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Stationary Markov Equilibria for Temporary Equilibrium Processes under Heterogeneous Expectations*

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Abstract

The purpose of this paper is to show the existence of stationary Markov processes of temporary equilibria within the framework of a stochastic overlapping generations model, considering the possibility that each generation may have heterogeneous expectations for future states. As the main theorem, we show that there exists a time-homogeneous Markov temporary equilibrium process whose transition admits an ergodic measure when the expectations of members in each generation are independently and identically distributed. The result may be regarded as an extension of the analysis in Grandmont and Hildenbrand (1974) and Blume (1982). To establish the result we enlarge sufficient conditions of theorem 1.1 of Duffie et al. (1994).

1 Introduction

In stochastic overlapping generations models, one can view the evolution of the economy as a sequence of equilibria. Particularly, since a stationary equilibrium process may be a focal point on dynamic economies, the problem of existence of stationary equilibrium processes has been studied in various frameworks. As seen in Spear and Srivastava [15] and Duffie et al. [5], almost literatures show the existence of rational expectations equilibria. The assumption of rational expectations imposes the restriction that agent's beliefs about future realizations be the true distributions of the relevant variables. Alternatively, the same problem has been studied within the framework of temporary general equilibrium models investigated by Grandmont [6]. Examples include Grandmont and Hildenbrand [7], Hellwig [8] and Blume [1]. In temporary equilibrium settings, agent's beliefs about future realizations are not necessarily consistent with the true distributions. However, all agents have same expectations in the model of Grandmont and Hildenbrand [7].

The purpose of this paper is to show the existence of a stochastic process of temporary equilibria whose transition admits an ergodic measure, considering the possibility that each agent in sequential economies may have a heterogeneous expectation for future states. Since ergodic measures are the notion of stochastic analogue of steady states in deterministic systems, the existence of ergodic Markov processes is important, still more on the economy in which heterogeneous agents exist. Intuitively, ergodicity shows that the long-term average behavior of any dynamic paths on the economy is stable. The main theorem shows that under standard assumptions there exists a time-homogeneous Markov temporary

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equilibrium process whose transition admits an ergodic measure if the expectations of members in each generation are independently and identically distributed. This result may be regarded as an extension of the analysis in Grandmont and Hildenbrand [7] and Blume [1]. The assumption that distributions on expectations are independent may be not so much restrictive from the economic point of view. Note that even if distributions on expectations are identical, the expectations of members in each generation may be in general heterogeneous on any dynamic paths of the temporary equilibrium process.

The proof of the main result depends heavily on the method developed in a work by Duffie et al. [5], where they established sufficient conditions under which an expectation correspondence has a time-homogeneous transition probability with an ergodic measure. We enlarge sufficient conditions of theorem 1.1 of them. In the case that the state space is a product of two Borel spaces and the expectations correspondence assigns each element of state space product measures whose marginal distributions to one space are arbitrary fixed, to ensure the existence of an invariant measure, we can take into account the noncompact self-justified set which is represented by the product of a compact set and a noncompact measurable set. This result is quite similar to Theorem 2.1 of Blume [1].

The remainder of this paper is organized as follows. In section 2, some definitions of stationary Markov equilibrium are introduced, adopting terminologies and notations similar to those in Duffie et al. [5]. In section 3, we analyze a temporary equilibrium model and prove the main result. The arguments in this section are mainly due to Grandmont and Hildenbrand [7], Christiansen and Majumdar [2] and Blume [1].

The following notational conventions, definitions, and facts will be employed in this paper. For a correspondence $F : X \to Y$, $f \overset{m}{\sim} F$ will denote a measurable selection from F and $\mathrm{Gr}(F)$ will denote the graph of F. The product of topological spaces will always be given the product topology. The σ-algebra over any topological space X is to be understood to be the Borel σ-algebra, denoted by $\mathscr{B}(X)$. Given a measurable space $(X, \mathscr{B}(X))$, the set of probability measures on X is denoted by $\mathscr{P}(X)$. Unless otherwise stated, $\mathscr{P}(X)$ will be endowed with the weak convergence topology (Parthasarathy [12, p.40]). Given the weak convergence topology, if X is a compact metric space, then $\mathscr{P}(X)$ is compact metric space (Parthasarathy [12, p.45]).

2 Stationary Markov Equilibria

We consider the method for constructing a stochastic process which describes equilibrium in each period. The descriptions and notations given here are for the most part consistent with Duffie et al. [5].

State space S is the space in which the equilibrium processes live. A state must therefore contain enough information to represent equilibrium conditions. State space may be constructed by both exogenous and endogenous spaces. We assume S is a nonempty Borel space[1] .

The equilibrating forces in the model are described by an *expectations correspondence* G defined by,

$$G : S \to \mathscr{P}(S).$$

$\mu \in G(s)$ is a distribution of tomorrow's state consistent with the constraints imposed by the current state s. If we regard S as the set of states of economy, then G can be interpreted as the constraint which restricts the relation between a current state and distributions of tomorrow's states.

[1] A Borel space is a measurable subset of a complete separable metric space, endowed with the relative topology and the σ-algebra generated by relatively open sets.

Now, consider a set $J \subset S$ such that if an arbitrary state in J is currently realized, then a corresponding distribution of tomorrow's states can be again a distribution on J. Such a subset is called a *self-justified set*. That is, a self-justified set is a nonempty measurable set $J \subset S$ such that $G(s) \cap \mathscr{P}(J) \neq \emptyset$ for all $s \in J$. Then we can obtain a selection Π from G on J such that $\Pi(s) \in G(s) \cap \mathscr{P}(J)$ for all $s \in J$. If Π is measurable, Π is what is called a *transition probability*. Given $\Pi : J \to \mathscr{P}(J)$ and an arbitrary initial distribution $\mu \in \mathscr{P}(J)$, we can construct a time-homogeneous Markov process $\{\tilde{s}_t\}_{t=1}^{\infty}$ on some probability space such that for all t it is almost surely the case that the conditional distribution of $\tilde{s}_{t+1}$ given $\tilde{s}_1, \cdots, \tilde{s}_t$ is in $G(\tilde{s}_t)$. (This is the usual theory of Markov processes. See e.g. Doob [3, p.190]. About conditional distribution, see e.g. Dudley [4, p.269].) Then the evolution of $\{\tilde{s}_t\}_{t=1}^{\infty}$ governed by Π satisfies the constraint on distributions of tomorrow's states embodied in G. Hence we adopt the following natural definition.

Definition 2.1 *A time-homogeneous Markov equilibrium (THME) for G is a nonempty measurable subset J and a transition probability $\Pi : J \to \mathscr{P}(J)$ with $\Pi(s) \in G(s)$ for all $s \in J$.*

For convenience, let (J, Π) denote a THME for G. We are interested in a THME which is able to sustain some notions of stationarity. An *invariant measure* for a THME $\Pi : J \to \mathscr{P}(J)$ is a measure $\mu \in \mathscr{P}(J)$ such that

$$\mu(A) = \int \Pi(s)(A)\mathrm{d}\mu(s), \text{ for all } A \in \mathscr{B}(J).$$

Definition 2.2 *A stationary Markov equilibrium for G is a THME which has an invariant measure.*

A more restrictive notion of stationarity is given by an ergodic measure. If μ is invariant for a transition probability $\Pi : J \to \mathscr{P}(J)$, then a *$\mu$-invariant set* is a measurable subset $A \subset J$ satisfying $\Pi(s) \in \mathscr{P}(A)$ for μ-a.e. $s \in A$. If A is a μ-invariant set, A^c is also a μ-invariant set. An invariant measure μ is *ergodic* for the transition Π if, for any μ-invariant set A, $\mu(A) = 0$, or $\mu(A) = 1$. Suppose that μ is an ergodic measure for Π. Let $\{\tilde{s}_t\}$ be a Markov process induced by initial distribution μ and transition Π. Let $(\Omega, \mathscr{F}, P)$ be the probability space which is the domain of $\{\tilde{s}_t\}$. Ergodicity is the following property(See e.g. Doob [3, p.219]). For any $h \in L^1(J, \mu)$,

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} h(\tilde{s}_t(\omega)) = \int h\mathrm{d}\mu, \; P - a.e. \; \omega \in \Omega.$$

Roughly speaking, the sample distribution generated by $\{\tilde{s}_t\}$ converges to the ergodic measure almost everywhere[2] .

