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On meromorphically convex and starlike functions

MAMORU NUNOKAWA

Abstract. The object of the present paper is to show that a meromorphically convex function is a meromorphically starlike function.

1 Introduction

Let A be the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_{n}z^{n}$$

which are analytic in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$.
If $f(z)\in A$ satisfies the condition

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0 \qquad (z\in U), \tag{1.1}$$

then $f(z)$ maps U onto a starlike domain $f(U)$ with respect to the origin. Further if $f(z)\in A$ satisfies the condition

$$1+\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}\right\}>0 \qquad (z\in U), \tag{1.2}$$

then $f(z)$ maps U onto a convex domain.
A function $f(z)\in A$ is said to be starlike if it satisfies the condition (1.1), and convex if it satisfies the condition (1.2).
Marx [3] and Strohhäcker [8] proved the following result independently:
If $f(z)\in A$ satisfies

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$$1 + \text{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} > 0 \qquad (z \in U),$$

then

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{2} \qquad (z \in U).$$

Robertson [7] introduced the concepts of functions starlike and convex of order α as the following:
If $f(z) \in A$ satisfies the condition

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in U)$$

for $0 \leq \alpha < 1$, then $f(z)$ is said to be starlike of order α in U, and if $f(z) \in A$ satisfies the condition

$$1 + \text{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (z \in U)$$

for $0 \leq \alpha < 1$, then $f(z)$ is said to be convex of order α in U.
From this definitions and Marx-Strohhäcker's theorem, a convex function of order 0 is a starlike function of order at least 1/2.
About the convex functions of order α, Jack [1], MacGregor [2] and Wilken and Feng [9] obtained some results and following sharp result was obtained: If $f(z) \in A$ satisfies

$$1 + \text{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (z \in U),$$

where $0 \leq \alpha < 1$, then it follows that

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta(\alpha) \qquad (z \in U),$$

where

$$\beta(\alpha) \geq \frac{2\alpha - 1 + \sqrt{9 - 4\alpha + 4\alpha^2}}{4} \geq \frac{1}{2}$$

and

$$\beta(\alpha) = \begin{cases} (1-2\alpha)/2^{2-2\alpha}(1-2^{2\alpha-1}) & \alpha \neq 1/2 \\ 1/2\log 2 \qquad \alpha = 1/2. \end{cases} \tag{1.3}$$

Next, let B be the class of functions of the form

$$g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n$$

which are analytic in the punctured disk $E = \{z \in \mathbb{C} : 0 < |z| < 1\}$.
If $g(z) \in B$ satisfies $g(z) \neq 0$ in E and

$$-\mathrm{Re}\left\{\frac{zg'(z)}{g(z)}\right\} > 0 \qquad (z \in U), \tag{1.4}$$

then $g(z)$ is said to be meromorphically starlike in this case, $g(z)$ is univalent in E and the complement of $g(E)$ is a starlike domain with respect to the origin [4].
If $g(z) \in B$ satisfies $g \neq 0$ in E and

$$-\left(1 + \mathrm{Re}\left\{\frac{zg''(z)}{g'(z)}\right\}\right) > 0 \qquad (z \in U), \tag{1.5}$$

then $g(z)$ is said to be meromorphically convex, and $g(z)$ is univalent in E and the complement of $g(E)$ ia a convex domain.
If $g(z) \in B$ satisfies $g(z) \neq 0$ in E and

$$-\mathrm{Re}\left\{\frac{zg'(z)}{g(z)}\right\} > \alpha \qquad (z \in U)$$

where $0 \leq \alpha < 1$,then $g(z)$ is said to be meromorphically starlike of order α, and if $g(z) \in B$ satisfies $g \neq 0$ in E and

$$-\left(1 + \mathrm{Re}\left\{\frac{zg''(z)}{g'(z)}\right\}\right) > \alpha \qquad (z \in U),$$

then $g(z)$ is said to be meromorphically convex of order α.
It is natural that we will expect to obtain almost same results between the meromorphically convex and starlike functions as the relationship between univalent convex and starlike functions.
Nevertheless, there is no interesting and important results between meromorphically convex and starlike functions of order α. After the Jack's result [1], the author tried it many times, but failed frustrated and abandoned again and again but at last, it will be settled. The result is very very easy and it will be settled by simply calculation.

