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A Variational Problem Governed by a Differential Inclusion in a Banach Space

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1 Introduction

Let $\mathcal{X}$ be a real separable reflexive Banach space. A correspondence (= multi-valued mapping) $\Gamma : [0,T] \times \mathcal{X} \twoheadrightarrow \mathcal{X}$ and a function $u : [0,T] \times \mathcal{X} \times \mathcal{X} \rightarrow \overline{\mathbb{R}}$ are assumed to be given. A double arrow $\twoheadrightarrow$ indicates the domain and the range of a correspondence. The compact interval $[0,T]$ is endowed with the Lebesgue measure dt. $\mathcal{L}$ denotes the σ-field of the Lebesgue-measurable sets of $[0,T]$.

Let $\mathcal{W}^{1,p}([0,T],\mathcal{X})$ be the Sobolev space consisting of functions of $[0,T]$ into $\mathcal{X}$ (cf. Appendix) And let $\triangle(a)$ be the set of all the solutions in the Sobolev space $\mathcal{W}^{1,p}([0,T],\mathcal{X})$ of a differential inclusion :

$$(*) \qquad \dot{x}(t) \in \Gamma(t, x(t)),\ x(0) = a,$$

where $\dot{x}$ denotes the derivative of x and a is a fixed vector in $\mathcal{X}$. And consider a variational problem :

$$(\sharp) \qquad \text{Minimize}_{x \in \triangle(a)} \int_0^T u(t, x(t), \dot{x}(t))dt.$$

The object of this paper is to discuss a couple of existence problems as follows :

(i) the existence of a solution for the differential inclusion $(*)$, and

(ii) the existence of an optimal solution for the variational problem $(\sharp)$.

In Maruyama [14] [15], I presented a solution of these problems in the special case $\mathcal{X} = \mathbb{R}^\ell$ by making use of the convenient properties of the weak convergence in the Sobolev space $\mathcal{W}^{1,2}([0,T],\mathbb{R}^\ell)$; i.e. if a sequence $\{x_n\}$ in $\mathcal{W}^{1,2}([0,T],\mathbb{R}^\ell)$, weakly converges to some $x^* \in \mathcal{W}^{1,2}([0,T],\mathbb{R}^\ell)$, then there exists a subsequence $\{z_n\}$ of $\{x_n\}$ such that

$$\begin{aligned} z_n &\longrightarrow x^* \quad \text{uniformly on } [0,T], \text{ and} \\ \dot{z}_n &\longrightarrow \dot{x}^* \quad \text{weakly in } \mathcal{L}^2([0,T],\mathbb{R}^\ell). \end{aligned} \qquad (W)$$

However it deserves a special notice that this property does not hold in the space $\mathcal{W}^{1,2}([0,T],\mathcal{X})$ if $\dim \mathcal{X} = \infty$. Taking account of this fact, I provided a new convergence result to overcome this difficulty in the case $\mathcal{X}$ is a real separable Hilbert space in Maruyama [17]. And I also gave a existence theory for the problems (i) and (ii) being based upon this new tool in the framework of a separable Hilbert space in Maruyama [17],[18].

The purpose of the present paper is to generalize my previous results to the case $\mathcal{X}$ is a real separable reflexive Banach space. Papageorgiou [19] also gave an elegant extension of my results in Maruyama [14],[15] to the infinite dimensional case. The present paper might be regarded as an alternative approach to Papageogiou's theory.

Let me mention about another improvement added on this occasion. In Maruyama [17], I imposed a very restrictive requirement on the continuity of the correspondence Γ ; i.e.

> the correspondence $x \mapsto\!\!\!\to \Gamma(t,x)$ is upper hemi-continuous for each fixed $t \in [0,T]$ with respect to the weak topology for the domain and the strong topology for the range.

I have to admit frankly that this is a very unpleasant assumption. In the present paper, I propose the upper hemi-continuity of $x \mapsto\!\!\!\to \Gamma(t,x)$ with respect to the "weak-weak" combination of topologies instead of the "weak-strong" combination.

2 A Convergence Theorem in $\mathcal{W}^{1,p}([0,T],\mathcal{X})$

As I have already said, any weakly convergent sequence $\{x_n\}$ in the Sobolev space $\mathcal{W}^{1,2}([0,T],\mathbb{R}^\ell)$ has a subsequence which satisfies the property (W) in section 1.

On the other hand, let $\mathcal{X}$ be a real Banach space with the Radon-Nikodým property (RNP). Then any absolutely continuous function $f : [0,T] \to \mathcal{X}$ is Fréchet-differentiable a.e. (If the Banach space $\mathcal{X}$ does not have RNP, this property does not hold. For a counter-example, see Komura [13].) Let $\{x_n\}$ be a sequence in $\mathcal{W}^{1,p}([0,T],\mathcal{X})$ which weakly converges to some $x^* \in \mathcal{W}^{1,p}([0,T],\mathcal{X})$.

We should keep in mind that it is not necessarily true that the sequence $\{x_n\}$ has a subsequence $\{z_n\}$ which satisfies the propery (W) if $\dim \mathcal{X} = \infty$ even in the case $p = 2$.

Counter-Example (Cecconi[9], pp.28-29) Let $\mathcal{H}$ be a real separable Hilbert space and $\{\varphi_n; n = 1, 2, \cdots\}$ a complete orthonormal system of $\mathcal{H}$. (cf. Yosida [28] P.89.) Define a sequence $\{x_n : [0, T] \to \mathcal{H}\}$ by

$$x_n(t) = t\varphi_n \quad (n = 1, 2, \cdots).$$

We also define the function $x^* : [0, 1] \to \mathcal{H}$ by $x^*(t) \equiv 0$. Then x_n's as well as x^* are elements of $\mathcal{W}^{1,2}([0, T], \mathcal{H})$. It follows from the Riemann-Lebesgue lemma that the sequence $\{x_n\}$ weakly converges to x^* in $\mathcal{W}^{1,2}([0, 1], \mathcal{H})$. However there is no subsequence of $\{x_n\}$ which converges strongly (hence uniformly) to x^* in $\mathcal{L}^2([0, 1], \mathcal{H})$.

The following theorem cultivated to overcome this difficulty is a generalization of Theorem 1 of Maruyama [18].

Henceforth we denote by $\mathcal{X}_s$ (resp.$\mathcal{X}_w$) a Banach space $\mathcal{X}$ endowed with the strong (resp. weak) topology.

THEOREM 1. Let $\mathcal{X}$ be a real separable reflexive Banach space. And consider a sequence $\{x_n\}$ in the Sobolev space $\mathcal{W}^{1,p}([0, T], \mathcal{X})(p \geqq 1)$. Assume that

(i) the set $\{x_n(t)\}_{n=1}^{\infty}$ is bounded (and hence relatively compact) in $\mathcal{X}_w$ for each $t \in [0, T]$, and

(ii) there exists some function $\psi \in \mathcal{L}^p([0, T], (0, +\infty))$ such that

$$\| \dot{x}_n(t) \| \leqq \psi(t) \quad \text{a.e.}$$

Then there exists a subsequence $\{z_n\}$ of $\{x_n\}$ and some $x^* \in \mathcal{W}^{1,p}([0, T], \mathcal{X})$ such that

(a) $z_n \to x^*$ uniformly in $\mathcal{X}_w$ on $[0, T]$, and

(b) $\dot{z}_n \to \dot{x}^*$ weakly in $\mathcal{L}^p(0, T], \mathcal{X})$.