Definition 2.3 *An ergodic Markov equilibrium for G is a THME which has an ergodic measure.*

For convenience, if μ is an invariant (ergodic) measure for a THME (J, Π), let (J, Π, μ) denote a stationary (ergodic) Markov equilibrium for G.

We summarize the result of Duffie et al. [5] in the following proposition.

Proposition 2.1 (Duffie et al. (1994)) *If G is convex-valued with a closed graph and there exists a compact self-justified set $J \subset S$, then there exists an ergodic Markov equilibrium (J, Π, μ) for G.*

[2] For any point $\omega \in \Omega$ let μ_n^{ω} denote the measure which has mass $\frac{1}{n}$ at each of the n points $\tilde{s}_1(\omega), \cdots, \tilde{s}_n(\omega)$. μ_n^{ω} is called the sample distribution based on the n random mapping $\tilde{s}_1, \cdots, \tilde{s}_n$ at ω(Parthasarathy [12, p.52]).

Now we wish extend their result. Consider the following situation.

(1) state space is the product of two Borel spaces S_1 and S_2.
(2) $\hat{G} : S_1 \times S_2 \to \mathscr{P}(S_1)$ is given.

The main result in this section is as follows.

Theorem 2.1 *If $\hat{G}$ is convex-valued with a closed graph and there exists a nonempty compact set $J_1 \subset S_1$ and a nonempty measurable set $J_2 \subset S_2$ such that $\hat{G}(s_1, s_2) \cap \mathscr{P}(J_1) \neq \emptyset$ for all $(s_1, s_2) \in J_1 \times J_2$, then for all $\lambda \in \mathscr{P}(J_2)$ there exists an ergodic Markov equilibrium $(J_1 \times J_2, \Pi, \mu \otimes \lambda)$ for G, defined by $G(s_1, s_2) = \{\mu \otimes \lambda | \mu \in \hat{G}(s_1, s_2)\}$.*

PROOF: See Appendix.

When $S_2 = \{s_2^0\}$, it can be reduced to the case in Duffie et al. [5]. On the contrary, since the self-justified set $J_1 \times J_2$ may not necessarily be a compact subset of $S_1 \times S_2$, Theorem 2.1 is not contained in Proposition 2.1. Our result is quite similar to Theorem 2.1 of Blume [1]. In the same setting, Blume shows the existence of stationary Markov equilibrium.

3 Ergodicity of Temporary Equilibrium Processes

3.1 The Primitives and Equilibrium Definition

We apply the result of section 2 to a temporary equilibrium model with the framework of an overlapping generations model. The fundamentals of the economy have the following conditions. In some case we use subscript t to indicate the period explicitly.

Y is a compact metric space of exogenous shocks. The random process $\{y_t\}$ of exogenous shocks is a time-homogeneous Markov process with transition $P : Y \to \mathscr{P}(Y)$.

Assumption 3.1 *P is continuous and $P(y)$ is atomless for all $y \in Y$.*

There are l perishable consumption goods in each period. The $(l+1)$-th good, money, can be stored from one period to the next. The commodity space is R^{l+1}. There are no future markets. We assume that the total stock of money is constant and equal to $M > 0$.

The number of agent types in each generation is n. Each agent lives for 2 periods. For agents of type i, the consumption set in a single period is $X^i = R^l_+$.

The private consumption endowments for the first (young) period of agents of type i is defined by $e_1^i : Y \to R^l_{++}$. $e_2^i : Y \to R^l_{++}$ gives the endowments of agents of type i for the second (old) period.

Assumption 3.2 *e_1^i and e_2^i are continuous for all i.*

Each agent has a von Neuman-Morgenstern utility function u^i defined over consumption goods in both periods.

Assumption 3.3 *$u^i : X^i \times X^i \to R$ is bounded continuous function which is strictly concave and strictly increasing for all i.*

With $e = (e_1^i, e_2^i)_{i=1}^n$ and $u = (u^i)_{i=1}^n$, the primitives of the economy is $\mathscr{E} = (P, e, u, M)$.

The set of admissible price systems is given by

$$\Delta = \{p = (q, r) \in R_+^l \times R_+ | \sum q_i + r = 1\}.$$

q is the vector of prices of consumption goods and r denotes the price of money. Let $\Delta^+ = \{p \in \Delta | q_i > 0, i = 1, \cdots, l\}$ and $\Delta^{++} = \{p \in \Delta^+ | r > 0\}$.

Let $A^i = R_+^{l+1}$ be the action space of agent i. For $a = (x, m) \in A^i$, x is current consumption and m is saving ,in terms of money, to be carried into the next period. Let $Z = \prod A^i \times \prod A^i \times \Delta$ be the endogenous space. We define the state space S as

$$S = \{(y, (a_1, a_2, p)) \in Y \times Z | \sum_i (x_1^i + x_2^i) = \sum_i (e_1^i(y) + e_2^i(y)), \ \sum_i m_1^i = M\}.$$

Next we consider the standard optimization problem of agents. Let us first consider an old agent at period t. Given the current exogenous shock y, price p and the action $\bar{a}_1^i \in A^i$ that he took at period $t-1$, an old agent chooses a current consumption $x_2^i \in X^i$ to maximize his utility $u^i(\bar{x}_1^i, \cdot)$ subject to

$$qx_2^i = qe_2^i(y) + r\bar{m}_1^i.$$

The solution of this problem is denoted $\phi_i(y, \bar{a}_1^i, p)$. Note that from Assumption 3.3 no solution exists if $p \in \Delta \setminus \Delta^+$. $\xi_2^i(y, \bar{a}_1^i, p) = (\phi^i(y, \bar{a}_1^i, p), 0)$ is the action chosen by the old agent i. This defines the old agent's demand function,

$$\xi_2^i : Y_t \times A_{t-1}^i \times \Delta_t^+ \to A_t^i.$$

Next consider a young agent at period t. A young agent makes a forecast of the equilibrium price system and of the exogenous shock in the next period. By assumption, the agent's expectation depends only on the current exogenous shock, price system which is currently quoted and the equilibrium price system that prevailed in the preceding period.

Assumption 3.4 *The expectation function $\psi^i : Y_t \times \Delta_{t-1} \times \Delta_t \to \mathscr{P}(Y_{t+1} \times \Delta_{t+1}^+)$ is continuous for all i. The set $\Psi^i \stackrel{\text{def}}{=} \{\psi^i | \psi^i$ is continuous.$\}$ is endowed with the compact-open topology*[3] .

Given the current exogenous shock y, the current price p_t, the previous equilibrium price p_{t-1} and the expectation function ψ^i, a young agent's choice among actions is made in the usual dynamic programming fashion. More precisely, the young agent chooses a_1^i to maximize, subject to his budget constraint $q_t x_1^i + r_t m_1^i = q_t e_1^i(y)$, the expected utility $v^i(y, \psi^i, p_{t-1}, p_t, a_1^i)$ defined by

$$v^i(y, \psi^i, p_{t-1}, p_t, a_1^i) = \int_{Y_{t+1} \times \Delta_{t+1}^+} u^i(x_1^i, \phi^i(\cdot, a_1^i, \cdot)) d\psi^i(y, p_{t-1}, p_t).$$

The optimal actions of this problem is denoted by $\xi_1^i(y, \psi^i, p_{t-1}, p_t)$.

