2.5) *holds true for the Koebe function $\kappa(z) = z/(1-z)^2$.*

3. Main Results

We begin by proving

Theorem 1. *Let $\alpha < 2$, $\beta < 2$ and $\gamma < 1$. If $f(z) \in \mathcal{A}(\alpha, \beta, \gamma)$, then*

$$|a_n| \leq \frac{2(1-\gamma)}{|\phi(n,\alpha) - \phi(n,\beta)|} \prod_{j=2}^{n-1} \left(1 + \frac{2(1-\gamma)\phi(j,\beta)}{|\phi(j,\alpha) - \phi(j,\beta)|}\right) \qquad (n \geq 2), \tag{3.1}$$

where $\phi(n,\alpha)$ and $\phi(n,\beta)$ are given by (2.3). *The result is sharp.*

Proof. If we set

$$p(z) = \frac{\frac{\mathcal{J}_z^{\alpha} f(z)}{\mathcal{J}_z^{\beta} f(z)} - \gamma}{1-\gamma} = 1 + c_1 z + c_2 z^2 + \cdots \qquad (f \in \mathcal{A}), \tag{3.2}$$

then $p(z)$ is analytic with $p(0) = 1$ and has positive real part in $\mathcal{U}$. Since $\mathcal{J}_z^{\alpha} f(z) = ((1-\gamma)p(z) + \gamma)\mathcal{J}_z^{\beta} f(z)$, by virtue of (2.2), we have

$$(\phi(n,\alpha) - \phi(n,\beta))\, a_n = (1-\gamma)\left\{c_{n-1} + \sum_{m=2}^{n-1} \phi(m,\beta) c_{n-m} a_m\right\} \qquad (n \geq 2).$$

By applying Carathéodory's Lemma (see [1, p.41]), we obtain

$$|\phi(n,\alpha)-\phi(n,\beta)||a_n| \leq 2(1-\gamma)\left\{1+\sum_{m=2}^{n-1}\phi(m,\beta)|a_m|\right\}. \tag{3.3}$$

We will prove, using mathematical induction, that the assertion (3.1) is satisfied for $n \geq 2$. If $n=2$, then

$$|a_2| \leq \frac{2(1-\gamma)}{|\phi(2,\alpha)-\phi(2,\beta)|}.$$

Now suppose that the assertion (3.1) is satisfied for $n \leq k$. Then, from (3.1) and (3.3) we have

$$\begin{aligned}
&|\phi(k+1,\alpha)-\phi(k+1,\beta)||a_{k+1}| \\
&\quad\leq 2(1-\gamma)\left\{1+\sum_{m=2}^{k}\phi(m,\beta)|a_m|\right\} \\
&\quad\leq 2(1-\gamma)\left\{1+\sum_{m=2}^{k}\phi(m,\beta)\frac{2(1-\gamma)}{|\phi(m,\alpha)-\phi(m,\beta)|}\prod_{j=2}^{m-1}\left(1+\frac{2(1-\gamma)\phi(j,\beta)}{|\phi(j,\alpha)-\phi(j,\beta)|}\right)\right\} \\
&\quad= 2(1-\gamma)\prod_{j=2}^{k}\left(1+\frac{2(1-\gamma)\phi(j,\beta)}{|\phi(j,\alpha)-\phi(j,\beta)|}\right).
\end{aligned}$$

Hence

$$|a_n| \leq \frac{2(1-\gamma)}{|\phi(n,\alpha)-\phi(n,\beta)|}\prod_{j=2}^{n-1}\left(1+\frac{2(1-\gamma)\phi(j,\beta)}{|\phi(j,\alpha)-\phi(j,\beta)|}\right)$$

for all $n \geq 2$.

Finally, the result is sharp for the function $f(z)$ given by

$$f(z) = z+\frac{2(1-\gamma)}{|\phi(n,\alpha)-\phi(n,\beta)|}\prod_{j=2}^{n-1}\left(1+\frac{2(1-\gamma)\phi(j,\beta)}{|\phi(j,\alpha)-\phi(j,\beta)|}\right)z^n \qquad (n \geq 2).$$

Remark 1. Letting $\alpha=1$, $\beta=0$ and $\gamma=0$ in Theorem 1, we immediately obtain that

$$f(z) \in \mathcal{S}^* \Rightarrow |a_n| \leq n$$

for all $n \geq 2$ ([1, Theorem 2.14]).