Remark Since $\mathcal{X}$ is separable and reflexive, the following results holds true. Assume that $p \geqq 1$.

[I] $\mathcal{L}^p([0, T], \mathcal{X})$ is separable.

[II] $\mathcal{L}^p([0,T],\mathcal{X})'$ is isomorphic to $\mathcal{L}^q([0,T],\mathcal{X}')$, where $1/p+1/q=1$ and " ' " denotes the dual space.

[III] Any absolutely continuous function $f:[0,T]\to\mathcal{X}$ is Fréchet-differentiable a.e. and the "fundamental theorem of calculus" , i.e.

$$f(t)=f(0)+\int_0^t \dot{f}(\tau)d\tau\,;\ t\in[0,T]$$

is valid.

Proof of Theorem 1. (a) To start with, we shall show the equicontinuity of $\{x_n\}$. Since ψ is integrable, there exists some $\delta>0$ for each $\varepsilon>0$ such that

$$\| x_n(t)-x_n(s)\| \leqq \int_s^t \|\dot{x}_n(\tau)\| d\tau \leqq \int_s^t \psi(\tau)d\tau \leqq \varepsilon \quad \text{for all} \quad n$$

provided that $|t-s|\leqq\delta$. This proves the equicontinuity of $\{x_n\}$ in the strong topology for $\mathcal{X}$. Hence $\{x_n\}$ is also equicontinuous in the weak topology for $\mathcal{X}$.

Taking account of this fact as well as the assumtion (i), we can claim, thanks to the Ascoli-Arzelà theorem (cf. Schwartz[21] p.78), that $\{x_n\}$ is relatively compact in $C([0,T],\mathcal{X}_w)$ (the set of continuous functions of $[0,T]$ into $\mathcal{X}_w$) with respect to the topology of uniform convergence.

By the assumption (i), $\{x_n(0)\}$ is bounded in $\mathcal{X}$, say

$$\sup_n \| x_n(0)\| \leqq C<+\infty.$$

And the assumption (ii) implies that

$$\| \int_0^t \dot{x}_n(\tau)d\tau \| \leqq \|\psi\|_1 \quad \text{for all} \quad t\in[0,T].$$

Hence

$$\sup_n \| x_n(t)\| = \sup_n \| x_n(0)+\int_0^t \dot{x}_n(\tau)d\tau \| \quad \leqq \quad C+\|\psi\|_1$$
$$\text{for all} \quad t\in[0,T].$$

Thus each x_n can be regarded as a mapping of $[0,T]$ into the set

$$M=\{w\in\mathcal{X}\mid \|w\|\leqq C+\|\psi\|_1\}.$$

The weak topology on M is metrizable because M is bounded and $\mathcal{X}$ is a separable reflexive Banach space. Hence if we denote by M_w the space M endowed with the weak topology, then the uniform convergence topology on $\mathcal{C}([0,T],M_w)$ is metrizable.

Since we can regard $\{x_n\}$ as a relatively compact subset of $\mathcal{C}([0,T],M_w)$, there exists a subsequence $\{y_n\}$ of $\{x_n\}$ which uniformly converges to some $x^* \in \mathcal{C}([0,T],\mathcal{X}_w)$.

(b) Since

$$\| \dot{y}_n(t) \| \leqq \psi(t) \quad \text{a.e.},$$

the sequence $\{w_n : [0,T] \to \mathcal{X}\}$ defind by

$$w_n(t) = \frac{\dot{y}_n(t)}{\psi(t)} \,;\, n = 1, 2, \cdots$$

is contained in the unit ball of $\mathcal{L}^\infty([0,T],\mathcal{X})$ which is weak*-compact (as the dual space of $\mathcal{L}^1([0,T],\mathcal{X}'))$ by Alaoglu's theorem. Note that the weak* topology on the unit ball of $\mathcal{L}^\infty([0,T],\mathcal{X})$ is metrizable since $\mathcal{L}^1([0,T],\mathcal{X}')$ is separable. Hence $\{w_n\}$ has a subsequence $\{w_{n'}\}$ which converges to some $w^* \in \mathcal{L}^\infty([0,T],\mathcal{X})$ in the weak* topology. We shall write $\dot{z}_n = \dot{y}_{n'} = \psi \cdot w_{n'}$.

If we define an operator $A : \mathcal{L}^\infty([0,T],\mathcal{X}) \to \mathcal{L}^p[0,T],\mathcal{X})$ by

$$A : g \mapsto \psi \cdot g,$$

then A is continuous in the weak* topology for $\mathcal{L}^\infty$ and the weak topology for $\mathcal{L}^p$. In order to see this, let $\{g_\lambda\}$ be a net in $\mathcal{L}^\infty([0,T],\mathcal{X})$ such that $w^* - \lim_\lambda g_\lambda = g^* \in \mathcal{L}^\infty([0,T],\mathcal{X})$; i.e.

$$\int_0^T \langle \alpha(t), g_\lambda(t) \rangle \, dt \to \int_0^T \langle \alpha(t), g^*(t) \rangle \, dt \quad \text{for all} \quad \alpha \in \mathcal{L}^1([0,T],\mathcal{X}').$$

Then it is quite easy to verify that

$$\begin{aligned} \int_0^T \langle \beta(t), \psi(t) g_\lambda(t) \rangle \, dt &= \int_0^T \langle \psi(t)\beta(t), g_\lambda(t) \rangle \, dt \\ &\to \int_0^T \langle \psi(t)\beta(t), g^*(t) \rangle \, dt \\ \text{for all} \quad & \beta \in \mathcal{L}^q([0,T],\mathcal{X}'), \ 1/p + 1/q = 1 \end{aligned}$$

since $\psi \cdot \beta \in \mathcal{L}^1([0,T],\mathcal{X}')$. This proves the continuity of A.

Hence

$$\dot{z}_n = \psi \cdot w_{n'} \to \psi \cdot w^* \quad \text{weakly in} \quad \mathcal{L}^p([0,T],\mathcal{X}), \tag{1}$$

which implies

$$\langle\theta, \int_s^t \dot{z}_n(\tau)d\tau\rangle = \int_s^t \langle\theta, \dot{z}_n(\tau)\rangle\, d\tau \longrightarrow \int_s^t \langle\theta, \psi(\tau)\cdot w^*(\tau)\rangle\, d\tau \quad \text{for all} \quad \theta \in \mathcal{X}'. \tag{2}$$

On the other hand, since

$$z_n(t) - z_n(s) = \int_s^t z_n(\tau)d\tau \quad \text{for all} \quad n,$$

and $z_n(t) - z_n(s) \longrightarrow x^*(t) - x^*(s)$ in $\mathcal{X}_w$, we get

$$\langle\theta, \int_s^t \dot{z}_n(\tau)d\tau\rangle = \langle\theta, z_n(t) - z_n(s)\rangle \longrightarrow \langle\theta, x^*(t) - x^*(s)\rangle \quad \text{for all} \quad \theta \in \mathcal{X}'. \tag{3}$$

(2) and (3) imply that

$$\langle\theta, x^*(t) - x^*(s)\rangle = \langle\theta, \int_s^t \psi(\tau)\cdot w^*(\tau)d\tau\rangle \quad \text{for all} \quad \theta \in \mathcal{X}',$$

from which we can deduce the equality

$$x^*(t) - x^*(s) = \int_s^t \psi(\tau)\cdot w^*(\tau)d\tau. \tag{4}$$

By (1) and (4), we get the desired result :

$$\dot{z}_n \longrightarrow \dot{x}^* = \psi\cdot w^* \quad \text{weakly in} \quad \mathcal{L}^p([0,T],\mathcal{X}).$$

□

In the proof of our Theorem 1, we are making use of some ideas of Aubin and Cellina [1] (pp.13-14). However their reasoning does not seem to be perfectly sound.

