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Various inverse problems for univalent functions

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Abstract

The purpose of this note is to try to find necessity for famous theorems on univalency and convexity by using the Poisson formula. In Section 1, such theorems are stated. Section 2 is devoted to studying the radius of univalence for several theorems, and the radius of convexity for the theorems is investigated in Section 3. Several results are improved in Section 4.

1. Introduction

Let $n \in \mathcal{N} = \{1, 2, 3, \cdots\}$ and let $\mathcal{A}(\rho)$ denote the class of functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open disk $\mathcal{U}(\rho) = \{z : |z| < \rho\}$ with the radius ρ, so when $\rho = 1$, $\mathcal{U}(1)$ is the unit disk. Let $\mathcal{R}(\rho)$ be the subclass of $\mathcal{A}(\rho)$ consisting of $\mathrm{Re}\{f'(z)\} > 0$ in $\mathcal{U}(\rho)$. A function $f(z) \in \mathcal{A}(\rho)$ is called *univalent* if it never takes the same value twice; that is, if $f(z_1) \neq f(z_2)$ for all points z_1 and z_2 in a domain $\mathcal{D}$ of complex plane with $z_1 \neq z_2$. We denote by $\mathcal{S}(\rho)$ the subclass of $\mathcal{A}(\rho)$ consisting of univalent functions. Further, a function in $\mathcal{A}(\rho)$ is said to be *convex* if

$$1 + \mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} > 0 \quad \text{in} \quad \mathcal{U}(\rho).$$

Let $\mathcal{C}(\rho)$ denote the subclass of $\mathcal{A}(\rho)$ of such convex functions in $\mathcal{U}(\rho)$.

In 1934, Noshiro showed that

Theorem A. [2] *If $f(z)$ is analytic in a convex domain $\mathcal{D}$ and $\mathrm{Re}\{f'(z)\} > 0$ there, then $f(z)$ is univalent in $\mathcal{D}$.*

In 1962, MacGregor proved that

Theorem B. [3] *If $f(z) \in \mathcal{R}(1)$, then we have*

$$|a_n| \leq \frac{2}{n} \quad \textit{for} \quad n = 2, 3, \cdots$$

and

$$|f'(z)| \leq \frac{1 + |z|}{1 - |z|} \quad \textit{in} \quad \mathcal{U}(1).$$

In 1925, Littlewood showed

Theorem C. [1] *For any* $f(z) \in \mathcal{S}(1)$, *the inequality* $|a_n| < en$ *for* $n = 2, 3, \cdots$ *holds.*

In [4,5,6], the following results are established.

Theorem D. [5,6] *For any* $f(z) \in \mathcal{S}(1)$, *the inequality* $|a_n| \le n$ *holds for all* $n = 2, 3, \cdots$. *The equality is valid if and only if* $f(z)$ *is the Koebe function or its rotation.*

Theorem E. [4,5,6] *For each* $f(z) \in \mathcal{S}(1)$, *there hold the inequalities*

$$\frac{1-r}{(1+r)^3} \le |f'(z)| \le \frac{1+r}{(1-r)^3} \quad in \quad \mathcal{U}(1)$$

and

$$\frac{r}{(1+r)^2} \le |f(z)| \le \frac{r}{(1-r)^2} \quad in \quad \mathcal{U}(1).$$

For a fixed $z \in \mathcal{U}(1)$, $z \neq 0$, *the equality occurs if and only if* $f(z)$ *is the Koebe function or its rotation.*

Theorem E is called *the distortion theorem.*
In order to establish our results, we need the following lemmas.

Lemma 1. [7] *For* $0 < r < 1$, *the equality*

$$\int_0^{\pi} \frac{2r \sin t}{1 - 2r \cos t + r^2} dt = 2 \log \frac{1+r}{1-r}$$

holds.

Lemma 2. [4,5,6] *Let* $u(z)$ *be harmonic in* $|z| \le R$ *and continuous in* $|z| < R$. *Then* $u(z)$ *is given by the equation*

$$u(z) = u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \text{Re}\left\{u(Re^{i\varphi})\right\} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\varphi - \theta) + r^2} d\varphi, \tag{1.2}$$

where $0 \le r < R$.

The equation (1.2) is called *the Poisson formula.*

2. Radius of univalence

Proposition 1. *Let* $f(z) \in \mathcal{A}(1)$ *and*

$$|a_n| \le \frac{2}{n} \quad for \quad n = 2, 3, \cdots,$$

then $f(z) \in \mathcal{R}(1/3)$.

