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Integral Means of the Fractional Derivative for Certain Starlike and Convex Functions of order α

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Abstract

In this paper we study a subclass of analytic functions consisting of functions of the form

$$f(z)=z-\sum_{k=n+1}^{\infty}e^{i(k-1)\vartheta}a_kz^k \quad (\vartheta \text{ real},\ a_k\geq 0;\ n\in N).$$

We show the integral means of the fractional derivative for starlike and convex functions of order $\alpha(0\leq\alpha<1)$ belonging to the subclass.

1 Introduction

Denote by A the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

that are analytic in the open unit disk $U=\{z:z\in C,\ |z|<1\}$, and by $A(n)$ the subclass of A consisting of all functions of the form

$$f(z)=z-\sum_{k=n+1}^{\infty}a_kz^k \quad (a_k\geq 0\ ;\ n\in N=\{1,2,3,\cdots\}). \tag{1.1}$$

We denote by $T(n)$ the subclass of $A(n)$ of univalent functions in U, further by $T_\alpha(n)$ and $C_\alpha(n)$ the subclasses of $T(n)$ consisting of functions which are starlike of order $\alpha(0\leq\alpha<1)$ and convex of order $\alpha(0\leq\alpha<1)$, respectively. These subclasses $T(n)$, $T_\alpha(n)$ and $C_\alpha(n)$ were introduced by Chatterjea[1]. When $n=1$ these notations are

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usually used as $T(1) = T$, $T_\alpha(1) = T^*(\alpha)$ and $C_\alpha(1) = C(\alpha)$, which were introduced earlier by Silverman[7]. Chatterjea[1] showed that a function $f(z)$ of the form (1.1) is in $T_\alpha(n)$ if and only if $\sum_{k=n+1}^{\infty}(k-\alpha)a_k \leq 1-\alpha$, and that a function $f(z)$ of the form (1.1) is in $C_\alpha(n)$ if and only if $\sum_{k=n+1}^{\infty} k(k-\alpha)a_k \leq 1-\alpha$. In the case of $n = 1$ these results coincide with Theorem 2 and Corollary 2 of Silverman[7], respectively.

Denote by $A(n, \vartheta)$ the subclass of A consisting of all functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} a_k z^k \quad (\vartheta \text{ real}, \ a_k \geq 0; \ n \in N)$$

(see, Sekine and Owa[6]).
We note that $A(n, 0) = A(n)$. We define the subclasses $T(n, \vartheta)$, $T_\alpha^*(n, \vartheta)$ and $C_\alpha(n, \vartheta)$ of $A(n, \vartheta)$ by the same way as those for the subclasses $T(n)$, $T_\alpha(n)$ and $C_\alpha(n)$ of $A(n)$, respectively. Then it is clear that $T(n, 0) = T(n)$, $T_\alpha^*(n, 0) = T_\alpha(n)$ and $C_\alpha(n, 0) = C_\alpha(n)$.

Sekine and Owa[6] proved that a function $f(z)$ in $A(n, \theta)$ is in $T_\alpha^*(n, \theta)$ if and only if

$$\sum_{k=n+1}^{\infty} (k-\alpha)a_k \leq 1-\alpha \tag{1.2}$$

and that a function $f(z)$ in $A(n, \vartheta)$ is in $C_\alpha(n, \vartheta)$ if and only if

$$\sum_{k=n+1}^{\infty} k(k-\alpha)a_k \leq 1-\alpha. \tag{1.3}$$

We note that the coefficient inequalities (1.2) and (1.3) do not contain ϑ and coincide with the coefficient inequalities for $T_\alpha(n)$ and $C_\alpha(n)$ of Chatterjea[1], respectively.

We have the following results needed later. Since the proofs are similar to those in [5], we omit the proofs(see, [5]).

