数理解析研究所講究録
1166 巻 2000 年 138-143

The Weierstrass semigroup of a pair and moduli in $\mathcal{M}_3$

Jiryo Komeda (米田二良)
(Collaboration with Seon Jeong Kim)
Kanagawa Institute of Technology

0. Introduction

Let N be the additive semigroup of non-negative integers. A subsemigroup H of N is called a *numerical semigroup* if $\sharp(\mathrm{N}\backslash H)<\infty$. The number $g(H):=\sharp(\mathrm{N}\backslash H)$ is called the *genus* of H. A certain numerical semigroup of genus g is constructed from a pointed complete non-singular curve of genus g. In this paper we will define a subsemigroup of the additive semigroup $\mathrm{N}\times\mathrm{N}$ like a numerical semigroup. For such a subsemigroup of $\mathrm{N}\times\mathrm{N}$ we can also define its genus. Moreover, in the case where H is such a semigroup of genus 3 we will count the number of the moduli $\mathcal{M}_H$ of curves with a pair of points whose semigroup is H.

1. Numerical semigroups and Weierstrass semigroups

In this section we will review some facts on numerical semigroups which are useful for defining a subsemigroup of $\mathrm{N}\times\mathrm{N}$ like a numerical semigroup. First we give the examples of numerical semigroups of lower genus. For elements $a_1,\ldots,a_n\in\mathrm{N}$ we denote by $<a_1,\ldots,a_n>$ the semigroup generated by $a_1,\ldots,a_n$.

Example 1.1. The semigroup $<2,3>$ is only one numerical semigroup of genus 1.

Examples 1.2. The semigroups $<3,4,5>$ and $<2,5>$ are the numerical semigroups of genus 2.

Examples 1.3. A numerical semigroup H of genus 3 is one of the following semigroups:

H	$\mathrm{N}\backslash H$
$<4,5,6,7>$	$\{1,2,3\}$
$<3,5,7>$	$\{1,2,4\}$
$<3,4>$	$\{1,2,5\}$
$<2,7>$	$\{1,3,5\}$

The following invariant of a numerical semigroup is important to define a subsemigroup of $\mathrm{N}\times\mathrm{N}$ like a numerical semigroup.

Definition 1.4. Let H be a numerical semigroup. We set

$$c(H)=\mathrm{Min}\{c\in\mathrm{N}\mid c+\mathrm{N}\subseteq H\},$$

This work has been supported by the Japan Society for the Promotion of Science and the Korea Science and Engineering Foundation (Project No. 976-0100-001-2).

which is called the *conductor* of H.

The number $c(H)$ satisfies the following inequality:

Remark 1.5. $c(H) \leq 2g(H)$ (For example, see Lemma 2.1 (3) in Komeda [5]).

To describe a connection between numerical semigroups and pointed curves we introduce the following notations: Let C be a complete nonsingular irreducible algebraic curve of genus g over an algebraically closed field of characteristic 0, which is called a *curve* in this paper, and $\mathbf{K}(C)$ the field of rational functions on C. For any point P of C we define the set $H(P)$ by

$$H(P) := \{\alpha \in \mathbf{N} \mid \text{there exists } f \in \mathbf{K}(C) \text{ with } (f)_\infty = \alpha P\}.$$

We have the following well-known fact:

Fact 1.6. $H(P)$ is a numerical semigroup of genus g.

Hence, for a fixed numerical semigroup H we consider the moduli $\mathcal{M}_H$ of curves with a point whose semigroup is H.

Definition 1.7. Let $\mathcal{M}_g$ be the moduli variety of curves of genus g. For a numerical semigroup H of genus g we set

$$\mathcal{M}_H = \{[C] \in \mathcal{M}_g \mid \text{there exists } P \in C \text{ such that } H(P) = H\}.$$

If $\mathcal{M}_H \neq \emptyset$, H is called a *Weierstrass semigroup.*

We have the following facts on what numerical semigroups are Weierstrass or not.

Fact 1.8. (1) If $g(H) \leq 3$, then H is Weierstrass (Classical).
(2) If $g(H) = 4$, then H is Weierstrass (Lax [7]).
(3) If $5 \leq g(H) \leq 7$, then H is Weierstrass (Komeda [6]).
(4) For any $g \geq 16$ there exists a non-Weierstrass numerical semigroup of genus g (Buchweitz [2]).