Proof. Differentiating (1.1), we find

$$\begin{aligned}\operatorname{Re}\{f'(z)\} &= \operatorname{Re}(1+2a_2z+3a_3z^2+\cdots+na_nz^{n-1}+\cdots)\\ &\geq 1-2|z|-2|z|^2-\cdots = 1-\frac{2|z|}{1-|z|}\\ &= \frac{1-|z|-2|z|}{1-|z|} = \frac{1-3|z|}{1-|z|}.\end{aligned}$$

To prove that $f(z) \in \mathcal{R}(\rho)$, we find a possible ρ under $\operatorname{Re}\{f'(z)\} > 0$, for which we put $1-3|z| > 0$. Then we have $|z| < 1/3$.

Proposition 2. *If $f(z) \in \mathcal{A}(1)$ and*

$$\frac{1-|z|}{1+|z|} \leq |f'(z)| \leq \frac{1+|z|}{1-|z|} \quad in \quad \mathcal{U}(1),$$

then we have $f(z) \in \mathcal{S}(\rho)$ with $\rho = 0.364184\cdots$.

Proof. Using the Poisson formula, we have

$$\log\{f'(z)\} = \frac{1}{2\pi}\int_0^{2\pi}\left(\log|f'(Re^{i\varphi})|\right)\left\{\frac{\xi+z}{\xi-z}\right\}d\varphi$$

for $z = re^{i\theta}$, $\xi = Re^{i\varphi}$, $0 < r < R < 1$. Taking the imaginary part in both sides of the equation, we have

$$\begin{aligned}|\arg\{f'(z)\}| &\leq \left\{\log\left(\frac{1+R}{1-R}\right)\right\}\frac{1}{2\pi}\int_0^{2\pi}\left|\operatorname{Im}\left\{\frac{\xi+z}{\xi-z}\right\}\right|d\varphi\\ &= \frac{2}{\pi}\left\{\log\left(\frac{1+R}{1-R}\right)\log\left(\frac{R+r}{R-r}\right)\right\}.\end{aligned}$$

To find a possible ρ under $|\arg f'(z)| < \pi/2$, we put

$$\frac{2}{\pi}\left\{\log\left(\frac{1+R}{1-R}\right)\log\left(\frac{R+r}{R-r}\right)\right\} = \frac{\pi}{2}. \tag{2.1}$$

Fixing various values of $0 < R < 1$, we solved the equation (2.1) for r by the aid of computer with the software 'Mathematica'. Then the maximum of such values r would be a possible ρ, and we found that $\rho = 0.364184\cdots$ is one of the value ρ.

Proposition 3. *If $f(z) \in \mathcal{A}(1)$ and $|a_n| \leq en$ ($n \in \mathcal{N}$), then we have $f(z) \in \mathcal{S}(\rho)$ with $\rho = 0.0766899\cdots$.*

Proof. By an easy calculation, we have

$$\begin{aligned}\operatorname{Re}\{f'(z)\} &> 1-2\times 2e|z|-3\times 3e|z|^2-\cdots-nen|z|^{n-1}-\cdots\\ &= 1-2^2e|z|-3^2e|z|^2-\cdots-n^2e|z|^{n-1}-\cdots\\ &= 1+e-e-2^2|z|-3^2|z|-\cdots-n^2e|z|^{n-1}-\cdots\\ &= \frac{e(-r^3+3r^2-4r)+(1-r)^3}{(1-r)^3}\end{aligned}$$

for $|z| = r$ $(0 \leq r < 1)$. To see that $f(z) \in \mathcal{S}(\rho)$, we find a possible ρ under $\mathrm{Re} f'(z) > 0$ for which we put $e(-r^3 + 3r^2 - 4r) + (1-r)^3 = 0$. In the same way as Proposition 2, ρ can be found by using a computer as $\rho = 0.0766899\cdots$.

Proposition 4. *If $f(z) \in \mathcal{A}(1)$ and $|a_n| \leq n$ $(n \in \mathcal{N})$, then we have $f(z) \in \mathcal{S}(\rho)$ with $\rho =$ $0.164878\cdots$.*

Proof. By the same way with Proposition 3, we obtain

$$\begin{aligned} \mathrm{Re}\{f'(z)\} &\geq 1 - 2|a_2||z| - 3|a_3||z|^2 - \cdots - n|a_n||z|^{n-1} - \cdots \\ &\geq 1 - 2\times 2|z| - 3\times 3|z|^2 - \cdots - n\times n|z|^{n-1} - \cdots \\ &= 1 - 2^2|z| - 3^2|z| - \cdots - n^2|z|^{n-1} - \cdots \\ &= 2 - \frac{1+|z|}{(1-|z|)^3} = \frac{1-7|z|+6|z|^2-2|z|^3}{(1-|z|)^3} \end{aligned}$$

for $|z| = r$ $(0 \leq r < 1)$. To find a possible ρ for $f(z) \in \mathcal{S}(\rho)$ under $\mathrm{Re} f'(z) > 0$, we put $1 - 7r + 6r^2 - 2r^3 = 0$. The same line with Proposition 2 leads that $\rho = 0.164878\cdots$.