If $p_t \in \Delta_t^+ \setminus \Delta_t^{++}$ and agent i has the expectation ψ^i such that for some (y, p_{t-1}), $\psi^i(y, p_{t-1}, p_t)(Y \times \Delta^{++}) > 0$, then clearly no optimal solution exists for agent i. Roughly speaking, if the price of money is zero in a period and a agent forecasts that it will be positive in the next period, he will infinitely

[3] A subbase for this topology is given by sets of the form $\{\psi | \psi(D) \subset U\}$ for any compact subset D of $Y \times \Delta \times \Delta$ and any open subset U of $\mathscr{P}(Y \times \Delta^+)$. Ψ is a separable metric space with this topology (Kuratowski [11], p.93-94).

demand money. Hence in the sequel we assume that all agents in the economy have the expectation $\psi^i \in \hat{\Psi}^i$ defined by

$$\hat{\Psi}^i = \{\psi^i \in \Psi^i | p_t \in \Delta_t^+ \setminus \Delta_t^{++} \Rightarrow \psi^i(y, p_{t-1}, p_t)(Y \times \Delta_{t+1}^{++}) = 0, \text{ for all } (y, p_{t-1}) \in Y \times \Delta_{t-1}\}.$$

Assumption 3.5 *$\psi^i \in \hat{\Psi}^i$ for all i.*

Let $\hat{\Psi} = \prod \hat{\Psi}^i$ be the space of expectation profiles for a single generation in a single period.

Definition 3.1 *A temporary equilibrium given $(y, s, \psi) \in Y_t \times S_{t-1} \times \hat{\Psi}_t$ is a pair of allocation and price $(a_1^*, a_2^*, p^*) \in Z_t$ satisfying the following conditions,*

(i) $a_1^{i*} \in \xi_1^i(y, \psi^i, p, p^*)$, for all i,
(ii) $a_2^{i*} = \xi_2^i(y, a_2^i, p^*)$, for all i,
(iii) $\sum_i (x_1^{i*} + x_2^{i*}) = \sum_i (e_1^i(y) + e_2^i(y))$,
(iv) $\sum_i m_1^{i*} = M$.

This is the standard temporary general equilibrium concept. Let $\{\tilde{\psi}_t\}_{t=1}^{\infty}$ be a $\hat{\Psi}$-valued stochastic process on some probability space $(\Omega, \mathscr{F}, Q)$.

Definition 3.2 *A temporary equilibrium process with respect to $\{\tilde{\psi}_t\}_{t=1}^{\infty}$ is a S-valued stochastic process $\{\tilde{s}_t\}_{t=0}^{\infty}$ on $(\Omega, \mathscr{F}, Q)$ such that it is almost surely the case that for all $t \geq 1$ (a_t^1, a_t^2, p_t) is a temporary equilibrium given (y_t, s_{t-1}, ψ_t).*

The main result of this paper is as follows.

Theorem 3.1 *Under assumptions from 3.1 to 3.5, if $\{\tilde{\psi}_t\}_{t=1}^{\infty}$ is an i.i.d. process, then for arbitrary economy $\mathscr{E}$ there exists a time-homogeneous Markov temporary equilibrium process $\{\tilde{s}_t\}_{t=0}^{\infty}$ with respect to $\{\tilde{\psi}_t\}_{t=1}^{\infty}$ whose transition admits an ergodic measure.*

The proof of the theorem is relegated to subsequent subsections. The result in Grandmont and Hildenbrand [7] follows immediately from Theorem 3.1 when $\tilde{\psi}_t$ is a constant mapping whose distribution is some Dirac measure δ_ψ for all t [4]. In this case there is no possibility that each generation may have heterogeneous expectations. While, in our setting, even if $\{\tilde{\psi}_t\}_{t=1}^{\infty}$ is i.i.d., the expectations of members in each generation are in general heterogeneous on a dynamic path $(s_{t-1}, \psi_t)_{t=1}^{\infty}$ of temporary equilibrium process $\{\tilde{s}_t\}_{t=0}^{\infty}$ with respect to $\{\tilde{\psi}_t\}_{t=1}^{\infty}$.

3.2 First Step: the Temporary Equilibrium Correspondence

In this subsection, from demand correspondences of agents we construct the temporary equilibrium correspondence and prove the properties.

Old agent's demand function has following properties.

Proposition 3.1 *Under Assumption 3.2 and 3.3,*

[4] the Dirac measure at ψ is the measure that assigns unit measure to the singleton $\{\psi\}$.

(i) $\phi^i : Y \times A_1^i \times \Delta^+ \to X_2^i$ *is a continuous function.*

(ii) *Let the sequence* $(y^n, \bar{a}_1^{in}, p^n) \in Y \times A_1^i \times \Delta^+$ *be convergent to* $(y, \bar{a}_1^i, p)$ *such that* $p \in \Delta \setminus \Delta^+$. *Then* $\|\phi^i(y^n, \bar{a}_1^{in}, p^n)\|$ *diverges to infinity.*

PROOF: The correspondence $B_2^i : Y \times A_1^i \times \Delta^+ \to X^i$, where $B_2^i(y, \bar{a}_1^i, p) = \{x_2^i \in X^i | qx_2^i = qe_2^i(y) + r\bar{m}_1^i\}$ is clearly nonempty, compact-valued and continuous correspondence. Since $u^i(\bar{x}_1^i, \cdot)$ is continuous and strictly concave, (i) follows from the Maximum theorem. (ii) is obvious from strict monotonicity of u^i. ∎

Clearly ξ_2^i is also continuous.

Next we consider the optimization problem of young agents. The expected utility v^i has the following properties.

Lemma 3.1 *Under assumptions from 3.2 to 3.4,* $v^i : Y \times \Psi^i \times \Delta_{t-1} \times \Delta_t^+ \times A_1^i \to R$ *is a continuous function.*

PROOF: Let $(y^n, \psi^{in}, p_{t-1}^n, p_t^n, a_1^{in}) \to (y, \psi^i, p_{t-1}, p_t, a_1^i)$. Clearly,

$$v^i(y, \psi^i, p_{t-1}, p_t, a_1^i) = \int_R u^i(x_1^i, \cdot) \mathrm{d}\psi^i(y, p_{t-1}, p_t)\phi^{i^{-1}}(a_1^i, \cdot, \cdot).$$

Since from theorem 7.5 of Kelley [10, p.223], compact-open topology and jointly continuous topology is coincident on Ψ^i, we have $\psi^{in}(y^n, p_{t-1}^n, p_t^n) \to \psi^i(y, p_{t-1}, p_t)$. Define $\mu^n = \psi^{in}(y^n, p_{t-1}^n p_t^n)\phi^{i^{-1}}(a_1^{in}, \cdot, \cdot)$ and $\mu = \psi^i(y, p_{t-1}, p_t)\phi^{i^{-1}}(a_1^i, \cdot, \cdot)$. Then we obtain $\mu^n \to \mu$ by theorem 68 of Hildenbrand [9, p.51].

$$\begin{aligned} \Big|\int u^i(x_1^{in}, \cdot)\mathrm{d}\mu^n - \int u^i(x_1^i, \cdot)\mathrm{d}\mu\Big| \;\le\; & \Big|\int u^i(x_1^{in}, \cdot)\mathrm{d}\mu^n - \int u^i(x_1^{in}, \cdot)\mathrm{d}\mu\Big| \\ & + \Big|\int u^i(x_1^{in}, \cdot)\mathrm{d}\mu - \int u^i(x_1^i, \cdot)\mathrm{d}\mu\Big|. \end{aligned} \tag{1}$$

Since $\{u^i(x_1^{in}, \cdot)\}$ is uniformly bounded and equicontinuous, the first term of the right-hand side of (1) converges to zero from theorem 6.8 of Parthasarathy [12, p.51]. The second term also converges to zero because $u^i(x_1^{in}, \cdot) \to u^i(x_1^i, \cdot)$. Hence $v^i(y^n, \psi^{in}, p_{t-1}^n, p_t^n, a_1^{in}) \to v^i(y, \psi^i, p_{t-1}, p_t, a_1^i)$. ∎

Lemma 3.2 *Consider* $a_1^i, a_1'^i \in A_1^i$ *such that* $a_1^i \neq a_1'^i$. *Under assumptions from 3.2 to 3.4, for any* $(y, \psi^i, p_{t-1}, p_t) \in Y \times \Psi^i \times \Delta_{t-1} \times \Delta_t^+$ *and* $t \in (0,1)$, $v^i(y, \psi^i, p_{t-1}, p_t, ta_1^i + (1-t)a_1'^i) \geq tv^i(y, \psi^i, p_{t-1}, p_t, a_1^i) + (1-t)v^i(y, \psi^i, p_{t-1}, p_t, a_1'^i)$. *Inequality is strict if and only if (1)* $x_1^i \neq x_1'^i$ *or (2)* $x_1^i = x_1'^i$ *and* $m_1^i \neq m_1'^i$ *and* $(y, \psi^i, p_{t-1}, p_t)$ *satisfies that* $\psi^i(y, p_{t-1}, p_t)(Y \times \Delta^{++}) > 0$.