Theorem 2. *Let $\lambda < 1$ and $-\lambda/(1-\lambda) \leq \gamma < 1$. Then $f(z) \in \mathcal{A}(\lambda+1, \lambda, \gamma)$ if and only if $\mathcal{J}_z^\lambda f(z) \in \mathcal{S}^*((1-\lambda)\gamma + \lambda)$.*

Proof. In view of Lemma 1, we have

$$\frac{z(\mathcal{J}_z^\lambda f(z))'}{\mathcal{J}_z^\lambda f(z)} = (1-\lambda)\frac{\mathcal{J}_z^{\lambda+1} f(z)}{\mathcal{J}_z^\lambda f(z)} + \lambda. \tag{3.4}$$

Assume that $f(z) \in \mathcal{A}(\lambda+1, \lambda, \gamma)$. Then, from (3.4) we obtain

$$\begin{aligned}\operatorname{Re}\left(\frac{z(\mathcal{J}_z^\lambda f(z))'}{\mathcal{J}_z^\lambda f(z)}\right) &= (1-\lambda)\operatorname{Re}\left(\frac{\mathcal{J}_z^{\lambda+1} f(z)}{\mathcal{J}_z^\lambda f(z)}\right) + \lambda \\ &> (1-\lambda)\gamma + \lambda.\end{aligned}$$

Thus $\mathcal{J}_z^\lambda f(z) \in \mathcal{S}^*((1-\lambda)\gamma + \lambda)$.

Conversely, suppose that $\mathcal{J}_z^\lambda f(z) \in \mathcal{S}^*((1-\lambda)\gamma + \lambda)$. In view of Lemma 1 and (3.4), it is clear that

$$\operatorname{Re}\left(\frac{\mathcal{J}_z^{\lambda+1} f(z)}{\mathcal{J}_z^\lambda f(z)}\right) > \gamma.$$

This completes the proof of Theorem 2.

By virtue of Theorem 2 and Lemma 4, we obtain

Corollary 1. *Let $\lambda < 1$ and $-\lambda/(1-\lambda) \leq \gamma < 1$. Then $zf'(z) \in \mathcal{A}(\lambda+1, \lambda, \gamma)$ if and only if $\mathcal{J}_z^\lambda f(z) \in \mathcal{K}((1-\lambda)\gamma + \lambda)$.*

Proof. By (1.8) and Theorem 2, it follows that

$$z(\mathcal{J}_z^\lambda f(z))' = \mathcal{J}_z^\lambda (zf'(z)) \in \mathcal{S}^*((1-\lambda)\gamma + \lambda).$$

Hence, from (1.4) we obtain $\mathcal{J}_z^\lambda f(z) \in \mathcal{K}((1-\lambda)\gamma + \lambda)$.

Corollary 2. *Let $\lambda < 1$ and let $f(z) \in \mathcal{A}(\lambda+1, \lambda, -\lambda/(1-\lambda))$. Then*

$$\left|\frac{z(\mathcal{J}_z^\lambda f(z))'}{\mathcal{J}_z^\lambda f(z)}\right| \leq 1 + \frac{|z| \ln\left(\frac{(1+|z|)^2 |\mathcal{J}_z^\lambda f(z)|}{|z|}\right)}{(1-|z|)\ln\left(\frac{1+|z|}{1-|z|}\right)} \qquad (z \in \mathcal{U}). \tag{3.5}$$

Equality in (3.5) holds true for the function $f(z) = h(z) * (z/(1-z)^2)$, where

$$h(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+\lambda-1)}{\Gamma(2-\lambda)n!} \quad z^n \qquad (z \in \mathcal{U})$$

and the operator $*$ stands for the Hadamard product or convolution of two regular functions.

With the aid of Lemma 1 and Lemma 2, we prove

Theorem 3. *Let $\lambda < 1$ and $0 \leq \delta < 1$. Suppose that $f(z) \in \mathcal{A}(\lambda+1, \lambda, \gamma(\delta))$, where*

$$\gamma(\delta) = \begin{cases} 1 - \frac{\delta}{2(1-\lambda)(1-\delta)} & (0 \leq \delta \leq \frac{1}{2}) \\ 1 - \frac{1-\delta}{2\delta(1-\lambda)} & (\frac{1}{2} \leq \delta < 1). \end{cases}$$

Then

$$\text{Re}\left(\frac{\mathcal{J}_z^{\lambda} f(z)}{z}\right) > \delta \qquad (z \in \mathcal{U}). \tag{3.6}$$

Proof. If we define the function ω by

$$\frac{\mathcal{J}_z^{\lambda} f(z)}{z} = \frac{1 + (2\delta - 1)\omega(z)}{1 + \omega(z)} \qquad (z \in \mathcal{U}), \tag{3.7}$$

then ω is analytic in $\mathcal{U}$ with $\omega(0) = 0$ and $\omega(z) \neq -1$. Making use of the logarithmic differentiation of both side in (3.7), we have

$$\frac{z(\mathcal{J}_z^{\lambda} f(z))'}{\mathcal{J}_z^{\lambda} f(z)} = 1 - \frac{2(1-\delta) z\omega'(z)}{(1+\omega(z))(1+(2\delta-1)\omega(z))}.$$