Proposition 5. *If $f(z) \in \mathcal{A}(1)$ and*

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3} \quad \text{in} \quad \mathcal{U}(1),$$

then we have $f(z) \in \mathcal{S}(\rho)$ with $\rho = 0.232019\cdots$.

Proof. From the Poisson formula, we obtain

$$\log\{f'(z)\} = \frac{1}{2\pi}\int_0^{2\pi} (\log|f'(\xi)|) \left\{\frac{\xi+z}{\xi-z}\right\} d\varphi$$

for $z = re^{i\theta}$, $\xi = Re^{i\varphi}$, $0 \leq r < R < 1$. Then we have

$$\arg\{f'(z)\} = \frac{1}{2\pi}\int_0^{2\pi} \left(\log|f'(Re^{i\varphi})|\right) \mathrm{Im}\left\{\frac{\xi+z}{\xi-z}\right\} d\varphi.$$

Taking the absolute value in both sides and applying Lemma 1, we simply have

$$\begin{aligned} |\arg\{f'(z)\}| &\leq \frac{1}{2\pi}\int_0^{2\pi} \left|\log|f'(Re^{i\varphi})|\right| \left|\mathrm{Im}\left\{\frac{\xi+z}{\xi-z}\right\}\right| d\varphi \\ &\leq \left\{\log\left(\frac{1+R}{(1-R)^3}\right)\right\} \frac{1}{2\pi}\int_0^{2\pi} \left|\mathrm{Im}\left\{\frac{\xi+z}{\xi-z}\right\}\right| d\varphi \\ &= \left\{\log\left(\frac{1+R}{(1-R)^3}\right)\right\} \frac{1}{2\pi} \times 4\left\{\log\left(\frac{R+r}{R-r}\right)\right\} \\ &= \frac{2}{\pi}\left\{\log\left(\frac{1+R}{(1-R)^3}\right)\right\}\left\{\log\left(\frac{R+r}{R-r}\right)\right\}. \end{aligned}$$

To prove that $f(z) \in \mathcal{S}(\rho)$, we find a possible ρ under $|\arg f'(z)| \leq \frac{\pi}{2}$ for which we put

$$\frac{2}{\pi}\left\{\log\left(\frac{1+R}{(1-R)^3}\right)\right\}\left\{\log\left(\frac{R+r}{R-r}\right)\right\} = \frac{\pi}{2}. \tag{2.2}$$

Thus $\rho = 0.232019\cdots$.

Proposition 6. *If $f(z) \in \mathcal{A}(1)$ and*

$$\frac{1}{(1+|z|)^2} \leq |f'(z)| \leq \frac{1}{(1-|z|)^2} \quad \text{in} \quad \mathcal{U}(1),$$

then we have $f(z) \in \mathcal{S}(\rho)$ with $\rho = 0.356919\cdots$.

Proof. From the Poisson formula, then

$$\arg\{f'(z)\} = \frac{1}{2\pi}\int_0^{2\pi} \left(\log|f'(Re^{i\varphi})|\right) \operatorname{Im}\left\{\frac{\xi+z}{\xi-z}\right\} d\varphi.$$

Taking the absolute value in both sides and applying Lemma 1, we obtain

$$|\arg\{f'(z)\}| \leq \frac{2}{\pi}\left\{\log\left(\frac{1}{(1-R)^2}\right)\right\}\left\{\log\left(\frac{R+r}{R-r}\right)\right\}.$$

In order to find a possible ρ under $|\arg f'(z)| \leq \frac{\pi}{2}$, we put

$$\frac{2}{\pi}\left\{\log\left(\frac{1}{(1-R)^2}\right)\right\}\left\{\log\left(\frac{R+r}{R-r}\right)\right\} = \frac{\pi}{2}. \tag{2.3}$$

Thus ρ can be found as $\rho = 0.356919\cdots$.