2. Numerical semigroups of a pair

In this section we define a subsemigroup of $\mathbf{N} \times \mathbf{N}$ like a numerical semigroup.

Definition 2.1. A subsemigroup H of $\mathbf{N} \times \mathbf{N}$ is called a *numerical semigroup of a pair of genus g* if it satisfies the following three conditions:
(1) $\mathbf{N} \backslash \{\gamma \in \mathbf{N} \mid (\gamma, 0) \notin H\}$ and $\mathbf{N} \backslash \{\delta \in \mathbf{N} \mid (0, \delta) \notin H\}$ are numerical semigroups of genus g,
(2) for any $(h_1, h_2) \in \mathbf{N} \times \mathbf{N}$ with $h_1 + h_2 \geq 2g$, we have $(h_1, h_2) \in H$, and
(3) we have a bijection

$$\sigma : \{\gamma \in \mathbf{N} \mid (\gamma, 0) \notin H\} \longrightarrow \{\delta \in \mathbf{N} \mid (0, \delta) \notin H\}$$

such that

$$\mathbf{N}\times\mathbf{N}\backslash H=\bigcup_{\alpha\in\{\gamma\in\mathbf{N}|(\gamma,0)\notin H\}}\Big(\{(\alpha,\beta)|\beta=0,1,\ldots,\sigma(\alpha)-1\}$$

$$\cup\{(\mu,\sigma(\alpha)|\mu=0,1,\ldots,\alpha-1\}\Big).$$

In this case the set $\{(\alpha,\sigma(\alpha))|\alpha\in\{\gamma|(\gamma,0)\notin H\}\}$ is called the *generating set* of H, which is denoted by Γ_H. Thus, if $\pi_i:\mathbf{N}\times\mathbf{N}\longrightarrow\mathbf{N}$ is the i-th projection for $i=1,2$, then $\mathbf{N}\backslash\pi_1(\Gamma_H)$ and $\mathbf{N}\backslash\pi_2(\Gamma_H)$ are numerical semigroups of genus g.

Example 2.2. (1) The semigroup H with generating set $\Gamma_H=\{(1,1)\}$ is only one numerical semigroup of a pair of genus 1.
(2) A numerical semigroup H of a pair of genus 2 is one of the following types:

Type	Γ_H	$\mathbf{N}\backslash\pi_1(\Gamma_H)$	$\mathbf{N}\backslash\pi_2(\Gamma_H)$
I	$\{(1,3),(3,1)\}$	$<2,5>$	$<2,5>$
II	$\{(1,2),(3,1)\}$	$<2,5>$	$<3,4,5>$
III	$\{(1,3),(2,1)\}$	$<3,4,5>$	$<2,5>$
IVa	$\{(1,1),(2,2)\}$	$<3,4,5>$	$<3,4,5>$
IVb	$\{(1,2),(2,1)\}$	$<3,4,5>$	$<3,4,5>$

Fact 2.3.(Kim [3]) Let C be a curve of genus g and P,Q two distinct points of C. We define

$$H(P,Q)=\{(\alpha,\beta)\in\mathbf{N}\times\mathbf{N}|\text{ there exists }f\in\mathbf{K}(C)\text{ with }(f)_\infty=\alpha P+\beta Q\}.$$

Then $H(P,Q)$ is a numerical semigroup of a pair of genus g.

Definition 2.4. Let H be a numerical semigroup of a pair of genus g. We set

$$\mathcal{M}_H=\{[C]\in\mathcal{M}_g\mid\text{there exist two distinct points }P\text{ and }Q\text{ of }C$$
$$\text{such that }H(P,Q)=H\}.$$

If $\mathcal{M}_H\neq\emptyset$, H is called a *Weierstrass semigroup of a pair.*

Fact 2.5. If H is a numerical semigroup of a pair of genus ≤ 2, then it is Weierstrass (Kim [3]).

In the case of genus 3 the statement like Fact 2.8 does not hold.