PROOF: By definition of ϕ^i

$$\begin{aligned} & v^i(y, \psi^i, p_{t-1}, p_t, ta_1^i + (1-t)a_1'^i) \\ = & \int u^i(tx_1^i + (1-t)x_1'^i, \phi^i(ta_1^i + (1-t)a_1'^i, \cdot, \cdot))\mathrm{d}\psi^i(y, p_{t-1}, p_t) \\ \geq & \int u^i(tx_1^i + (1-t)x_1'^i, t\phi^i(a_1^i, \cdot, \cdot) + (1-t)\phi^i(a_1'^i, \cdot, \cdot))\mathrm{d}\psi^i(y, p_{t-1}, p_t). \end{aligned}$$

(1) is obvious from strict concavity of u^i. Let $m_1^i > m_1'^i$. For any $(\hat{y}, \hat{p}_{t+1}) \in Y \times \Delta_{t+1}^+ \setminus \Delta_{t+1}^{++}$ the budget constraints $B_2^i(\hat{y}, \hat{p}_{t+1}, a_1^i)$ and $B_2^i(\hat{y}, \hat{p}_{t+1}, a_1'^i)$ are coincident and hence $\phi^i(\hat{y}, \hat{p}_{t+1}, a_1^i) = \phi^i(\hat{y}, \hat{p}_{t+1}, a_1'^i)$.

For any $(\hat{y}, \hat{p}_{t+1}) \in Y \times \Delta_{t+1}^{++}$ we have $\phi^i(\hat{y}, \hat{p}_{t+1}, a_1^i) \neq \phi^i(\hat{y}, \hat{p}_{t+1}, a'^i_1)$ from strict monotonicity of u^i. Hence when $\psi^i(y, p_{t-1}, p_t)(Y \times \Delta^{++}) > 0$, (2) follows from strict concavity of u^i. ∎

We can prove the following properties of demand correspondence ξ_1^i.

Proposition 3.2 *Under assumptions from 3.2 to 3.5,*

(i) $\xi_1^i : Y \times \hat{\Psi}^i \times \Delta_{t-1} \times \Delta_t^+ \to A_1^i$ *has a closed graph.*
(ii) $\xi_1^i : Y \times \hat{\Psi}^i \times \Delta_{t-1} \times \Delta_t^{++} \to A_1^i$ *is a continuous function.*
(iii) *Let* $(y, \psi^i, p_{t-1}, p_t) \in Y \times \hat{\Psi}^i \times \Delta_{t-1} \times (\Delta_t^+ \setminus \Delta_t^{++})$. $\xi_1^i(y, \psi^i, p_{t-1}, p_t)$ *is nonempty. If* $a_1^i = (x_1^i, m_1^i) \in \xi_1^i(y, \psi^i, p_{t-1}, p_t)$, *then* $(x_1^i, t m_1^i) \in \xi_1^i(y, \psi^i, p_{t-1}, p_t)$ *for all* $t \geq 0$.

PROOF:
(i) Let $(y^n, \psi^{in}, p_{t-1}^n, p_t^n) \to (y, \psi^i, p_{t-1}, p_t)$ and $a_1^{in} \to \bar{a}_1^i$ with $a_1^{in} \in \xi_1^i(y^n, \psi^{in}, p_{t-1}^n, p^n)$. The correspondence $B_1^i : Y \times \hat{\Psi}^i \times \Delta_t^+ \to A_1^i$, where $B_1^i(y, \psi^i, p_t) = \{a_1^i \in A_1^i | p_t a_1^i = q_t e_1^i(y)\}$ is clearly nonempty-valued and continuous correspondence. Since v^i is continuous from Lemma 3.1, $v^i(y^n, \psi^{in}, p_{t-1}^n, p_t^n, a_1^{in}) \geq v^i(y^n, \psi^{in}, p_{t-1}^n, p_t^n, a_1^i)$, for all $a_1^i \in B_1^i(y^n, \psi^{in}, p_t^n)$ implies $v^i(y, \psi^i, p_{t-1}, p_t, \bar{a}_1^i) \geq v^i(y, \psi^i, p_{t-1}, p_t, a_1^i)$, for all $a_1^i \in B_1^i(y, \psi^i, p_t)$. Hence $\bar{a}_1^i \in \xi_1^i(y, \psi^i, p_{t-1}, p_t)$. ∎

(ii) $B_1^i : Y \times \hat{\Psi}^i \times \Delta_t^{++} \to A_1^i$ is clearly nonempty, compact-valued and continuous correspondence. Then ξ_1^i is nonempty-valued and u.s.c. on $Y \times \hat{\Psi}^i \times \Delta_{t-1} \times \Delta_t^{++}$ by maximum theorem. Let a_1^i ,$a'^i_1 \in \xi_1^i(y, \psi^i, p_{t-1}, p_t)$. From Lemma 3.2(i) $x_1^i = x'^i_1$. But this implies $m_1^i = m'^i_1$ since $q_t x_1^i + r_t m_1^i = q_t x'^i_1 + r_t m'^i_1$ and $r_t > 0$. Hence $\xi_1^i(y, \psi^i, p_{t-1}, p_t)$ contains only one point. Hence ξ_1^i is a continuous function.

(iii) From Assumption 3.5 we may restrict our attention to the state $(y, p) \in Y \times \Delta_{t+1}^+ \setminus \Delta_{t+1}^{++}$. While in this case for any a_1^i and $a'^i_1 \in A_1^i$ such that $x_1^i = x'^i_1$, we have $\phi^i(a_1^i, y, p) = \phi^i(a'^i_1, y, p)$. Now consider the problem of maximizing $v^i(y, \psi^i, p_{t-1}, p_t, \cdot)$ under $B_{10}^i(y, \psi^i, p_t) \stackrel{\text{def}}{=} \{a_1^i \in A_1^i | p_t a_1^i = q_t e_1^i(y), m_1^i = 0\}$. Let $\xi_{10}^i(y, \psi^i, p_{t-1}, p_t)$ be the optimal actions of this problem. We define the set

$$\hat{\xi}_1^i(y, \psi^i, p_{t-1}, p_t) = \{a_1^i = (x_1^i, m_1^i) | \text{for some } (x_1^i, 0) \in \xi_{10}^i(y, \psi^i, p_{t-1}, p_t),\ (x_1^i, m_1^i) \text{ for all } m_1^i \geq 0\}.$$

Then clearly $\hat{\xi}_1^i(y, \psi^i, p_{t-1}, p_t) = \xi_1^i(y, \psi^i, p_{t-1}, p_t)$. While $\xi_{10}^i(y, \psi^i, p_{t-1}, p_t)$ is nonempty from compactness of $B_{10}^i(y, \psi^i, p_t)$. Therefore $\xi_1^i(y, \psi^i, p_{t-1}, p_t)$ is also nonempty. ∎

We shall denote by $V(y, s, \psi)$ the set of temporary equilibria given (y, s, ψ). Now we define the *temporary equilibrium correspondence* as

$$W : Y_t \times S_{t-1} \times \hat{\Psi}_t \ni (y, s, \psi) \mapsto (y, V(y, s, \psi)) \subset S_t.$$

We can prove the following proposition.