By Lemma 1, we obtain

$$\frac{\mathcal{J}_z^{\lambda+1} f(z)}{\mathcal{J}_z^{\lambda} f(z)} = 1 - \frac{2(1-\delta)}{1-\lambda} \frac{z\omega'(z)}{(1+\omega(z))(1+(2\delta-1)\omega(z))}.$$

Suppose that there exists a point $z_0 \in \mathcal{U}$ such that $\max_{|z| \leq |z_0|} |\omega(z)| = |\omega(z_0)| = 1$. Then, by using Lemma 2, we get

$$\text{Re}\left(\frac{\mathcal{J}_z^{\lambda+1} f(z_0)}{\mathcal{J}_z^{\lambda} f(z_0)}\right) = 1 - \frac{2k(1-\delta)}{1-\lambda} \frac{\delta}{|1+(2\delta-1)\omega(z_0)|^2}.$$

When $0 \leq \delta \leq 1/2$,

$$\text{Re}\left(\frac{\mathcal{J}_z^{\lambda+1} f(z_0)}{\mathcal{J}_z^{\lambda} f(z_0)}\right) \leq 1 - \frac{\delta}{2(1-\lambda)(1-\delta)}.$$

When $1/2 \leq \delta < 1$,

$$\text{Re}\left(\frac{\mathcal{J}_z^{\lambda+1} f(z_0)}{\mathcal{J}_z^{\lambda} f(z_0)}\right) \leq 1 - \frac{1-\delta}{2\delta(1-\lambda)}.$$

These contradict the hypothesis that $f(z) \in \mathcal{A}(\lambda+1, \lambda, \gamma(\delta))$. Hence $|\omega(z)| < 1$ for all $z \in \mathcal{U}$. Thus, from (3.7) we obtain the desired result.

Setting $\delta = 1/2$ in Theorem 3, we have

Corollary 3. *Let* $\lambda < 1$. *If* $f(z) \in \mathcal{A}(\lambda+1, \lambda, (1-2\lambda)/2(1-\lambda))$, *then*

$$\mathrm{Re}\left(\frac{\mathcal{J}_z^{\lambda} f(z)}{z}\right) > \frac{1}{2} \qquad (z \in \mathcal{U}).$$

Remark 2. Taking $\lambda = 0$ in Corollary 3, we see that $f(z) \in \mathcal{S}^*(1/2)$ implies $\mathrm{Re}(f(z)/z) > 1/2$. Since $\mathcal{K} \subset \mathcal{S}^*(1/2)$, Corollary 3 is a generalization of the result due to Strohhäcker [9] (see also Duren [1, p.72]).

Next, by using Lemma 3 and Theorem 3, we have

Corollary 4. *Under the hypotheses of Theorem 3, let the function* $f(z)$ *is defined by* (1.1). *Then*

$$|z| - (1-\delta)|z|^2 \leq |\mathcal{J}_z^{\lambda} f(z)| \leq |z| + (1-\delta)|z|^2 \tag{3.8}$$

for $z \in \mathcal{U}$. *Equality in all cases occurs for the function*

$$f(z) = z + \frac{(2-\lambda)(1-\delta)}{2} z^2 \exp(i\theta) \tag{3.9}$$

at $z = \pm|z|\exp(-i\theta)$.

Proof. Notice from (2.2), (3.6) and Lemma 3 that

$$\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} |a_n| \leq 1-\delta. \tag{3.10}$$

By using (2.2) and (3.10), we have

$$\begin{aligned} |\mathcal{J}_z^{\lambda} f(z)| &\geq |z| - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} |a_n||z|^n \\ &\geq |z| - |z|^2 \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} |a_n| \\ &\geq |z| - (1-\delta)|z|^2 \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} |\mathcal{J}_z^{\lambda} f(z)| &\leq |z| + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} |a_n||z|^n \\ &\leq |z| + (1-\delta)|z|^2 \end{aligned} \tag{3.12}$$

for $z \in \mathcal{U}$. Combining the inequalities in (3.11) and (3.12), we obtain Corollary 4.

Corollary 5. *Under the hypotheses of Theorem 3, let the function $f(z)$ is defined by* (1.1) *and let* $0 \leq \lambda < 1$. *Then*

$$|z| - \frac{(2-\lambda)(1-\delta)}{2}|z|^2 \leq |f(z)| \leq |z| + \frac{(2-\lambda)(1-\delta)}{2}|z|^2 \tag{3.13}$$

for $z \in \mathcal{U}$. This result is sharp with an extremal function $f(z)$ given by (3.9).