Counterexample 2.6. The numerical semigroup H of a pair of genus 3 with $\Gamma_H=\{(1,5),(3,2),(5,1)\}$ is not Weierstrass.

Proof. We note that $\mathrm{N}\backslash\pi_1(\Gamma_H) =< 2,7 >$ and $\mathrm{N}\backslash\pi_2(\Gamma_H) =< 3,4 >$. Suppose that H were Weierstrass. Then there exist a curve C and its two distinct points P, Q such that $H(P,Q) = H \supset (< 2,7 > \times\{0\})\cup(\{0\}\times < 3,4 >)$. Thus, $H(P) =< 2,7 >$, hence C is hyperelliptic, and $H(Q) =< 3,4 >$, hence C is non-hyperelliptic. (It means that

$$\mathcal{M}_{\mathrm{N}\backslash\pi_1(\Gamma_H)} \cap \mathcal{M}_{\mathrm{N}\backslash\pi_2(\Gamma_H)} = \mathcal{M}_{<2,7>} \cap \mathcal{M}_{<3,4>} = \emptyset.)$$

This is a contradiction. Q.E.D.

But we obtain the following result:

Theorem 2.7. *Let H be a numerical semigroup of a pair of genus 3 such that*

$$\mathcal{M}_{\mathrm{N}\backslash\pi_1(\Gamma_H)} \bigcap \mathcal{M}_{\mathrm{N}\backslash\pi_2(\Gamma_H)} \neq \emptyset.$$

Then the semigroup H is Weierstrass.

Proof. If $\mathrm{N}\backslash\pi_i(\Gamma_H) =< 2,7 >$ for some i, this result is due to Kim [3]. Suppose that $\mathrm{N}\backslash\pi_i(\Gamma_H) \neq< 2,7 >$ for $i = 1,2$. Then we have the following table up to symmetries.

Type	Γ_H	$\mathrm{N}\backslash\pi_1(\Gamma_H)$	$\mathrm{N}\backslash\pi_2(\Gamma_H)$
I	$\{(1,5),(2,2),(5,1)\}$	$<3,4>$	$<3,4>$
IIa	$\{(1,2),(2,4),(5,1)\}$	$<3,4>$	$<3,5,7>$
IIb	$\{(1,4),(2,2),(5,1)\}$	$<3,4>$	$<3,5,7>$
IIIa	$\{(1,2),(2,3),(5,1)\}$	$<3,4>$	$<4,5,6,7>$
IIIb	$\{(1,3),(2,2),(5,1)\}$	$<3,4>$	$<4,5,6,7>$
IVa	$\{(1,2),(2,4),(4,1)\}$	$<3,5,7>$	$<3,5,7>$
IVb	$\{(1,4),(2,2),(4,1)\}$	$<3,5,7>$	$<3,5,7>$
Va	$\{(1,3),(2,1),(4,2)\}$	$<3,5,7>$	$<4,5,6,7>$
Vb	$\{(1,2),(2,3),(4,1)\}$	$<3,5,7>$	$<4,5,6,7>$
Vc	$\{(1,3),(2,2),(4,1)\}$	$<3,5,7>$	$<4,5,6,7>$
VIa	$\{(1,2),(2,1),(3,3)\}$	$<4,5,6,7>$	$<4,5,6,7>$
VIb	$\{(1,3),(2,1),(3,2)\}$	$<4,5,6,7>$	$<4,5,6,7>$
VIc	$\{(1,3),(2,2),(3,1)\}$	$<4,5,6,7>$	$<4,5,6,7>$
VId	$\{(1,1),(2,2),(3,3)\}$	$<4,5,6,7>$	$<4,5,6,7>$

We note that every non-hyperelliptic curve of genus 3 can be expressed by a non-singular curve of degree 4 in the projective 2-space Proj $k[x,y,z]$ through a canonical embedding. For curves C with its points P and Q in the below table we have $H = H(P,Q)$. In fact, the case of Type VIc is trivial, for example, due to Arbarello, Cornalba, Griffiths and Harris [1, VIII Exercises B.7]. Using the Bertini's theorem and elementary calculation, we can easily prove that each curve is nonsingular for general constants a and b, and that the given points P and Q satisfy $H = H(P, Q)$. Note that the canonical series on each curve in the table are cut out by lines on the plane.