Proposition 3.3 *Under assumptions from 3.2 to 3.5, W is (i) nonempty-valued, (ii) compact-valued and u.s.c.*

PROOF:
(i) Fix (y_0, s_0, ψ_0). For any $p \in \Delta^+$, consider the set $\zeta(p)$ of aggregate excess demand,

$$\zeta(p) = \sum_i \Big(\xi_1^i(y_0, \psi_0^i, p_0, p) + \{\xi_2^i(y_0, a_{10}^i, p)\} - \{(e_1^i(y_0) + e_2^i(y_0), M)\} \Big).$$

We wish to find a p^* such that $0 \in \zeta(p^*)$. Now from Proposition 3.1 and 3.2, ζ is a continuous function on Δ^{++}. From Walras law, $p \in \Delta^+$, $z \in \zeta(p)$ imply $pz = 0$. Next consider a increasing sequence of compact, convex subsets Δ^n of Δ^{++} such that $\Delta^{++} \subset \bigcup \Delta^n$. For each n we define Q^n as the convex hull of $\zeta(\Delta^n)$. Then Q^n is convex compact set. For any $z \in Q^n$, let $\theta^n(z)$ be the set of prices $p \in \Delta^n$ which maximize $p \cdot z$. To each $(p,z) \in \Delta^n \times Q^n$, associate the set $\theta^n(z) \times \{\zeta(p)\}$. This correspondence has a fixed point. That is, there exists $p^n \in \Delta^n$ and $z^n = \zeta(p^n)$ such that $pz^n \leq p^n z^n = 0$, for all $p \in \Delta^n$. This implies that $\{z^n\}$ is bounded. Indeed $\{z^n\}$ is clearly bounded below. And since $p^1 z^n \leq 0$ with $0 \ll p^1 \in \Delta^1$, $\{z^n\}$ is bounded above. We may therefore obtain convergent subsequences $z^{n_i} \to z^* \in R^{l+1}$ and $p^{n_i} \to p^* \in \Delta$. Clearly $p^* \in \Delta^+$, for otherwise, one could contradict Proposition 3.1(i). Since ζ has a closed graph on Δ^+, this implies $z^* \in \zeta(p^*)$. Finally $pz^* \leq p^* z^* = 0$ for all $p \in \Delta^+$. It follows that $z^* \leq 0$. Hence if $p^* \in \Delta^{++}$, $0 = z^* \in \zeta(p^*)$. When $p^* \in \Delta^+ \setminus \Delta^{++}$, we have $z_g^* = 0$ and $z_m^* \leq 0$, where $z^* = (z_g^*, z_m^*) \in R^l \times R$. From Proposition 3.2(iii), this implies $0 \in \zeta(p^*)$ in this case, too.

(ii) Let us first prove the following Lemma.

Lemma 1 Let X and Y be metric spaces. $\phi : X \to Y$ is compact-valued and u.s.c. if and only if $x^n \to x^0$, $y^n \in \phi(x^n)$ implies there exists a subsequence y^{n_i} converging to some $y \in \phi(x)$.

PROOF: (only if part) Since $\bigcup_{n=0}^{\infty}\{x^n\}$ is compact, $\bigcup_{n=0}^{\infty}\phi(x^n)$ is also compact. Hence y^n has a convergent subsequence $y^{n_i} \to y$. Then $y \in \phi(x^0)$ since ϕ has a closed graph. (if part) ϕ is clearly compact-valued. Suppose that for some x^0, ϕ is not u.s.c. There exists an open set $U \supset \phi(x^0)$ and a sequence $x^n \in V^n(x^0)$ such that $V^n(x^0) = \{x \in X | d(x^0, x) < \frac{1}{n}\}$ and $\phi(x^n) \not\subset U$. We have a sequence $y^n \in \phi(x^n) \bigcap U^c$. Then there exists a subsequence $y^{n_i} \to y \in \phi(x^0)$. This contradicts $y \in U^c$. ∎

Hence it is sufficient to show that $(y^n, s^n, \psi^n) \to (y, s, \psi)$ and $\bar{s}^n \in W(y^n, s^n, \psi^n)$ implies that there exists a subsequence $\bar{s}^{n_i} \to \bar{s} \in W(y, s, \psi)$. Since ξ_1^i has a closed graph and ξ_2^i is continuous, $\zeta(p, y, s, \psi)$ has a closed graph. it is easy to see that W itself has a closed graph. We claim that $\bar{s}^n = (\bar{y}^n, \bar{a}_1^n, \bar{a}_2^n, \bar{p}^n)$ has a convergent subsequence. Since $\bar{p}^n \in \Delta$, there is a subsequence $\bar{p}^{n_i} \to p \in \Delta$. Clearly $p \in \Delta^+$, for otherwise, one could contradict Proposition 3.1(ii). Since compactness of $\sum(e_1^i(Y) + e_2^i(Y))$, there exists $k \in R^l$ such that $\sum(e_1^i(y) + e_2^i(y)) \ll k$ for all $y \in Y$. Hence we have $0 \leq x_1^{in} \leq \sum(e_1^i(y^n) + e_2^i(y^n)) \ll k$ and $0 \leq m_1^{in} \leq M$ for all n and i. Then there is a convergent subsequence of a_1^n. Therefore There exists a subsequence $(\bar{y}^{n_i}, \bar{a}_1^{n_i}, \bar{a}_2^{n_i}, \bar{p}^{n_i}) \to (\bar{y}, \bar{a}_1, \bar{a}_2, \bar{p}) \in Y \times R^{l+1} \times R^{l+1} \times \Delta^+$. Hence we have $(\bar{y}, \bar{a}_1, \bar{a}_2, \bar{p}) \in W(y, s, \psi)$ since W has a closed graph. ∎

3.3 Second Step: the Expectations Correspondence

First, we can prove the following lemma. This lemma shows the existence of a self-justified set.

Lemma 3.3 *Under assumptions from 3.2 to 3.5, For any state $s \in S$ there is a compact subset J of S containing s such that $W(Y, J, \hat{\Psi}) \subset J$.*

PROOF: Assume the proposition is false. Then there exists $s_0 = (y_0, a_{10}, a_{20}, p_0) \in S$ such that for every compact subset J containing s_0 we have $W(Y, J, \hat{\Psi}) \not\subset J$. Choose $p \in \Delta \setminus \Delta^+$ such that $p \neq p_0$. Let $U_n(p) = \{z \in R^{l+1} | d(p, z) < \frac{1}{n}\}$. For some n we have $p_0 \notin U_n(p)$. $\Delta^m \stackrel{\text{def}}{=} \Delta \bigcap U_{n+m}(p)^c$ is a compact subset of Δ for all $m \in N$. Then $J^m = \{(y_0, a_{10}, a_{20})\} \times \Delta^m$ is a sequence of compact subset of S containing s_0. By assumption there exists $(y^m, s^m, \psi^m) \in Y \times J^m \times \hat{\Psi}$ and there is $\bar{s}^m \in S$ such that (1)$\bar{s}^m \in W(y^m, s^m, \psi^m)$ and (2)$\bar{s}^m \notin J^m$. Clearly $\bar{p}^m \to p \in \Delta \setminus \Delta^+$. While the sequence $\{\bar{a}_2^m\}$ is bounded. This is a contradiction to Proposition 3.1(ii). ∎

Next, we concretely construct the expectations correspondence $\hat{G}$. From Lemma 3.3 there exists a nonempty compact set $J \subset S$ such that $W(Y, J, \hat{\Psi}) \subset J$. Let W_J denote the restriction W to $Y \times J \times \hat{\Psi}$, that is,

$$W_J : Y_t \times J_{t-1} \times \hat{\Psi}_t \to J_t.$$

J_{t-1} and J_t are identical. Fix (s_{t-1}, ψ_t). This point (s_{t-1}, ψ_t) and an exogenous shock y_t determine corresponding t-period's temporary equilibria, $W_J(y_t, s_{t-1}, \psi_t)$. Now choose a selection f from W_J. Then $f(\cdot, s_{t-1}, \psi_t)$ is a function which assigns a exogenous shock a corresponding temporary equilibrium. Meanwhile, since t-period's distribution of exogenous shocks is determined to $P(y_{t-1})$, Y_t is regarded as the probability space $(Y_t, \mathscr{B}(Y_t), P(y_{t-1}))$. In addition, if f is a measurable selection from W_J, we can obtain a distribution of temporary equilibrium, $P(y_{t-1})f(\cdot, s_{t-1}, \psi_t)^{-1} \in \mathscr{P}(J_t)$. $\hat{G}(s_{t-1}, \psi_t)$ is constructed as the set of all t-period's distributions of temporary equilibrium related to (s_{t-1}, ψ_t). That is,

$$\hat{G} : J_{t-1} \times \hat{\Psi}_t \ni (s, \psi) \mapsto \{P(y)f(\cdot, s, \psi)^{-1} \in \mathscr{P}(J_t) | f \overset{m}{\sim} W_J\} \subset \mathscr{P}(J_t).$$

Note that $\hat{G}$ is nonempty-valued. Indeed, since W_J is clearly nonempty-valued, compact-valued and u.s.c. from Proposition 3.3, there exists a measurable selection f from W_J by the Kuratowski-Ryll-Nardzewski Theorem (Hildenbrand [9],p.55).