3 Differential Inclusions (1)

In this section, we prepare several lemmas which are to play crucial roles in the existence theory for differential inclusions.

Throughout this section, $\mathcal{X}$ **is assumed to be a real separable reflexive Banach space.**

Let us begin by specifying some assumptions imposed on the correspondence $\Gamma : [0,T] \times \mathcal{X}_w \twoheadrightarrow \mathcal{X}_w$. Special attentions should be paid to the fact that both of the domain and the range of Γ are endowed with the weak topologies.

Assumption 1. Γ is compact-convex-valued ; i.e. $\Gamma(t,x)$ is a non-empty, compact and convex subset of $\mathcal{X}_w$ for all $t \in [0,T]$ and all $x \in \mathcal{X}$.

Assumption 2. The correspondence $x \mapsto\!\!\!\rightarrow \Gamma(t,x)$ is upper hemi-continuous (abbreviated as u.h.c.) for each fixed $t \in [0,T]$; i.e. for any fixed $(t,x) \in [0,T] \times \mathcal{X}_w$ and for any neighborhood V of $\Gamma(t,x) \subset \mathcal{X}_w$, there exists some neighborhood U of x such that $\Gamma(t,z) \subset V$ for all $z \in U$.

Assumption 3. The graph of the correspondence $t \mapsto\!\!\!\rightarrow \Gamma(t,x)$ is $(\mathcal{L}, \mathcal{B}(\mathcal{X}_w))$-measurable for each fixed $x \in \mathcal{X}$ where $\mathcal{B}(\mathcal{X}_w)$ denotes the Borel σ-field on $\mathcal{X}_w$. (For the concept of "measurability" of a correspondence, the best reference is Castaing-Valadier [8] Chap.III.)

Assumption 4. Γ is $\mathcal{L}^p$-integrably bounded ; i.e. there exists $\psi \in \mathcal{L}^p([0,T],(0,+\infty))(p > 1)$ such that $\Gamma(t,x) \subset S_{\psi(t)}$ for every $(t,x) \in [0,T] \times \mathcal{X}$, where $S_{\psi(t)}$ is the closed ball in $\mathcal{X}$ with the center 0 and the radius $\psi(t)$.

The following lemma is essentially due to Castaing [5].

LEMMA 1 (Castaing [5]) Suppose that a correspondence $\Gamma : \mathcal{X} \twoheadrightarrow \mathcal{X}$ satisfies the Assumptions 1-3, and that a function $x : [0,T] \to \mathcal{X}$ is Bochner-integrable. Then there exists a closed-valued correspondence $\Sigma : [0,T] \twoheadrightarrow \mathcal{X}_w$ such that

$$\Sigma(t) \subset \Gamma(t, x(t)) \quad \text{for all} \quad t \in [0,T],$$

and the graph $G(\Sigma)$ of Σ is $(\mathcal{L}, \mathcal{B}(\mathcal{X}_w))$-measurable.

Proof. Let $\{x_n : [0,T] \to \mathcal{X}\}$ be a sequence of simple functions which satisfies that

$$\| x_n(t) - x(t) \| \to 0 \quad \text{for each} \quad t \in [0,T] \quad \text{as} \quad n \to \infty.$$

(For the existence of such a sequence, see Yosida [28] p.133.)

Define a correspondence $\Gamma_n : [0,T] \twoheadrightarrow \mathcal{X}_w$ by

$$\Gamma_n : t \mapsto\!\!\!\rightarrow \Gamma(t, x_n(t))\,;\; n = 1, 2, \cdots.$$

Then it can be shown that the graph $G(\Gamma_n)$ of each Γ_n is $(\mathcal{L}, \mathcal{B}(\mathcal{X}_w))$-measurable. In order to confirm it, we denote by $\{y_1, y_2, \cdots, y_k\}$ the image of $[0,T]$ by the simple function x_n ; i.e.

$$x_n([0,T]) = \{y_1, y_2, \cdots, y_k\}.$$

Furthermore if we define a correspondence $\Phi_j : [0,T] \twoheadrightarrow \mathcal{X}_w$ $(j = 1, 2, \cdots, k)$ by

$$\Phi_j : t \mapsto\!\!\!\to \Gamma(t, y_j),$$

then the graph $G(\Phi_j)$ of Φ_j is obviously $(\mathcal{L}, \mathcal{B}(\mathcal{X}_w))$-measurable. The graph $G(\Gamma_n)$ of Γ_n can be expressed as

$$G(\Gamma_n) = \cup_{j=1}^{k} G\,[\,\Phi_j\,|_{x_n^{-1}(\{y_j\})}],$$

where $\Phi_j\,|_{x_n^{-1}(\{y_j\})}$ is the restriction of the correspondence Φ_j to the set $x_n^{-1}(\{y_j\}) = \{t \in [0,T]\,|\,x_n(t) = y_j\}$. Since $G\,[\,\Phi_j|_{x_n^{-1}(\{y_j\})}](j = 1, 2, \cdots, k)$ is $(\mathcal{L}, \mathcal{B}(\mathcal{X}_w))$-measurable, so is $G(\Gamma_n)$.

Since $\| x_n(t) - x(t) \| \to 0$ for each $t \in [0,T]$ as $n \to \infty$, the set $\{x_1(t), x_2(t), \cdots; x(t)\}$ is weakly compact for each $t \in [0,T]$. Furthermore, by the Assumptions 1-2, the correspondence Γ is compact-valued and u.h.c. in the second variable. Consequently the set

$$\cup_{n=1}^{\infty} \Gamma(t, x_n(t))$$

is relatively compact in $\mathcal{X}_w$ (for each $t \in [0,T]$). Taking account of the fact that the weak topology of a weakly compact subset of a separable Banach space is metrizable, we can conclude, by Baire's category theorem, that the set

$$\Sigma(t) \equiv \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} \overline{\Gamma(t, x_m(t))}^{w}$$

is non-empty (for each $t \in [0,T]$), where $-\,w$ denotes the closure operation with respect to the weak topology.

The correspondence $\Sigma : [0,T] \twoheadrightarrow \mathcal{X}_w$ is closed-valued and its graph is $(\mathcal{L}, \mathcal{B}(\mathcal{X}_w))$-measurable. Finally the inclusion

$$\Sigma(t) \subset \Gamma(t, x(t)) \quad \text{for each} \quad t \in [0,T]$$

is clear because Γ is compact-valued and u.h.c. □

We can show the Next lemma in a similar way as in Maruyama[17], taking account of [III] of the Remark on page 4.