2 Main theorem

Lemma 1. *Let $p(z)$ be analytic in U, $p(0)=1$ and suppose that*

$$\mathrm{Re}\left\{p(z) - \frac{zp'(z)}{p(z)}\right\} > 0 \qquad (z \in U). \tag{2.1}$$

Then we have $\mathrm{Re}p(z) > 0$ *in* U.

Proof. Applying the same method as [5, Lemma 1] and from hypothesis (2.1), we have $p(z) \neq 0$ in U. If there exists a point $z_0 \in U$ such that

$$\mathrm{Re}p(z) > 0 \qquad \text{for } |z| < |z_0| < 1$$

and $\mathrm{Re}p(z_0) = 0$, then from [6], we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik,$$

where k is real and $|k| \geq 1$. Then we have

$$\mathrm{Re}\left\{p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)}\right\} = 0.$$

This contradicts (2.1) and therefore we have $\mathrm{Re}p(z) > 0$ in U. □

Theorem 1. *A meromorphically convex function in U is a meromorphically starlike function in U.*
There exists a function $g(z) \in B$ which is a meromorphically convex function of order 0 and simultaneously meromorphically starlike function of order 0.
On the contrary of the results (1.3) for analytic convex and starlike functions, for arbitrary $\alpha, 0 \leq \alpha < 1$, there exists a function $g(z) \in B$ which is a meromorphically convex function of order α and simultaneously meromorphically starlike function of order 1/2.

Proof. Let us put $g(z) \in B, g(z) \neq 0$ in E and

$$p(z) = -\frac{zg'(z)}{g(z)}.$$

Then it follows that

$$-\left(1 + \frac{zg''(z)}{g'(z)}\right) = p(z) - \frac{zp'(z)}{p(z)}.$$

From Lemma 1, if $g(z)$ is meromorphically convex, then $g(z)$ is simultaneously meromorphically starlike.
Next, let us put

$$p(z) = \frac{1+z}{1-z},$$

then it follows that

$$\mathrm{Re}\,p(z) > 0 \qquad (z \in U), \quad \mathrm{Re}\,p(z) = 0, \quad (|z| = 1, z \neq 1), \tag{2.2}$$

$$p(z) - \frac{zp'(z)}{p(z)} = \frac{1+z^2}{1-z^2}, \tag{2.3}$$

$$\mathrm{Re}\left\{p(z) - \frac{zp'(z)}{p(z)}\right\} > 0 \qquad (z \in U), \tag{2.4}$$

and

$$\mathrm{Re}\left\{p(z) - \frac{zp'(z)}{p(z)}\right\} = \mathrm{Re}\left(\frac{1+z^2}{1-z^2}\right) = 0 \quad (|z| = 1, z \neq \pm 1). \tag{2.5}$$

Putting $p(z) = 1/(1-z)$, we have

$$\mathrm{Re}\,p(z) > \frac{1}{2} \quad (z \in U), \qquad \mathrm{Re}\,p(z) = \frac{1}{2} \quad (|z| = 1, z \neq 1), \tag{2.6}$$

and

$$p(z) - \frac{zp'(z)}{p(z)} = \frac{1-z}{1-z} = 1. \tag{2.7}$$

From (2.2), (2.3), (2.4), (2.5), (2.6), and (2.7), we complete the proof of the theorem. □

Remark 1. *For $0 < \alpha < 1$, from (1.3), we have $\beta(\alpha) > 1/2$.*

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Mamoru Nunokawa
Department of mathematics
University of Gunma
Aramaki, Maebashi, Gunma 371-8510
Japan