3. Radius of convexity

Proposition 7. *If $f(z) \in \mathcal{A}(1)$ and*

$$\frac{1-|z|}{1+|z|} \leq |f'(z)| \leq \frac{1+|z|}{1-|z|} \quad \text{in} \quad \mathcal{U}(1),$$

then we have $f(z) \in \mathcal{C}(\rho)$ with $\rho = 0.194537\cdots$.

Proof. By means of the Poisson formula, we obtain

$$\log\{f'(z)\} = \frac{1}{2\pi}\int_0^{2\pi} \log|f'(Re^{i\varphi})| \left\{\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right\} d\varphi$$

for $z = re^{i\theta}$, $0 < r < R < 1$. Taking the differentiation in both sides of this equation and Lemma 2, and an easy calculation leads

$$\begin{aligned}
\left|\frac{zf''(z)}{f'(z)}\right| &= \left|\frac{1}{2\pi}\int_0^{2\pi} \log|f'(Re^{i\varphi})| \left\{\frac{2zRe^{i\varphi}}{(Re^{i\varphi}-z)^2}\right\} d\varphi\right| \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} \left|\log|f'(Re^{i\varphi})|\right| \left\{\frac{2|z|R}{|Re^{i\varphi}-z|^2}\right\} d\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left|\log|f'(Re^{i\varphi})|\right| \frac{2Rr}{R^2-2Rr\cos(\varphi-\theta)+r^2} d\varphi \\
&\leq \frac{2Rr}{2\pi}\left\{\log\left(\frac{1+R}{1-R}\right)\right\}\int_0^{2\pi} \frac{d\varphi}{R^2-2Rr\cos(\varphi-\theta)+r^2} \\
&= 2Rr\left\{\log\left(\frac{1+R}{1-R}\right)\right\}\left(\frac{1}{R^2-r^2}\right),
\end{aligned}$$

where we employed the formulas

$$\frac{1}{R^2 - 2Rr\cos(\varphi - \theta) + r^2} = \frac{1}{|Re^{i\varphi} - z|^2}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\varphi - \theta) + r^2} d\varphi = 1.$$

To show that $f(z) \in \mathcal{C}(\rho)$, let us find a possible ρ under $\left|\frac{zf''(z)}{f'(z)}\right| \leq 1$.

(3.1)
$$2Rr\left\{\log\left(\frac{1+R}{1-R}\right)\right\}\left(\frac{1}{R^2 - r^2}\right) = 1.$$

Then ρ is calculated to be $\rho = 0.194537\cdots$.

Proposition 8. *If $f(z) \in \mathcal{A}(1)$, $f'(z) \neq 0$ in $\mathcal{U}(1)$ and*

$$|\arg\{f'(z)\}| \leq \frac{1+|z|}{1-|z|} \quad in \quad \mathcal{U}(1),$$

then we have $f(z) \in \mathcal{C}(\rho)$ with $\rho = 0.329296\cdots$.

Proof. From the Poisson formula, we have

$$i\log\{f'(z)\} = \frac{1}{2\pi}\int_0^{2\pi} \arg\left\{f'(Re^{i\varphi})\right\}\left\{\frac{Re^{i\varphi} + z}{Re^{i\varphi} - z}\right\} d\varphi.$$

The differentiation in both sides of this equation and Lemma 2 lead

$$\begin{aligned}
-\frac{izf''(z)}{f'(z)} &= \frac{1}{2\pi}\int_0^{2\pi} \arg\left\{f'(Re^{i\varphi})\right\}\left\{\frac{2Re^{i\varphi}z}{(Re^{i\varphi} - z)^2}\right\} d\varphi \\
\left|\frac{zf''(z)}{f'(z)}\right| &\leq \frac{1}{2\pi}\int_0^{2\pi} \left|\arg\left\{f'(Re^{i\varphi})\right\}\right|\left\{\frac{2Rr}{|Re^{i\varphi} - z|^2}\right\} d\varphi \\
&\leq \frac{1}{2\pi}\left\{\frac{1+R}{1-R}\right\}\int_0^{2\pi} \frac{2Rr}{R^2 - 2Rr\cos(\varphi - \theta) + r^2} d\varphi \\
&= \frac{1}{2\pi}\left\{\frac{1+R}{1-R}\right\}\left\{\frac{2Rr}{R^2 - r^2}\right\}.
\end{aligned}$$

To show that $f(z) \in \mathcal{C}(\rho)$, we find a possible ρ under $\left|\frac{zf''(z)}{f'(z)}\right| \leq 1$ for which we put

(3.2)
$$\frac{1}{2\pi}\left\{\frac{1+R}{1-R}\right\}\left\{\frac{2Rr}{R^2 - r^2}\right\} = 1.$$

Then $\rho = 0.329296\cdots$.