Theorem 1. 1 *The extremal points of $T_\alpha^*(n, \vartheta)$ are functions*

$$f_1(z) = z \text{ and } f_k(z) = z - e^{i(k-1)\vartheta}\frac{1-\alpha}{k-\alpha}z^k \quad (k \geq n+1). \tag{1.4}$$

Theorem 1. 2 *The extremal points of $C_\alpha(n, \vartheta)$ are functions*

$$f_1(z) = z \text{ and } f_k(z) = z - e^{i(k-1)\vartheta}\frac{1-\alpha}{k(k-\alpha)}z^k \quad (k \geq n+1). \tag{1.5}$$

2 Fractional derivative and Subordination

In this section we recall the concepts of fractional derivative and subordination. Further we give several known results needed later.

Definition 2. 1 ([4]) *The fractional derivative of order* λ *is defined by*

$$D_z^\lambda f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^\lambda} d\xi \quad (0 \leq \lambda < 1),$$

where $f(z)$ *is an analytic function in a simple connected region of the z-plane containing the origin and the many-values of* $(z-\xi)^{-\lambda}$ *is removed by requiring* $\log(z-\xi)$ *to be real when* $z - \xi > 0$.

Remark 2. 1

(2.1) $$D_z^\lambda z^m = \frac{\Gamma(m+1)}{\Gamma(m+1-\lambda)} z^{m-\lambda} \quad (m \in N),$$

where $0 \leq \lambda < 1$.

For analytic functions $g(z)$ and $h(z)$ in U with $g(0) = h(0)$, $g(z)$ is said to be subordinate to $h(z)$ if exists an analytic function $w(z)$ so that $w(0) = 0, |w(z)| < 1 (z \in U)$ and $g(z) = h(w(z))$, we denote this subordination by $g(z) \prec h(z)$.

In 1925, Littlewood[3] proved the following subordination theorem.

Theorem 2. 1 ([3]) *If* g *and* f *are analytic in* U *with* $g \prec f$, *then for* $\lambda > 0$ *and* $0 < r < 1$,

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^\lambda d\theta \leq \int_0^{2\pi} \left|f(re^{i\theta})\right|^\lambda d\theta.$$

Making use of Theorem 2.1, Silverman[8] proved the following integral means for univalent function with negative coefficients.

Theorem 2. 2 ([8]) *Suppose* $f(z) \in T, \lambda > 0$, *and* $f_2(z) = z - z^2/2$. *Then for* $z = re^{i\theta}, 0 < r < 1$,

$$\int_0^{2\pi} |f(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_2(z)|^\lambda d\theta.$$

Further, Kim and Choi[2] showed the integral means of the fractional derivative for T, C, $T^*(\alpha)$ and $C(\alpha)$. In this paper, we show the integral means of the fractional derivative of order λ for the functions belonging to $T_\alpha^*(n;\vartheta)$ and $C_\alpha(n;\vartheta)$.

3 Results

Theorem 3. 1 *Suppose* $f(z) \in T_\alpha^*(n;\vartheta)$, $\beta > 0$, *and* $f_{n+1}(z)$ *is defined by* (1.4). *Then for* $z = re^{i\theta}$ *and* $0 < r < 1$,

$$\int_0^{2\pi} \left|D_z^\lambda f(z)\right|^\beta d\theta \leq \int_0^{2\pi} \left|D_z^\lambda f_{n+1}(z)\right|^\beta d\theta \quad (0 \leq \lambda < 1).$$

Proof. If $f(z) \in T^*_\alpha(n;\vartheta)$, then we have $f(z) = \sum_{k=0}^{\infty} e^{i(k-1)\vartheta} a_k z^k \quad (a_k \geq 0)$. By Remark 2.1 for the function $f(z)$, we have

$$D_z^\lambda f(z) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} \left(1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} k a_k \Phi(k) z^{k-1}\right),$$

where

$$\Phi(k) = \frac{\Gamma(k)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)} \quad (k \geq n+1).$$

Since $\Phi(k)$ is a non-increasing function of k, it follows that

$$0 < \Phi(k) \leq \Phi(n+1) = \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)}.$$

On the other hand, for the function

$$f_{n+1}(z) = z - e^{in\vartheta} \frac{1-\alpha}{n+1-\alpha} z^{n+1},$$

we have

$$D_z^\lambda f_{n+1}(z) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} \left(1 - \frac{e^{in\vartheta}(1-\alpha)\Gamma(2-\lambda)\Gamma(n+2)}{(n+1-\alpha)\Gamma(n+2-\lambda)} z^n\right).$$