LEMMA 2 Let A be a non-empty compact and convex set in $\mathcal{X}_w$, and X a subset of $\mathcal{W}^{1,p}([0,T], \mathcal{X})(p > 1)$ defined by

$$X = \{x \in \mathcal{W}^{1,p} | \parallel \dot{x}(t) \parallel \leqq \psi(t) \quad \text{a.e.,} \quad x(0) \in A\},$$

where $\psi \in \mathcal{L}^p([0,T],(0,+\infty))$. Then X is non-empty convex and compact in $\mathcal{X}_w$.

Proof. Since it is obvious that X is non-empty and convex, we have only to show the weak compactness of X.

It is not hard to show the boundedness of X. Let x be any element of X. Then x can be represented in the form

$$x(t) = a + \int_0^t \dot{x}(\tau)d\tau \,;\, t \in [0,T]$$

(a is a point of A) by [III] of the Remark on page 3. It follows that

$$\begin{aligned} \parallel x(t) \parallel &= \parallel a + \int_0^t \dot{x}(\tau)d\tau \parallel \leqq \parallel a \parallel + \int_0^t \parallel \dot{x}(\tau) \parallel d\tau \\ &\leqq \parallel a \parallel + \int_0^t \psi(\tau)d\tau \leqq B + \int_0^T \psi(\tau)d\tau, \end{aligned}$$

where $B = \sup_{a \in A} \parallel a \parallel < +\infty$. Consequently we have the evaluation :

$$\sup_{x \in X} \parallel x \parallel_p^p \leqq [B + \int_0^T \psi(\tau)d\tau]^p \cdot T < +\infty,$$

where $\parallel \cdot \parallel_p$ denotes the $\mathcal{L}^p$-norm. Since the right-hand side is independent of x, X is bounded in $\mathcal{L}^p$. On the other hand, the set $\{\dot{x} \,|\, x \in X\}$is also bounded by $\parallel \psi \parallel_p$. Therefore we can claim that X is bounded in $\mathcal{W}^{1,p}$.

$\mathcal{W}^{1,p}$ is reflexive because $\mathcal{X}$ is reflexive and $p > 1$. Hence the bounded set X is weakly relatively compact in $\mathcal{W}^{1,p}$.

To show the weak compactness of X, we need only to show the weak closedness of X. However X is weakly closed if and only if X is strongly closed since X is convex. Let $\{x_n\}$ be a sequence in X which strongly converges to x^* in $\mathcal{W}^{1,p}$. Then $\{\dot{x}_n\}$ has a subsequence, say $\{\dot{x}_{n'}\}$, which converges to $\dot{x}^*$ a.e. Since $\parallel \dot{x}_{n'}(t) \parallel \leqq \psi(t)$ a.e., it follows that

$$\parallel \dot{x}^*(t) \parallel \leqq \psi(t) \quad \text{a.e.}$$

Finally it is clear that $x^*(0) \in A$. Then we obtain $x^* \in X$. This proves that X is strongly closed in $\mathcal{W}^{1,p}$. □

We denote by $\mathcal{B}(0\,;\mathcal{X}_w)$ a neighborhood base of the zero element of $\mathcal{X}_w$ which consists of conves sets. The following lemma plays a crucial role in the subsequent arguments although its proof is easy.

LEMMA 3 Suppose that the Assumptions 1-2 are satisfied. Let (t^*, x^*) be any point of $[0,T] \times \mathcal{X}$. Define, for any $V \in \mathcal{B}(0; \mathcal{X}_w)$, a subset $K(t^*; x^*, V)$, of $[0,T] \times \mathcal{X}$ by

$$K(t^*; x^*, V) = \{(t,x) \in [0,T] \times \mathcal{X} \mid x \in x^* + V,\ t = t^*\}.$$

Then we have

$$\Gamma(t^*, x^*) = \cap_{V \in \mathcal{B}(0;\mathcal{X}_w)} \overline{\mathrm{co}}\, \Gamma(K(t^*\,;\, x^*, V)).$$

(Here we do not have to distinguish the convex closure with respect to the strong topology and that with respect to weak topology. So I simply denote it by $\overline{\mathrm{co}}$.)

LEMMA 4 Suppose that the Assumptions 1,2 and 4 (with $p > 1$) are satisfied. Let A be a non-empty convex compact subset of $\mathcal{X}_w$. Then the set

$$H \equiv \{(a,x,y) \in A \times X \times X \mid \dot{y}(t) \in \Gamma(t, x(t)) \quad \text{a.e. and} \quad x(0) = y(0) = a\}$$

is weakly compact in $A \times X \times X$. (The set X is defined in Lemma 2.)

Proof. Since we have already known that $A \times X \times X$ is weakly compact in $\mathcal{X} \times \mathcal{W}^{1,p} \times \mathcal{W}^{1,p}$, it is enough to show that H is a weakly closed subset of $A \times X \times X$.

Since $\mathcal{W}^{1,p}$ is a reflexive Banach space, the dual of which is separable, the weak topology on the bounded set X is metrizable. So we are permitted to use a sequence argument.

Let $\{q_n \equiv (a_n, x_n, y_n)\}$ be a sequence in H which weakly converges to some $q^* = (a^*, x^*, y^*)$ in $A \times X \times X$. We have to show that $q^* \in H$. And it is enough to check that

$$\dot{y}^*(t) \in \Gamma(t, x^*(t)) \quad \text{a.e.}$$

The set $\{x_n(t)\}$ is relatively compact in $\mathcal{X}_w$ (for each $t \in [0,T]$) since we have the evaluation:

$$\| x_n(t) \| \leqq \| a \| + \int_0^t \| \dot{x}_n(\tau) \| d\tau \leqq \| a \| + \int_0^T \psi(\tau) d\tau$$

by the Assumption 4. Hence, thanks to Theorem 1, $\{q_n\}$ has a subsequence (no change in notation) such that

$$x_n(t) \to x^*(t) \quad \text{uniformly in} \quad \mathcal{X}_w, \quad \text{and} \tag{1}$$

$$\dot{y}_n(t) \to \dot{y}^*(t) \quad \text{weakly in} \quad \mathcal{L}^p. \tag{2}$$

Then applying Mazur's theorem, we can choose, for each $j \in \mathbb{N}$, some finite elements

$$\dot{y}_{n_j+1}, \dot{y}_{n_j+2}, \cdots, \dot{y}_{n_j+m(j)}$$

of $\{\dot{y}_n\}$ and numbers

$$\alpha_{ij} \geqq 0,\ 1 \leqq i \leqq m(j),\ \sum_{i=1}^{m(j)} \alpha_{ij} = 1$$

such that

$$\| \dot{y}^* - \sum_{i=1}^{m(j)} \alpha_{ij} y_{n_j} +_i \|_p \leqq \frac{1}{j}, n_{j+1} > n_j + m(j).$$