Proposition 9. *$f(z) \in \mathcal{A}(1)$, $f'(z) \neq 0$ in $\mathcal{U}(1)$ and*

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3} \quad in \quad \mathcal{U}(1),$$

then we have $f(z) \in \mathcal{C}(\rho)$ *with* $\rho = 0.376934\cdots$.

Proof. From a simple calculation, we have

$$\frac{zf''(z)}{f'(z)} = \frac{1}{2\pi}\int_0^{2\pi} (\log|f'(\xi)|)\left\{\frac{2\xi z}{(\xi - z)^2}\right\} d\varphi.$$

and

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{1}{2\pi}\left\{\log\left(\frac{1+R}{(1-R)^3}\right)\right\}\left\{\frac{2Rr}{R^2 - r^2}\right\}.$$

In order to attempt for the left side of this inequalities to be less than 1, we put

(3.3) $$\frac{1}{2\pi}\left\{\log\left(\frac{1+R}{(1-R)^3}\right)\right\}\left\{\frac{2Rr}{R^2 - r^2}\right\} = 1,$$

which yields $\rho = 0.376934\cdots$.

Proposition 10. *If* $f(z) \in \mathcal{A}(1)$ *and*

$$\frac{1}{(1+|z|)^2} \leq |f'(z)| \leq \frac{1}{(1-|z|)^2} \quad \text{in} \quad \mathcal{U}(1),$$

then we have $f(z) \in \mathcal{C}(\rho)$ *with* $\rho = 0.489936\cdots$.

Proof. By the same way with the above Proposition, we simple have

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{1}{2\pi}\left\{\log\left(\frac{1}{(1-R)^2}\right)\right\}\left\{\frac{2Rr}{R^2 - r^2}\right\}.$$

Putting

(3.4) $$\frac{1}{2\pi}\left\{\log\left(\frac{1}{(1-R)^2}\right)\right\}\left\{\frac{2Rr}{R^2 - r^2}\right\} = 1,$$

we have $\rho = 0.489936\cdots$.

4. Improvement of Proposition 1

We now consider improvement of Proposition 1.

Proposition 11. *If* $f(z) \in \mathcal{A}(1)$ *and*

$$|a_n| \leq \frac{2}{n} \quad for \quad n = 2, 3, \cdots,$$

then

$$|f'(z) - k| < m \quad \text{in} \quad |z| < \frac{m - k + 1}{m - k + 3},$$

where $k \geq 1$, $1/2 < m \leq k$.

Proof. By means of (1.1), then we have

$$\begin{aligned} |f'(z)-k| &= \left|1-k+\sum_{n=2}^{\infty} na_n z^{n-1}\right| \\ &\leq |k-1|+\sum_{n=2}^{\infty} 2|z|^{n-1} \\ &= |k-1|+2|z|\frac{1}{1-|z|} \\ &= k-1+\frac{2r}{1-r} \\ &= \frac{k-1-(k-3)r}{1-r} < m \end{aligned}$$

where $r < (m-k+1)/(m-k+3)$, and we have the result.

Proposition 12. *If* $f(z) \in \mathcal{A}(1)$ *and*

$$|a_n| \leq \frac{2}{n} \quad for \quad n = 2, 3, \cdots,$$

then

$$|f'(z)-k| < m \quad in \quad |z| < \frac{m-k+1}{m-k+1},$$

where $1/2 < m \leq k \leq 1$.

Proof. By the similar way as Proposition 11, we have

$$\begin{aligned} |f'(z)-k| &= 1-k+\frac{2r}{1-r} \\ &= \frac{1-k+(1+k)r}{1-r} < m, \end{aligned}$$

where $r < (m-k+1)/(m+k+1)$.

For the case of $k = 1$ in Propositions 11 and 12, we obtain

Corollary *If* $f(z) \in \mathcal{A}(1)$ *and*

$$|a_n| \leq \frac{2}{n} \quad for \quad n = 2, 3, \cdots,$$

then

$$|f'(z)-1| < m \quad in \quad |z| < \frac{m}{m+2},$$

where $1 \geq m > 1/2$.

Remark If we put $m = 1$ in Corollary, we find that $f(z) \in \mathcal{R}(1/3)$, which means that Propositions 11 and 12 are improvements of Proposition 1.

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