Finally, we can prove the following property of $\hat{G}$.

Proposition 3.4 *Under assumptions from 3.1 to 3.5, (i) $\hat{G}$ has a closed graph and (ii) $\hat{G}$ is convex-valued.*

PROOF: We claim that $\hat{\Psi} \subset \Psi$ is closed. it suffices to show that $\hat{\Psi}^i \subset \Psi^i$ is closed. Let $\psi^{in} \to \psi^i$ with $\psi^{in} \in \hat{\Psi}^i$. Choose $p_t \in \Delta^+ \setminus \Delta^{++}$. For all (y, p_{t-1}), we have $\psi^{in}(y, p_{t-1}, p_t) \to \psi^i(y, p_{t-1}, p_t)$. (See the proof of Lemma 3.1.) Since $Y \times \Delta^{++} \subset Y \times \Delta^+$ is open, by the theorem 6.1 of Parthasarathy [12, p.40],

$$0 = \liminf_n \psi^{in}(y, p_{t-1}, p_t)(Y \times \Delta^{++}) \geq \psi^i(y, p_{t-1}, p_t)(Y \times \Delta^{++}).$$

Hence $\psi^i(y, p_{t-1}, p_t)(Y \times \Delta^{++}) = 0$. Now Proposition 3.4 follows from theorem 3.1 and 3.2 of Blume [1]. ∎

Now Theorem 3.1 follows directly from Theorem 2.1. From Proposition 3.4 $\hat{G}$ is convex-valued with a closed graph. J is compact and for all $(s, \psi) \in J \times \hat{\Psi}$, $\hat{G}(s, \psi) \subset \mathscr{P}(J)$. The assumptions of Theorem 2.1 is all satisfied. Let $\lambda \in \mathscr{P}(\hat{\Psi})$ be the identical distribution of $\{\tilde{\psi}_t\}_{t=1}^{\infty}$. When $G : J \times \hat{\Psi} \to \mathscr{P}(J \times \hat{\Psi})$ is defined by $G(s, \psi) = \{\mu \otimes \lambda | \mu \in \hat{G}(s, \psi)\}$, we can obtain an ergodic Markov equilibrium $(J \times \hat{\Psi}, \Pi, \mu \otimes \lambda)$ for G. The Markov process $\{\tilde{s}_{t-1}, \tilde{\psi}_t\}_{t=1}^{\infty}$ which is constructed by $(J \times \hat{\Psi}, \Pi, \mu \otimes \lambda)$ is clearly the required temporary equilibrium process $\{\tilde{s}_t\}_{t=0}^{\infty}$ with respect to $\{\tilde{\psi}_t\}_{t=1}^{\infty}$. This completes the proof of Theorem 3.1.

Appendix

PROOF OF THEOREM 2.1 : (This proof is adapted from Duffie et al. [5, Theorem 1.1 and Corollary 1.1].)

The map j is defined by $j : \mathscr{P}(S_1) \ni \mu \mapsto \mu \otimes \lambda \in \mathscr{P}(S_1) \otimes \lambda$[5] . Clearly j is bijective and linear. Since G is the composite correspondence of j with $\hat{G}$, G is convex-valued with a closed graph. Let G_J be the restriction of G to $J_1 \times J_2$ in both domain and range, that is, $G_J : J_1 \times J_2 \to \mathscr{P}(J_1) \otimes \lambda$, where $G_J(s_1, s_2) = \hat{G}(s_1, s_2) \otimes \lambda$. G_J is also convex-valued with a closed graph. Let $m_1 : \mathscr{P}(\mathrm{Gr}(G_J)) \to \mathscr{P}(J_1 \times J_2)$ and $m_2 : \mathscr{P}(\mathrm{Gr}(G_J)) \to \mathscr{P}(\mathscr{P}(J_1) \otimes \lambda)$ be the restrictions to $\mathscr{P}(\mathrm{Gr}(G_J))$ of the functions that give the marginals of distributions on $J_1 \times J_2 \times \mathscr{P}(J_1) \otimes \lambda$.

Lemma 1 For arbitrary $\theta \in \mathscr{P}(J_1 \times J_2)$, there exists $\nu \in \mathscr{P}(\mathrm{Gr}(G_J))$ such that $m_1(\nu) = \theta$.

PROOF: There exists a measurable selection $\Pi \overset{m}{\sim} G_J$ from the Kuratowski-Ryll-Nardzewski Theorem (Hildenbrand [9],p.55). We define the function

$$k : J_1 \times J_2 \ni (s_1, s_2) \mapsto (s_1, s_2, \Pi(s_1, s_2)) \in J_1 \times J_2 \times \mathscr{P}(J_1) \otimes \lambda.$$

Clearly k is measurable. For arbitrary $\theta \in \mathscr{P}(J_1 \times J_2)$, we have a distribution $\theta k^{-1} \in \mathscr{P}(J_1 \times J_2 \times \mathscr{P}(J_1) \otimes \lambda)$. Then θk^{-1} is obviously a measure in $\mathscr{P}(\mathrm{Gr}(G_J))$. Since for all $A \in \mathscr{B}(J_1 \times J_2)$, $\theta k^{-1}(A \times \mathscr{P}(J_1) \otimes \lambda) = \theta(A)$, we have $m_1(\theta k^{-1}) = \theta$. ∎

From Lemma 1 we can define the nonempty-valued correspondence $m_1^{-1} : \mathscr{P}(J_1 \times J_2) \to \mathscr{P}(\mathrm{Gr}(G_J))$. m_1^{-1} is clearly convex-valued with a closed graph. Let $C(J_1)$ be the set of all continuous functions on J_1. Let $\Gamma : C(J_1) \to C(J_1 \times J_2)$ be the bounded linear operator which is defined by $\Gamma(f)(s_1, s_2) = f(s_1)$. For any $\eta \in \mathscr{P}(\mathscr{P}(J_1) \otimes \lambda)$, we define the bounded linear functional

$$\Lambda : C(J_1) \ni f \mapsto \int_{\mathscr{P}(J_1)\otimes\lambda} \int_{J_1 \times J_2} \Gamma(f)(s_1, s_2) \mathrm{d}\mu' \otimes \lambda(s_1, s_2) \mathrm{d}\eta(\mu' \otimes \lambda) \in R.$$

By the Reisz representation theorem (Royden [13], p. 357), there exists $E\eta \in \mathscr{P}(J_1)$ such that for all $f \in C(J_1)$

$$\Lambda(f) = \int_{J_1} f \mathrm{d}E\eta.$$

This function $E : \mathscr{P}(\mathscr{P}(J_1) \otimes \lambda) \to \mathscr{P}(J_1)$ is continuous and linear. Since $E \circ m_2 \circ m_1^{-1} \circ j : \mathscr{P}(J_1) \to \mathscr{P}(J_1)$ is convex-valued with a closed graph, by Fan-Glicksberg fixed point theorem this correspondence has a fixed point. Let M be the set of fixed points of this correspondence. M is clearly convex and compact.