Denoting

$$\eta_j(t) = \sum_{i=1}^{m(j)} \alpha_{ij} \dot{y}_{n_j+i}(t),$$

we obtain

$$\eta_j(t) \in \mathrm{co}(\cup_{i=1}^{m(j)} \Gamma(t, x_{n_j+i}(t)).$$

Since $\{\eta_j\}$ has a subsequence which converges to y^* a.e., we may assume, without loss of generality, that

$$\| \eta_j(t) - y^*(t) \| \to 0 \quad \text{a.e.} \tag{3}$$

On the other hand, for each $V \in \mathcal{B}(0\,; \mathcal{X}_w)$, there exists some $n_0(V) \in \mathbb{N}$ such that

$$x_n(t) \in x^*(t) + V$$
$$\text{for all} \quad n \geqq n_0(V) \quad \text{and for all} \quad t \in [0, T].$$

That is ,

$$(t, x_n(t)) \in K(t\,; x^*(t), V) \quad \text{for all} \quad n \geqq n_0(V) \quad \text{and for all} \quad t \in [0, T].$$

Hence we have

$$\eta_j(t) \in \mathrm{co}\Gamma(K(t\,; x^*(t), V) \quad \text{a.e.}$$

for sufficiently large j. Passing to the limit, we obtain

$$\dot{y}^*(t) \in \overline{\mathrm{co}}\,\Gamma(K(t\,; x^*(t), V)) \quad \text{a.e.} \tag{4}$$

by (3). Since (4) holds true for all $V \in \mathcal{B}(0\,;\mathcal{X}_w)$, it follows that

$$y^*(t) \in \cap_{V\in\mathcal{B}(0;\mathcal{X}_w)}\overline{\text{co}}\Gamma(K(t\,;x^*(t),V) = \Gamma(t,x^*(t)) \quad \text{a.e.}$$

The last equality in (5) comes from Lemma 3. Thus we have proved that $(a^*,x^*,y^*) \in H$. □

4 Differential Inclusions (2)

$\mathcal{X}$ is still assumed to be a real separable reflexive Banach space in this section.

We are now going to find out a solution of $(*)$ in the Sobolev space $\mathcal{W}^{1,p}([0,T],\mathcal{X}), p > 1$. Define a set $\Delta(a)$ in $\mathcal{W}^{1,p}$ by

$$\Delta(a) = \{x \in \mathcal{W}^{1,p}\,|\,x \quad \text{satisfies} \quad (*) \quad \text{a.e.}\}$$

for a fixed $a \in \mathcal{X}$. The following theorem tells us that $\Delta(a) \neq \emptyset$ and that Δ depends continuously, in some sense, upon the initial value a.

THEOREM 2. Suppose that the correspondence Γ satisfies the Assumptions 1-4. Let A be a non-empty, convex and compact subset of $\mathcal{X}_w$. Then

(i) $\Delta(a^*) \neq \emptyset$ for any $a^* \in A$, and

(ii) the correspondence $\Delta : A \twoheadrightarrow \mathcal{W}^{1,p}$ is compact-valued and u.h.c. on A_w, in the weak topology for $\mathcal{W}^{1,p}$.

The proof is essentially the same as in Maruyama [17].

Proof. (i) Fix any $a^* \in A$. If we define a set $X(a^*) \subset X$ by $X(a^*) = \{x \in X\,|\,x(0) = a^*\}$, then $X(a^*)$ is convex and weakly compact in $\mathcal{W}^{1,p}$. Furthermore we define a correspondence $\Phi : X(a^*)_w \twoheadrightarrow X(a^*)_w$ by

$$\Phi(x) = \{y \in X(a^*)\,|\,\dot{y}(t) \in \Gamma(t,x(t)) \quad \text{a.e.}\}.$$

Then the problem is simply reduced to finding out a fixed point of Φ.

1° $\Phi(x) \neq \emptyset$ for every $x \in X(a^*)$ — This fact can be proved through the Measurable Selection Theorem.

Let x be any element of $X(a^*)$. Then by Lemma 1, there exists a clsoed-valued correspondence $\Sigma : [0,T] \twoheadrightarrow \mathcal{X}_w$ such that $\Sigma(t) \subset \Gamma(t,x(t))$ for all $t \in [0,T]$, and its graph is $(\mathcal{L},\mathcal{B}(\mathcal{X}_w))$-measurble. We also note that $\mathcal{X}_w$ is a Souslin space. Thanks to Saint-Beuve's measurable selection theorem (Saint-Beuve [20]), Σ admits a $(\mathcal{L},\mathcal{B}(\mathcal{X}_w))$-measurable selection $\sigma : [0,T] \to \mathcal{X}$. Since

$\mathcal{X}$ is separable, σ is $(\mathcal{L}, \mathcal{B}(\mathcal{X}_s))$-mesurable. (cf. Yosida [28] p.131.) By the Assumption 4, σ is clearly integrable. If we define a function $y : [0,T] \to \mathcal{X}$ by

$$y(t) = a^* + \int_o^t \sigma(\tau) d\tau,$$

then $y \in \Phi(x)$.

2° Φ is convex-compact-valued. — This is not hard.

3° Φ is u.h.c. — If we define the a^*-selection H_{a^*} of H by $H_{a^*} = \{(a,x,y) \in H \mid a = a^*\}$, then H_{a^*} is obviously weakly compact in $A \times X \times X$. And the graph $G(\Phi)$ of Φ is expressed as $G(\Phi) = \text{proj}_{X \times X} H_{a^*}$, the projection of H_{a^*} into $X \times X$, which is also closed.

Summing up — Φ is convex-compact-valued and u.h.c. Applying now the Fan-Glicksberg Fixed-Point Theorem to the correspondence Φ, we obtain an $x^* \in X(a^*)$ such that $x^* \in \Phi(x^*)$; i.e.

$$\dot{x}^*(t) \in \Gamma(t, x^*(t)) \quad \text{a.e.} \quad \text{and} \quad x^*(0) = a^*.$$

This proves (i).

(ii) Since the compactness of $\Delta(a)(a \in A)$ can be verified by applying Mazur's theorem and making use of the Assumptions 1-2, we may omit the details. Hence we have only to show the u.h.c. of Δ. However it is also obvious because the graph $G(\Delta)$ of Δ can be expressed as

$$G(\Delta) = \text{proj}_{A \times X}\{(a,x,y) \in H \mid x = y\},$$

which is closed in $A \times X$.

□

I am much indebted to Castaing-Valadier [7] for various important ideas embodied in the proof of Theorem 2.

Remark. Among other things, the assumption that the set $\Gamma(t,x)$ is always convex is seriously restrictive, especially from the viewpoint of applications. However there seems to be no easy way to wipe out the convexity assumption. (See De Blasi [10] and Tateishi [23].)

Here it may be suggestive for us to glimpse the special case in which Γ is a (single-valued) mapping. A related result was obtained by Szep [23]. (I am indebted to Professor Tosio Kato for this reference.)

COROLLARY 1. Let $f : [0,T] \times \mathcal{X}_w \to \mathcal{X}_w$ be a (single-valued) mapping which satisfies the following three conditions.

(i) The function $x \mapsto f(t,x)$ is continuous for each fixed $t \in [0,T]$.

(ii) The function $t \mapsto f(t,x)$ is measurable for each fixed $x \in \mathcal{X}$.

(iii) There exists $\psi \in \mathcal{L}^p([0,T],(0,+\infty)), p > 1$ such that $f(t,x) \in S_{\psi(t)}$ for every $(t,x) \in [0,T] \times \mathcal{X}$; i.e. $\sup_{x\in\mathcal{X}} \| f(t,x) \| \leqq \psi(t)$ for all $t \in [0,T]$.