To prove this theorem we must show that

$$\int_0^{2\pi} \left|1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} k a_k \Phi(k) z^{k-1}\right|^\beta d\theta \leq \int_0^{2\pi} \left|1 - \frac{e^{in\vartheta}(1-\alpha)\Gamma(2-\lambda)\Gamma(n+2)}{(n+1-\alpha)\Gamma(n+2-\lambda)} z^n\right|^\beta d\theta.$$

Since

$$\int_0^{2\pi} \left|1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} k a_k \Phi(k) z^{k-1}\right|^\beta d\theta \leq \int_0^{2\pi} \left|1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} (k-\alpha) a_k \Phi(k) z^{k-1}\right|^\beta d\theta,$$

by virtue of Theorem 2.1, it suffices to show that

$$1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} (k-\alpha) a_k \Phi(k) z^{k-1} \prec 1 - \frac{e^{in\vartheta}(1-\alpha)\Gamma(2-\lambda)\Gamma(n+2)}{(n+1-\alpha)\Gamma(n+2-\lambda)} z^n. \tag{3.1}$$

If we put

$$1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} (k-\alpha) a_k \Phi(k) z^{k-1} = 1 - \frac{e^{in\vartheta}(1-\alpha)\Gamma(2-\lambda)\Gamma(n+2)}{(n+1-\alpha)\Gamma(n+2-\lambda)} (w(z))^n,$$

then we have

$$(w(z))^n = \frac{(n+1-\alpha)\Gamma(n+2-\lambda)}{e^{in\vartheta}(1-\alpha)\Gamma(2-\lambda)\Gamma(n+2)} \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} (k-\alpha) a_k \Phi(k) z^{k-1}.$$

Therefore we have

$$\begin{aligned}|w(z)|^n &\leq \frac{(n+1-\alpha)\Gamma(n+2-\lambda)}{(1-\alpha)\Gamma(2-\lambda)\Gamma(n+2)}\sum_{k=n+1}^{\infty}(k-\alpha)a_k\Phi(k)|z|^{k-1}\\ &\leq \frac{(n+1-\alpha)\Gamma(n+2-\lambda)}{(1-\alpha)\Gamma(2-\lambda)\Gamma(n+2)}\Phi(n+1)|z|\sum_{k=n+1}^{\infty}(k-\alpha)a_k\\ &\leq \frac{n+1-\alpha}{(n+1)(1-\alpha)}|z|\sum_{k=n+1}^{\infty}(k-\alpha)a_k\\ &\leq \frac{n+1-\alpha}{n+1}|z|\sum_{k=n+1}^{\infty}\frac{k-\alpha}{1-\alpha}a_k.\end{aligned}$$

By applying the coefficient inequality (1.2) to the inequality above we have

$$|w(z)|^n \leq |z| < 1,$$

that is, $|w(z)| < 1$. Therefore we have the subordination (3.1).

Theorem 3. 2 *Suppose $f(z) \in C_\alpha(n;\vartheta)$, $\beta > 0$, and $f_{n+1}(z)$ is defined by* (1.5). *Then for $z = re^{i\theta}$ and $0 < r < 1$,*

$$\int_0^{2\pi}\left|D_z^\lambda f(z)\right|^\beta d\theta \leq \int_0^{2\pi}\left|D_z^\lambda f_{n+1}(z)\right|^\beta d\theta \quad (0 \leq \lambda < 1).$$

Proof. By the assumption, we note

$$f_{n+1}(z) = z - e^{in\vartheta}\frac{1-\alpha}{(n+1)(n+1-\alpha)}.$$

Also we note that

$$(n+1)\sum_{k=n+1}^{\infty}(k-\alpha)a_k \leq \sum_{k=n+1}^{\infty}k(k-\alpha)a_k \leq 1-\alpha,$$

that is,

$$\sum_{k=n+1}^{\infty}\frac{k-\alpha}{1-\alpha} \leq \frac{1}{n+1}.$$

By means of two notes above, we can prove this theorem by an argument similar to that in Theorem 3.1.

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