We claim that for each $\mu \in M$ there is $\Sigma \overset{m}{\sim} \hat{G}$ such that $\mu = \int \Sigma \mathrm{d}\mu \otimes \lambda$. For each $\mu \in M$ there exists $\nu \in \mathscr{P}(\mathrm{Gr}(G_J))$ such that $E \circ m_2(\nu) = j^{-1} \circ m_1(\nu) = \mu$. Then there exists a measurable $P : J_1 \times J_2 \to \mathscr{P}(\mathscr{P}(J_1) \otimes \lambda)$ such that

$$\nu(E \times F) = \int_E P(s_1, s_2)(F) \mathrm{d}\mu \otimes \lambda, \text{ for all } E \in \mathscr{B}(J_1 \times J_2) \text{ and } F \in \mathscr{B}(\mathscr{P}(J_1) \otimes \lambda).$$

The existence of P can be verified from Dudley [4](p.269, theorem 10.2.1).

Lemma 2 $\mu = \int E \circ P \mathrm{d}\mu \otimes \lambda$.

[5] $\mathscr{P}(J_1) \otimes \lambda \overset{\mathrm{def}}{=} \{\mu \otimes \lambda | \mu \in \mathscr{P}(J_1)\}$.

PROOF: From Reisz representation theorem, for $f \in C(J_1)$, we have two equations

$$\int_{J_1\times J_2}\int_{J_1} f\mathrm{d}E\circ P\mathrm{d}\mu\otimes\lambda = \int_{J_1\times J_2}\int_{\mathscr{P}(J_1)\otimes\lambda}\int_{J_1\times J_2}\Gamma(f)\mathrm{d}\mu'\otimes\lambda\mathrm{d}P(s_1,s_2)(\mu'\otimes\lambda)\mathrm{d}\mu\otimes\lambda,$$

$$\int_{J_1} f\mathrm{d}E\left(\int P(s_1,s_2)\mathrm{d}\mu\otimes\lambda\right) = \int_{\mathscr{P}(J_1)\otimes\lambda}\int_{J_1\times J_2}\Gamma(f)\mathrm{d}\mu'\otimes\lambda\mathrm{d}\left(\int P(s_1,s_2)\mathrm{d}\mu\otimes\lambda\right).$$

Then right-hand sides of this equations are equal. For, in general, we have for an arbitrary measurable function g on $\mathscr{P}(J_1)\otimes\lambda$

$$\int_{J_1\times J_2}\int_{\mathscr{P}(J_1)\otimes\lambda} g\mathrm{d}P(s_1,s_2)\mathrm{d}\mu\otimes\lambda = \int_{\mathscr{P}(J_1)\otimes\lambda} g\mathrm{d}\left(\int P(s_1,s_2)\mathrm{d}\mu\otimes\lambda\right). \tag{2}$$

Indeed, it is easy to see that (2) holds for any characteristic functions and simple functions. For an arbitrary measurable g, there is an increasing sequence of simple functions which pointwisely converges to g. From monotone convergence theorem(Royden [13], p.265), (2) holds for g. Hence for each $f \in C(J_1)$, we have

$$\int_{J_1\times J_2}\int_{J_1} f\mathrm{d}E\circ P\mathrm{d}\mu\otimes\lambda = \int_{J_1} f\mathrm{d}E\left(\int P\mathrm{d}\mu\otimes\lambda\right).$$

For any closed set F of J_1, we define $B_n(F) = \{s_1 \in J_1 | d(F,s_1) < \frac{1}{n}\}$, $n \geq 1$. There is a continuous function f_n on J_1 to $[0,1]$ such that f_n is zero on $J_1 \setminus B_n(F)$ and one on F. Clearly $\lim f_n = \chi_F$. From bounded convergence theorem(Royden [13], p.267),

$$\begin{aligned}
&\lim \textstyle\int_{J_1\times J_2}\int_{J_1} f_n\mathrm{d}E\circ P\mathrm{d}\mu\otimes\lambda = \lim\int_{J_1} f_n\mathrm{d}E(\int P\mathrm{d}\mu\otimes\lambda)\\
\Leftrightarrow\ &\textstyle\int_{J_1\times J_2}\int_{J_1}\lim f_n\mathrm{d}E\circ P\mathrm{d}\mu\otimes\lambda = \int_{J_1}\lim f_n\mathrm{d}E(\int P\mathrm{d}\mu\otimes\lambda)\\
\Leftrightarrow\ &\textstyle\int_{J_1\times J_2} E\circ P(F)\mathrm{d}\mu\otimes\lambda = E(\int P\mathrm{d}\mu\otimes\lambda)(F).
\end{aligned}$$

Since every measure on a metric space is regular (Parthasarathy [12], p.27), we have $\int_{J_1\times J_2} E\circ P(A)\mathrm{d}\mu\otimes\lambda = E(\int P\mathrm{d}\mu\otimes\lambda)(A)$ for all $A \in \mathscr{B}(J_1)$. Further $E(\int P\mathrm{d}\mu\otimes\lambda) = E(m_2(\nu)) = \mu$ implies the required result. ∎

Next we will show that $E\circ P(s_1,s_2) \in \hat{G}(s_1,s_2)$, $\mu\otimes\lambda$-a.e. Consider $f \in C(J_1)$ and a constant $\varepsilon \in R$. We define

$$\begin{aligned}
A &\stackrel{\text{def}}{=} \left\{(s_1,s_2)\Big|\max_{\rho\in\hat{G}(s_1,s_2)}\int f\mathrm{d}\rho \leq \varepsilon < \int f\mathrm{d}E\circ P(s_1,s_2)\right\},\\
B &\stackrel{\text{def}}{=} \left\{\mu\otimes\lambda \in \mathscr{P}(J_1)\otimes\lambda\Big|\int\Gamma(f)\mathrm{d}\mu\otimes\lambda > \varepsilon\right\},\\
A' &\stackrel{\text{def}}{=} \left\{(s_1,s_2)\Big|\max_{\rho\in\hat{G}(s_1,s_2)}\int f\mathrm{d}\rho \leq \varepsilon\ ,\ P(s_1,s_2)(B) > 0\right\}.
\end{aligned}$$

Lemma 3 $A \subset A'$.

PROOF: Let $(s_1,s_2) \in A$.

$$\begin{aligned}
\varepsilon < \int_{J_1} f\mathrm{d}E\circ P(s_1,s_2) &= \int_{\mathscr{P}(J_1)\otimes\lambda}\int_{J_1\times J_2}\Gamma(f)\mathrm{d}\mu'\otimes\lambda\mathrm{d}P(s_1,s_2)\\
&= \int_B\int_{J_1\times J_2}\Gamma(f)\mathrm{d}\mu'\otimes\lambda\mathrm{d}P(s_1,s_2) + \int_{B^c}\int_{J_1\times J_2}\Gamma(f)\mathrm{d}\mu'\otimes\lambda\mathrm{d}P(s_1,s_2)\\
&\leq \int_B\int_{J_1\times J_2}\Gamma(f)\mathrm{d}\mu'\otimes\lambda\mathrm{d}P(s_1,s_2) + \int_{B^c}\varepsilon\mathrm{d}P(s_1,s_2)\\
&= \int_B\int_{J_1\times J_2}\Gamma(f)\mathrm{d}\mu'\otimes\lambda\mathrm{d}P(s_1,s_2) + \varepsilon P(s_1,s_2)(B^c).
\end{aligned}$$

Thus we have

$$\int_B \int_{J_1 \times J_2} \Gamma(f) \mathrm{d}\mu' \otimes \lambda \mathrm{d}P(s_1, s_2) > (1 - P(s_1, s_2)(B^c))\varepsilon.$$

Suppose $P(s_1, s_2)(B) = 0$. Since $P(s_1, s_2)(B^c) = 1$, we have $0 = \int_B \int_{J_1 \times J_2} \Gamma(f) \mathrm{d}\mu' \otimes \lambda \mathrm{d}P(s_1, s_2) > 0$. This is a contradiction. Hence $(s_1, s_2) \in A'$. ∎

Suppose that $\mu \otimes \lambda(A) > 0$. Then $\mu \otimes \lambda(A') > 0$ from Lemma 3. Since $\nu \in \mathscr{P}(\mathrm{Gr}(G_J))$ and $G_J(s_1, s_2) \cap B = \emptyset$ for any $(s_1, s_2) \in A'$, we now have

$$0 = \nu(A' \times B) = \int_{A'} P(s_1, s_2)(B) \mathrm{d}\mu \otimes \lambda > 0.$$