Then the differential equation

$$(**) \qquad \dot{x} = f(t,x),\ x(0) = a \,(\text{fixed vector in } \mathcal{X})$$

has at least a solution in $\mathcal{W}^{1,p}([0,T],\mathcal{X})$. (A solution of $(**)$ is a function $x \in \mathcal{W}^{1,p}$ which satisfies $(**)$ a.e.)

5 Variational problem governed by an Differential Inclusion

Let $\mathcal{X}$ be a real separable reflexive Banach space throughout this section, too. Assume that $u : [0,T] \times \mathcal{X}_w \times \mathcal{X}_s \to (-\infty,+\infty]$ is a given proper function. Consider a variational problem :

$$(\sharp) \qquad \text{Minimize}_{x\in\Delta(a)} J(x) = \int_0^T u(t,x(t),\dot{x}(t))dt,$$

where $\Delta(a)$ is the set of all the solutions of the differential inclusion $(*)$ discussed in the preceding sections.

In order to examine the existence of a solution of the problem $(\sharp)$, we have to check a couple of points as usual ; i.e.

(I) the compactness of $\Delta(a)$ for some suitable topology, and

(II) the lower semi-continuity of the functional J for the same topology.

Since we have already proved that $\Delta(a)$ is weakly compact in $\mathcal{W}^{1,p}([0,T],\mathcal{X})$ under certain conditions, we are concentrating on the second point (II) in this section. In this context, the theorem due to Castaing-Clauzure [6] provides the most crucial key. Related results are also obtained by Balder [2], Maruyama [16] and Valadier [25].

DEFINITION Let (Ω,ξ,μ) be a measure space, S a topological space, and $\mathcal{V}$ a real Banach space. A function $f : \Omega \times S \times \mathcal{V} \to \overline{\mathbb{R}}$ is assumed to be given. We denote by $\mathcal{M}(\Omega,S)$ the set of all the $(\xi \otimes \mathcal{B}(S))$-measurable functions. ($\mathcal{B}(S)$ denotes the Borel σ-field on S.) f is said to have the **lower compactness property** if $\{f^-(\omega,\varphi_n(\omega),\theta_n(\omega))\}$ is weakly relatively compact in $\mathcal{L}^1(\Omega,\overline{\mathbb{R}})$ for any sequence $\{(\varphi_n,\theta_n)\}$ in $\mathcal{M}(\Omega,S) \times \mathcal{L}^p(\Omega,\mathcal{V})(p \geqq 1)$ which satisfies the following three conditions:

(a) $\{\varphi_n\}$ converges in measure to some $\varphi^* \in \mathcal{M}(\Omega, S)$,

(b) $\{\theta_n\}$ converges weakly to some $\theta^* \in \mathcal{L}^p(\Omega, \mathcal{V})$, and

(c) there exists some $C < +\infty$ such that

$$\sup_n \int_\Omega f(\omega, \varphi_n(\omega), \theta_n(\omega)) d\mu \leqq C.$$

The following theorem is a variation of a result due to Castaing-Clauzure [6] in the spirit of Ioffe [12]. See also Valadier [27].

THEOREM 3 Let (Ω, ξ, μ) be a finite complete measure space, S a metrizable Souslin space, and $\mathcal{V}$ a separable reflexive Banach space. Suppose that a proper function $f : \Omega \times S \times \mathcal{V} \to \overline{\mathbb{R}}$ satisfies the following conditions:

(i) f is a normal integrand ; i.e.

  (a) f is $(\xi \otimes \mathcal{B}(S) \otimes \mathcal{B}(\mathcal{V}), \mathcal{B}(\overline{\mathbb{R}}))$-measurable, and

  (b) the function $(\xi, v) \mapsto f(\omega, \xi, v)$ is lower semi-continuous for any fixed $\omega \in \Omega$,

(ii) the function $v \mapsto f(\omega, \xi, v)$ is convex for any fixed $(\omega, \xi) \in \Omega \times S$, and

(iii) f has the lower compactness property.

Let $\{\varphi_n\}$ be a sequence in $\mathcal{M}(\Omega, S)$ which converges in measure to some $\varphi^* \in \mathcal{M}(\Omega, S)$. Let $\{\theta_n\}$ be a sequence in $\mathcal{L}^p(\Omega, \mathcal{V})(1 \leqq p < +\infty)$ which converges weakly to some $\theta^* \in \mathcal{L}^p(\Omega, \mathcal{V})$. Then we have

$$\int_\Omega f(\omega, \varphi^*(\omega), \theta^*(\omega)) d\mu \leqq \liminf_n \int_\Omega f(w, \varphi_n(\omega), \theta_n(\omega)) d\mu.$$

Remark 1° A normal integrand $f : \Omega \times S \times \mathcal{V} \to \overline{\mathbb{R}}$ which also satisfies the condition (ii) is called a *convex normal integrand.*

2° Ioffe [8] established a fundamental theorem on the lower semi-continuity of a nonlinear integral functional as above in the case both of S and $\mathcal{V}$ are finite dimensional Euclidean spaces. Theorem 3 is an extension of Ioffe's result to the case of nonlinear integral functional defined on the space of Bochner integreble functions.

LEMMA 5 Suppose that the Assumptions 1-4 are satisfied. Let $\{x_n\}$ be a sequence in $\Delta(a) \subset \mathcal{W}^{1,p}([0,T], \mathcal{X})$ $(p > 1)$. Let $u : [0,T] \times \mathcal{X}_w \times \mathcal{X}_s \to \overline{\mathbb{R}}$ be a proper convex normal integrand with the lower compactness property. Then there exists a subsequence $\{z_n\}$ of $\{x_n\}$ and $x^* \in \Delta(a)$ such that

$$J(x^*) \leqq \liminf_n J(z_n), \tag{1}$$

where

$$J(x) = \int_0^T u(t, x(t), \dot{x}(t))dt.$$

Proof. By the Assumption 4, all the images of x_n's are contained in some closed ball $\overline{B}$ with the center 0 ; i.e.

$$x_n(t) \in \overline{B} \quad \text{for all} \quad t \in [0,T] \quad \text{and} \quad n.$$

Hence we may restrict the domain of u to $[0,T] \times \overline{B}_w \times \mathcal{X}_s$ provided that the sequence $\{x_n\}$ is concerned. Denoting $\overline{u} = u|_{[0,T]\times\overline{B}\times\mathcal{X}}$, (restriction of u to $[0,T] \times \overline{B} \times \mathcal{X}$) we have to show that there exists a subsequence $\{z_n\}$ of $\{x_n\}$ and some $x^* \in \Delta(a)$ such that

$$\int_0^T \overline{u}(t, x^*(t), \dot{x}^*(t))dt \leqq \liminf_n \int_0^T \overline{u}(t, z_n(t), \dot{z}_n(t))dt,$$

which is equivalent to (1).