Type	C	P	Q
I	$y^3z - yz^3 - x^4 = 0$	$(0:0:1)$	$(0:1:0)$
IIa	$-x^4 + xy^3 + 2yz^3 = 0$	$(0:0:1)$	$(0:1:0)$
IIb	$-(x-z)^4 + xy^3 + 2yz^3 = 0$	$(1:0:1)$	$(0:1:0)$
IIIa	$yz^3 - x^4 + xy^3 - 2y^2z^2 = 0$	$(0:0:1)$	$(0:1:0)$
IIIb	$a(yz^3 - (x-z)^4) + b(xy^3 + y^2z^2) = 0$	$(1:0:1)$	$(0:1:0)$
IVa	$-x^3z + xy^3 + 2yz^3 = 0$	$(0:0:1)$	$(0:1:0)$
IVb	$-(x-z)^3z + xy^3 + 2yz^3 = 0$	$(1:0:1)$	$(0:1:0)$
Va	$a(yz^3 - x^3(x-z)) + by^4 = 0$	$(0:0:1)$	$(1:0:1)$
Vb	$a(yz^3 - x^3z) + b(xy^3 + y^2z^2) = 0$	$(0:0:1)$	$(0:1:0)$
Vc	$a(yz^3 - (x-z)^3z) + b(xy^3 + y^2z^2) = 0$	$(1:0:1)$	$(0:1:0)$
VIa	$a(yz^3 - x^2(x-z)^2) + by^4 = 0$	$(0:0:1)$	$(1:0:1)$
VIb	$a(yz^3 - x^2(x-z)(x-2z)) + by^4 = 0$	$(0:0:1)$	$(1:0:1)$
VIc	any curve	general	general

Q.E.D.

Theorem 2.8. *We can count the dimension of the moduli $\mathcal{M}_H$ of curves of genus 3 with a fixed Weierstrass semigroup H of a pair as follows:*

Type	Γ_H	dim $\mathcal{M}_H$
I	$\{(1,5),(2,2),(5,1)\}$	4
IIa	$\{(1,2),(2,4),(5,1)\}$	4
IIb	$\{(1,4),(2,2),(5,1)\}$	5
IIIa	$\{(1,2),(2,3),(5,1)\}$	5
IIIb	$\{(1,3),(2,2),(5,1)\}$	5
IVa	$\{(1,2),(2,4),(4,1)\}$	5
IVb	$\{(1,4),(2,2),(4,1)\}$	6
Va	$\{(1,3),(2,1),(4,2)\}$	6
Vb	$\{(1,2),(2,3),(4,1)\}$	6
Vc	$\{(1,3),(2,2),(4,1)\}$	6
VIa	$\{(1,2),(2,1),(3,3)\}$	6
VIb	$\{(1,3),(2,1),(3,2)\}$	6
VIc	$\{(1,3),(2,2),(3,1)\}$	6
VId	$\{(1,1),(2,2),(3,3)\}$	5

Proof. See Kim-Komeda[4]. Q.E.D.

References

[1] E. Arbarello, M. Cornalba, P.A. Griffiths and J. Harris, *Geometry of algebraic curves* Vol.I, Springer-Verlag, 1985.

[2] R.O. Buchweitz, *On Zariski's criterion for equisingularity and non-smoothable monomial curves,* preprint 113, University of Hannover, 1980.

[3] S.J. Kim, *On the index of the Weierstrass semigroup of a pair of points on a curve,* Arch. Math. **62** (1994), 73-82.

[4] S.J. Kim and J. Komeda, *The Weierstrass semigroup of a pair and moduli in* $\mathcal{M}_3$, preprint.

[5] J. Komeda, *On Weierstrass points whose first non-gaps are four,* J. reine angew. Math. **341** (1983), 68-86.

[6] J. Komeda, *On the existence of Weierstrass gap sequences on curves of genus* ≤ 8, J. Pure Appl. Algebra **97** (1994), 51-71.

[7] R.F. Lax, *Gap sequences and moduli in genus 4,* Math. Z. **175** (1980), 67-75.