This is a contradiction. By similar arguments, for each $f \in C(J_1)$ we obtain

$$\min_{\rho \in \hat{G}(s_1, s_2)} \int f \mathrm{d}\rho \leq \int f \mathrm{d}E \circ P(s_1, s_2) \leq \max_{\rho \in \hat{G}(s_1, s_2)} \int f \mathrm{d}\rho, \ \mu \otimes \lambda - a.e.$$

Suppose that for $\mu \otimes \lambda$ -a.e.(s_1, s_2), $E \circ P(s_1, s_2) \notin \hat{G}(s_1, s_2)$. Let $\mathscr{P}(J_1)'$ be the dual of $\mathscr{P}(J_1)$. Since $\hat{G}(s_1, s_2) \subset \mathscr{P}(J_1)$ is a convex set, by separation theorem there is an element F of $\mathscr{P}(J_1)'$ such that $F(E \circ P(s_1, s_2)) > F(\rho)$ for all $\rho \in \hat{G}(s_1, s_2)$. While $C(J_1)$ is a weak* dense subset of $\mathscr{P}(J_1)'$(Schaefer [14, p.?]). Hence there exists $\bar{f} \in C(J_1)$ such that $\int \bar{f} \mathrm{d}E \circ P(s_1, s_2) > \int \bar{f} \mathrm{d}\rho$ for all $\rho \in \hat{G}(s_1, s_2)$. This is a contradiction. Hence $E \circ P(s_1, s_2) \in \hat{G}(s_1, s_2)$, $\mu \otimes \lambda$ -a.e.

Choose arbitrary $\Sigma' \overset{m}{\sim} \hat{G}$. Define the function Σ on $J_1 \times J_2$ whose value equals to $E \circ P(s_1, s_2)$ on $\mu \otimes \lambda$ -a.e.(s_1, s_2) and $\Sigma'(s_1, s_2)$ on the complement. Then $\Sigma \overset{m}{\sim} \hat{G}$. From Lemma 2, we have $\mu = \int \Sigma \mu \otimes \lambda$. Define $\Pi(s_1, s_2) = \Sigma(s_1, s_2) \otimes \lambda$, and now we have $\Pi \overset{m}{\sim} G_J$ such that $\mu \otimes \lambda = \int \Pi \mu \otimes \lambda$.

Finally we show that there exists $\mu \in M$ such that $\mu \otimes \lambda$ is an ergodic measure for some measurable selection Π from G_J. Let $\mu \otimes \lambda$ be an extreme point of $M \otimes \lambda$ [6] and Π be the measurable selection from G_J associated with $\mu \otimes \lambda$. Extreme points exist by the Krein-Milman theorem(Royden [13, p.242]). Let $F \subset J_2$ be the support of λ. We first prove the following lemma.

Lemma 4 If for some $E \subset J_1$, $E \times F$ is a $\mu \otimes \lambda$-invariant set, then $\mu \otimes \lambda(E \times F) = 1$ or 0.

PROOF: Suppose that $\mu \otimes \lambda(E \times F) \in (0, 1)$. Since $E \times F$ is $\mu \otimes \lambda$-invariant, $E^c \times F$ is also $\mu \otimes \lambda$-invariant. Let ν and η be the conditional probability of $\mu \otimes \lambda$ relative to $E \times F$ and $E^c \times F$ respectively. That is,

$$\nu(A) \overset{\text{def}}{=} \frac{\mu \otimes \lambda(A \cap (E \times F))}{\mu \otimes \lambda(E \times F)}, \ \eta(A) \overset{\text{def}}{=} \frac{\mu \otimes \lambda(A \cap (E^c \times F))}{\mu \otimes \lambda(E^c \times F)}, \text{ for all } A \in \mathscr{B}(J_1 \times J_2).$$

Clearly ν and η are elements of $\mathscr{P}(J_1) \otimes \lambda$. We claim that ν and η are invariant measures for Π. Since $\mu \otimes \lambda$ is an invariant measure for Π, we have $\mu \otimes \lambda(A \cap (E \times F)) = \int \Pi(A \cap (E \times F))(s_1, s_2) \mathrm{d}\mu \otimes \lambda$ for all $A \in \mathscr{B}(J_1 \times J_2)$. Hence

$$\nu(A) = \int \frac{\Pi(s_1, s_2)(A \cap (E \times F))}{\mu \otimes \lambda(E \times F)} \mathrm{d}\mu \otimes \lambda.$$

[6] $M \otimes \lambda \overset{\text{def}}{=} \{\mu \otimes \lambda | \mu \in M\}$.

Since $\mu \otimes \lambda$ is an invariant measure and $E \times F$ is a $\mu \otimes \lambda$-invariant set, we have

$$\begin{aligned}
\mu \otimes \lambda(A \cap (E \times F)) &= \int\int \Pi(s_1, s_2)(A \cap (E \times F))\Pi(s_1', s_2')(ds_1, ds_2)d\mu \otimes \lambda(s_1', s_2') \\
&= \int\int_{E \times F} \Pi(s_1, s_2)(A \cap (E \times F))\Pi(s_1', s_2')(ds_1, ds_2)d\mu \otimes \lambda \\
&= \int\int_{E \times F} \Pi(s_1, s_2)(A)\Pi(s_1', s_2')(ds_1, ds_2)d\mu \otimes \lambda \\
&= \int\int \Pi(s_1, s_2)(A)\Pi(s_1', s_2')((E \times F) \cap (ds_1, ds_2))d\mu \otimes \lambda.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\mu \otimes \lambda(A \cap (E \times F))}{\mu \otimes \lambda(E \times F)} &= \int\int \Pi(s_1, s_2)(A)\frac{\Pi(s_1', s_2')((E \times F) \cap (ds_1, ds_2))}{\mu \otimes \lambda(E \times F)}d\mu \otimes \lambda \\
&= \int \Pi(s_1, s_2)(A)d\left(\int \frac{\Pi(s_1', s_2')((E \times F) \cap (\cdot))}{\mu \otimes \lambda(E \times F)}d\mu \otimes \lambda\right).
\end{aligned}$$

Therefore we have $\nu(A) = \int \Pi(s_1, s_2)(A)d\nu$. In the same way we can prove that η is an invariant measure for Π. ν and η turn out to be elements of $M \otimes \lambda$. Now we have $\mu \otimes \lambda = \mu(E)\nu + (1 - \mu(E))\eta$, which contradicts the fact that $\mu \otimes \lambda$ is an extreme point of $M \otimes \lambda$. ∎

Let $D \subset J_1 \times J_2$ be a $\mu \otimes \lambda$-invariant set. Let F be the support of λ. Then there exists an $A \times F \subset D$ such that $\Pi(s_1, s_2) \in A \times F$ for all $\mu \otimes \lambda$-a.e.$(s_1, s_2) \in D$. From Lemma 4, $\mu \otimes \lambda(A \times F)$ is zero or one. If $\mu \otimes \lambda(A \times F) = 1$, we have $\mu \otimes \lambda(D) = 1$. Suppose that $\mu \otimes \lambda(((J_1 \times F) \setminus (A \times F)) \cap D) > 0$ when $\mu \otimes \lambda(A \times F) = 0$. We have

$$\begin{aligned}
1 &= \mu \otimes \lambda((J_1 \times F) \setminus (A \times F)) \\
&= \int_{(J_1 \times F \setminus A \times F) \cap D} \Pi(s_1, s_2)(J_1 \times F \setminus A \times F)d\mu \otimes \lambda \\
&\quad + \int_{(J_1 \times F \setminus A \times F) \setminus D} \Pi(s_1, s_2)(J_1 \times F \setminus A \times F)d\mu \otimes \lambda \\
&< 1.
\end{aligned}$$

This is a contradiction. Hence if $\mu \otimes \lambda(A \times F) = 0$, we have $\mu \otimes \lambda(D) = 0$. Therefore $\mu \otimes \lambda$ is an ergodic measure for Π. ∎

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