The set $\overline{B}$ endowed with the weak topology is metrizable and compact. Hence it is a Polish space. According to Theorem 1, there exists a subsequence $\{z_n\}$ of $\{x_n\}$ and $x^* \in \mathcal{W}^{1,p}([0,T],\mathcal{X})$ such that

(a) $z_n \longrightarrow x^*$ uniformly in $\overline{B}_w$, and

(b) $\dot{z}_n \longrightarrow \dot{x}^*$ weakly in $\mathcal{L}^p([0,T],\mathcal{X})$.

(a) implies, of course, that $z_n \longrightarrow x^*$ in measure. Thus applying Theorem 3, we obtain the relation

$$\int_0^T \overline{u}(t, x^*(t), \dot{x}^*(t))dt \leqq \liminf_n \int_0^T \overline{u}(t, z_n(t), \dot{z}_n(t))dt.$$

Finally we have to prove that $x^* \in \Delta(a)$. By (a), it follows that

$$\lim_{n\to\infty} \langle z_n(t), \eta(t)\rangle = \langle x^*(t), \eta(t)\rangle$$

for any $t \in [0,T]$ and $\eta \in \mathcal{L}^q([0,T],\mathcal{X}')$, where $1/p + 1/q = 1$. Since $z_n(t) \in \overline{B}$, there exists some positive constant $C < \infty$ such that

$$|\langle z_n(t), \eta(t)\rangle| \leqq C \,\|\eta(t)\|.$$

Hence we have, by the Bounded Convergence Theorem, that

$$\lim_{n\to\infty}\int_0^T \langle z_n(t),\eta(t)\rangle\, dt = \int_0^T \langle x^*(t),\eta(t)\rangle\, dt$$

$$\text{for any} \quad \eta \in \mathcal{L}^q([0,T],\mathcal{X}').$$

This proves that $z_n \longrightarrow x^*$ weakly in $\mathcal{L}^p$.

Combining this result with (b), we can conclude that $\{z_n\}$ weakly converges to x^* in $\mathcal{W}^{1,p}$. Since $\Delta(a)$ is weakly closed, $x^* \in \Delta(a)$. □

Let $\{x_n\}$ be a minimizing sequence of the problem (♯). Then, by Lemma 5, $\{x_n\}$ has a subsequence (without change of notaion) such that

$$J(x^*) \leqq \liminf_n J(x_n)$$

for some $x^* \in \Delta(a)$. It is also obvious that

$$\inf_{x\in\Delta(a)} J(x) = \liminf_n j(x_n) \leqq J(x^*).$$

Thus we have proved that x^* is a solution of the problem (♯). Summing up

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THEOREM 4 Suppose that Assumptions 1-4 with $p > 1$ are satisfied for a correspondence $\Gamma : [0,T]\times\mathcal{X} \longrightarrow\!\!\!\rightarrow \mathcal{X}$. Furthermore let $u : [0,T]\times\mathcal{X}_w\times\mathcal{X}_s \longrightarrow \overline{\mathbb{R}}$ be a normal convex integrand with the lower compactness property. Then the problem (♯) has a solution.

Appendix

Banach Space-valued Sobolev Spaces

This appendix aims at a brief summary of the concepts and basic facts in the theory of Banach space-valued Sobolev spaces. (cf. Schwartz [22], Barbu [3].)

1. Let $p = (p_1, p_2, \cdots, p_\ell)$ be an ℓ-tuple of non-negative integers. The number $|p| = p_1 + p_2 + \cdots + p_\ell$ is called the **order** of p. We denote by D^p the differential operator

$$D^p = \frac{\partial^{p_1+p_2+\cdots+p_\ell}}{\partial x_1^{p_1}\partial x_2^{p_2}\cdots\partial x_\ell^{p_\ell}}$$

Let Ω be an open set of $\mathbb{R}^\ell$ and K a compact subset of Ω. We denote by $\mathcal{D}_K(\Omega)$ the set of all the infinitely differentiable real-valued functions $\varphi : \Omega \longrightarrow \mathbb{R}$ whose supports are contained in K ; i.e.

$$\mathcal{D}_K(\Omega) = \{\varphi \in C^\infty(\Omega, \mathbb{R}) | \text{supp}\, \varphi \subset K\}.$$

Under the topology generated by the family of seminorms :

$$p_{K,m}(\varphi) = \sup_{\substack{x \in K \\ |p| \leqq m}} |D^p \varphi(x)|, \; m = 1, 2, \cdots,$$

$\mathcal{D}_K(\Omega)$ becomes a locally convex Hausdorff topological vector space (LCHTVS).

The space $\mathcal{D}(\Omega) = \cup\{\mathcal{D}_K(\Omega) \,|\, K$ is a compact subset of $\Omega\}$ is also a vector space. And the space $\mathcal{D}(\Omega)$ endowed with the strict inductive limit topology defined by $\{\mathcal{D}_K(\Omega)|K$ is a compact subset of $\Omega\}$ is a LCHTVS, called the **Schwartz space.** It is well-known that a net $\{\varphi_\alpha\}$ in $\mathcal{D}(\Omega)$ converges to some $\varphi^* \in \mathcal{D}(\Omega)$ if and only if there exists some compact subset K of Ω with

$$\text{supp}\, \varphi_\alpha \subset K \quad \text{for all} \quad \alpha,$$

and

$$D^p \varphi_\alpha \longrightarrow D^p \varphi^* \quad \text{uniformly on} \quad \Omega$$

for every index $p = (p_1, p_2, \cdots, p_\ell)$

2. Let $\mathcal{X}$ be a real Banach space. Any continuous linear operator $S : \mathcal{D}(\Omega) \to \mathcal{X}$ is called a **$\mathcal{X}$-valued distribution** and the set of all the $\mathcal{X}$-valued distributions is denoted by $\mathcal{D}'(\Omega \,|\, \mathcal{X})$.

If $f : \Omega \to \mathcal{X}$ is a locally Bochner-integrable function, the operator $S_f : \mathcal{D}(\Omega) \to \mathcal{X}$ defined by

$$S_f : \varphi \mapsto \int_\Omega f(\omega)\varphi(\omega)d\omega, \; \varphi \in \mathcal{D}(\Omega)$$

is an $\mathcal{X}$-valued distribution. ($d\omega$ is the Lebesgue measure on Ω.) Identifying f and S_f , we can safely say that any locally Bochner-integrable function is an $\mathcal{X}$-valued distribution.

The value of $S \in \mathcal{D}'(\Omega \,|\, \mathcal{X})$ at $\varphi \in \mathcal{D}(\Omega)$ is sometimes denoted by $\langle S, \varphi \rangle$ instead of $S(\varphi)$.

Let S be an $\mathcal{X}$-valued distribution and D^p an differential operator. Then the operator $D^p S : \mathcal{D}(\Omega) \to \mathcal{X}$ defined by

$$\varphi \mapsto (-1)^{|p|} \langle S, D^p \varphi \rangle, \; \varphi \in \mathcal{D}(\Omega)$$

is also an $\mathcal{X}$-valued distribution, called the **distributional derivative** (or the **derivative in sense of distribution**) of S ; i.e.

$$\langle D^p S, \varphi \rangle = (-1)^{|p|} \langle S, D^p \varphi \rangle, \; \varphi \in \mathcal{D}(\Omega).$$

An $\mathcal{X}$-valued distribution is infinitely differentiable in the sense of distribution.