Proof. Observing that $\phi(n, \lambda)$ given by (2.3) is non-decreasing of n for fixed λ $(0 \leq \lambda < 1)$, we find from (3.10) that

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{(2-\lambda)(1-\delta)}{2}.$$

Hence, by using the same technique as detailed in the proof of Corollary 4, we obtain the assertion (3.13) of Corollary 5.

Finally, we state and prove

Theorem 4. *Let* $0 \leq \delta < 1$, $c \geq -\delta$, $c^2 + 2\delta(1+c) \geq 1$ *and* $1/(2(c+\delta)+1) \leq \gamma < 1$. *If* $f \in \mathcal{A}((\delta - 2\gamma + 1)/(1-\gamma), (\delta-\gamma)/(1-\gamma), \gamma - (1-\gamma)/2(c+\delta))$, *then the function* $F_c(z)$, *defined by*

$$F_c(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\,dt \qquad (f \in \mathcal{A}; z \in \mathcal{U}) \tag{3.14}$$

belongs to $\mathcal{A}((\delta - 2\gamma + 1)/(1-\gamma), (\delta-\gamma)/(1-\gamma), \gamma)$.

Proof. Let $\lambda = (\delta - \gamma)/(1-\gamma)$. From (3.14), we obtain

$$z\left(\mathcal{J}_z^{\lambda} F_c(z)\right)' + c\mathcal{J}_z^{\lambda} F_c(z) = (c+1)\mathcal{J}_z^{\lambda} f(z). \tag{3.15}$$

Define the function $\omega(z)$ by

$$\frac{z\left(\mathcal{J}_z^{\lambda} F_c(z)\right)'}{\mathcal{J}_z^{\lambda} F_c(z)} = \frac{1 + (2\delta - 1)\omega(z)}{1 + \omega(z)} \qquad (0 \leq \delta < 1; z \in \mathcal{U}). \tag{3.16}$$

Here $\omega(z)$ is analytic in $\mathcal{U}$ with $\omega(0) = 0$ and $\omega(z) \neq -1$. In view of (3.15), the assertion (3.16) yields

$$\frac{\mathcal{J}_z^{\lambda} f(z)}{\mathcal{J}_z^{\lambda} F_c(z)} = \frac{(1+c) + (2\delta - 1 + c)\omega(z)}{(1+c)(1+\omega(z))}. \tag{3.17}$$

Differentiating both side of (3.17) logarithmically, it follows that

$$\frac{z\left(\mathcal{J}_z^{\lambda} f(z)\right)'}{\mathcal{J}_z^{\lambda} f(z)} = \delta + (1-\delta)\frac{1-\omega(z)}{1+\omega(z)} - \frac{2(1-\delta)z\omega'(z)}{(1+\omega(z))(1+c+(2\delta-1+c)\omega(z))}. \tag{3.18}$$

By assuming $\max_{|z|\le|z_0|}|\omega(z)| = |\omega(z_0)| = 1$ for $z_0 \in \mathcal{U}$ and using the same technique as in the proof of Theorem 3, we find that (3.18) yields

$$\begin{aligned}\mathrm{Re}\left(\frac{z_0\left(\mathcal{J}_z^\lambda f(z_0)\right)'}{\mathcal{J}_z^\lambda f(z_0)}\right) &= \delta - \frac{2k(1-\delta)(c+\delta)}{|1+c+(2\delta-1+c)\omega(z_0)|^2}\\ &\le \delta - \frac{1-\delta}{2(c+\delta)}.\end{aligned}$$

This contradicts the assumption that $f(z) \in \mathcal{A}(\lambda+1,\lambda,\gamma-(1-\gamma)/2(c+\delta))$, that is,

$$\mathrm{Re}\left(\frac{z\mathcal{J}_z^{\lambda+1}f(z)}{\mathcal{J}_z^\lambda f(z)}\right) = \frac{1}{(1-\lambda)}\left[\mathrm{Re}\left(\frac{z\left(\mathcal{J}_z^\lambda f(z)\right)'}{\mathcal{J}_z^\lambda f(z)}\right) - \lambda\right] > \gamma - \frac{1-\gamma}{2(c+\delta)}$$

for $\lambda = (\delta-\gamma)/(1-\gamma)$. Therefore $\omega(z)$ has to satisfy that $|\omega(z)| < 1$ for all $z \in \mathcal{U}$. Hence, by (3.16) and Theorem 2, $\mathcal{J}_z^\lambda F_c(z) \in \mathcal{S}^*(\delta)$ and $F_c(z) \in \mathcal{A}(\lambda+1,\lambda,\gamma)$ for $\lambda = (\delta-\gamma)/(1-\gamma)$.

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