3. **The $\mathcal{X}$-valued Sobolev space** $\mathcal{W}^{k,p}(\Omega,\mathcal{X})(p \geqq 1)$ is the set of all the functions $f:\Omega \longrightarrow \mathcal{X}$ such that its distributional derivative $D^s f$ exists and belongs to $\mathcal{L}^p(\Omega,\mathcal{X})$ for all $s=(s_1,s_2,\cdots,s_\ell)$ with $|s| \leqq k$.

$\mathcal{W}^{k,p}(\Omega,\mathcal{X})$ is clearly a vector space. In fact, it becomes a Banach space under the norm :

$$\| f \|_{k,p} = \Big(\sum_{|s| \leqq k} \int_\Omega \| D^s f(\omega) \|^p \, d\omega \Big)^{1/p}$$

If $\mathcal{X}$ is a Hilbert space and $p=2$, $\mathcal{W}^{k,2}(\Omega,\mathcal{X})$ is also a Hilebert space under the inner product :

$$\langle f,g \rangle_{k,p} = \sum_{|s| \leqq k} \int_\Omega \langle D^s f(\omega), D^s g(\omega) \rangle \, d\omega.$$

Finally, we state three results which are to play some roles in this paper.

FACT 1 If $\mathcal{X}$ is a separable Banach space, then $\mathcal{W}^{k,p}(\Omega,\mathcal{X})(p \geqq 1)$ is also separable.

FACT 2 If $\mathcal{X}$ is a separable reflexive Banach space and $p > 1$, then $\mathcal{W}^{k,p}(\Omega,\mathcal{X})$ is reflexive.

Let $\Omega = (0,T)$. We denote by $\mathcal{W}^{k,p}([0,T],\mathcal{X})$ the set ι l the functions $f:[0:T] \rightarrow \mathcal{X}$ such that

a The derivatives $D^j f$ (defined a.e.) are absolutely conιιnuous for $j = 1,2,\cdots,k-1$, and

b $D^j f \in \mathcal{L}^p([0,T],\mathcal{X})$ for $j=0,1,2,\cdots,k$.

FACT 3 Let $\mathcal{X}$ be a Banach space with the Radon-Nikodým property. Then the following two statements are equivalent for a function $f \in \mathcal{L}^p([0,T],\mathcal{X})(p \geqq 1)$.

(i) $f \in \mathcal{W}^{k,p}([0,T],\mathcal{X})$.

(ii) There exists some $f_1 \in \mathcal{W}^{k,p}([0,T],\mathcal{X})$ such that $f(t) = f_1(t)$ a.e. $\omega \in (0,T)$.

Thus we may assume, without loss of generality, that each element of $\mathcal{W}^{k,p}((0,T),\mathcal{X})$ is defined on the closed interval $[0,T]$ rather than $(0,T)$. When we wish to emphasize this aspect, we use the notation $\mathcal{W}^{k,p}([0,T],\mathcal{X})$ rather than $\mathcal{W}^{k,p}((0,T),\mathcal{X})$.

References

[1] Aubin,J.P. and Cellina,A., *Differential Inclusions*. Springer, Berlin, 1984.

[2] Balder,E.J., An extension of Prohorov's theorem for transition probabilities with applications to infinite dimensional lower closure problem, *Rendiconti del Circolo Matematico di Parermo*, Serie II, **34** (1985), 427-447.

[3] Barbu,V., *Nonlinear Semigroups and Differential Equations in Banach Spaces*. Noordhoff, Leyden, 1976.

[4] Buttazo,G., *Semicontinuity, Relaxation and Integral Representation in the Calculus of Variations*. Longman Scientific & Technical, Harlow, 1989.

[5] Castaing,C., Sur les équations différentielles multivoques, *C.R.Acad.Sc. Paris*, Serie A, **263** (1966), 63-66.

[6] Castaing,C. and Clauzure,P., Semi continuité des fonctionelles intégrales, *Acta Math. Vietnam.*, **7** (1982), 139-170.

[7] Castaing,C. and Valadier,M., Equations différentielles multivoques dans les éspaces vectoriels localement convexes, *Rev. Française Informat. Recherche Opérationelles*, **3** (1969), 3-16.

[8] Castaing,C. and Valadier,M., *Convex Analysis and Measurable Multifunctions*, Springer, Berlin, 1977.

[9] Cecconi,J.P., Problems of the calculus of variations in Banach spaces and classes BV, L.Cesari ed., *Contributions to Modern Calculus of Variations*, Longman Scientific & Technical, Harlow, 1987.

[10] De Blasi, F.S., Differential inclusions in Banach spaces, *J.Differential Equations*, **66** (1987) 208-229.

[11] Diestel,J. and Uhl,J.J., *Vector Measures*, Amer.Math.Soc., Providence. 1977.

[12] Ioffe,A.D., On lower semicontinuity of integral functionals I, *SIAM J.Control*, **15** (1977) 521-538.

[13] Komura,Y., Nonlinear semi-groups in Hilbert space, *J.Math.Soc.Japan*, **19** (1967), 493-507.

[14] Maruyama,T., On a multi-valued differential equation : an existence theorem, *Proc.Japan Acad.*, **60A** (1984), 161-164.

[15] Maruyama,T., Variational problems governed by a multi-valued differential equation, *Proc.Japan Acad.*, **60A** (1984), 212-214.

[16] Maruyama,T., Lower semi-continuity theorems for nonlinear integral functionals on spaces of Bochner-integrable functions, preprint (1989).

[17] Maruyama,T., Weak convergence theorem in Sobolev spaces with application to Filippov's evolution equations, *Proc.Japan Acad.*, **66A** (1990) 217-221.

[18] Maruyama,T., A variational problem governed by a multi-valued differential equation in a separable Hilbert space, S.Elaydi ed., *Differential Equations — Stability and Control*, Marcel Dekker, New York, 1990.

[19] Papageorgiou,N.S., Existence and variational problems for nonlinear evolution inclusions, *Math.Japonica*, **38** (1993).433-443.

[20] Saint-Beuve,M.F., On the extension of von Neumann-Aumann's Theorem, *J.Functional Anal.*, **17** (1974), 112-129.

[21] Schwartz,L., *Functional Analysis*, Couran Institute of Mathematical Sciences, New York, 1964.

[22] Schwartz.L., *Théorie des distributions*, Hermann, Paris, 1966.

[23] Szep,A., Existence theorem for weak solutions of ordinary differential equations in reflexive Banach spaces, *Studia Scientiarum Mathematicarum Hungarica*, 6 (1971), 197-203.

[24] Tateishi,H., Nonconvex-valued differential inclusions in a separable Hilbert space, *Proc.Japan Acad.*, **68A** (1992) 296-301.

[25] Tolstonogov,A.A., On the structure of the solution set for diffenretial inclusions in a Banach space, *Math. USSR Sbornik*, **46** (1983), 1-15.

[26] Valadier,M., Application des mesures de Young aux suites uniformément intégrables dans un Banach séparable, *Sém.Anal.Convexe Montpellier*, exposé **3**, 1990.

[27] Valadier,M., Young measures, A.Cellina ed., *Methods of Nonconvex Analysis*, Springer, Berlin / Heidelberg / New York, 1990

[28] Yosida,K., *Functional Analysis*, 3rd ed., Springer, Berlin, 1971.