数理解析研究所講究録
1166 巻 2000 年 52-58

Free algebras for generalized automata and language theory

Z. Ésik*
Dept. of Computer Science
University of Szeged
P.O.B. 652
6701 Szeged, Hungary
esik@inf.u-szeged.hu

Abstract

We give a geometric representation of the free algebras equipped with m associative and n associative and commutative operation.

1 Introduction

Complement reducible graphs (or cographs, for short) and series-parallel graphs and partial orders have arisen in several fields of mathematics and computer science. Efficient sequential and parallel algorithms have been developed for the recognition of both cographs and series-parallel graphs, and to solve many problems that are intractable for graphs in general. Such problems include chromatic number, clique, isomorphism, Hamiltonicity, etc. Several characterizations of cographs are known. It was shown in [CLB81] that a graph is a cograph iff it has no induced subgraph which is a P_4, i.e., a path on 4 vertices. Moreover, it was shown in [CPB85] that a graph is a cograph iff it has a (necessarily unique) representation by a cotree. An alternative formulation of the latter result may be given using the language of universal algebra: For any set A, (isomorphism classes of) cographs whose vertices are labeled in A form the free algebra generated by A equipped with two associative and commutative operations. The two operations are disjoint union and its dual. The corresponding facts for (labeled) posets are the characterization of the series-parallel posets by the N-condition, and the fact that for any set A, the A-labeled series-parallel posets form the free algebra on A equipped with an associative and an associative and commutative operation, viz. series and parallel product,

*Partially supported by grant no. T30511 from the National Foundation of Hungary for Scientific Research.

see [Gra81, VTL82, Gis88]. The N-condition is of course related to the P_4-condition: A partial order is N-free iff its symmetric closure is P_4-free, moreover, a graph is P_4-free iff it has an orientation which an N-free poset, i.e., an N-free transitive digraph, cf. [VTL82].

A common generalization of labeled cographs and series-parallel posets appears in [Win86, BC88]. The event structures defined in [BC88] are labeled sets equipped with a partial order, called causality, and a disjoint irreflexive symmetric relation, called conflict. Each event structure gives rise to a further irreflexive symmetric relation, called concurrency, which complements the other two. The three relations determine three associative operations. The operations corresponding to the symmetric relations are also commutative. It is shown in [BC88] that the A-labeled event structures generated from the singletons are freely generated by A in the variety of those algebras equipped with three associative operations, two of which are commutative. Generalizing the P_4-condition and the N-condition, a graph theoretic characterization of event structures in the free algebras is also given in [BC88]. Further examples of algebras with several associative operations related to concurrency may be found in [ZJ94]. For generalizations of automata and language theory over free algebras with two or more associative operations see [LW98, LWa, LWb] and [Kus00].

In this note, extending the above ideas, we give a concrete geometric description of the free algebras with m associative and n associative and commutative operations, where m, n are any nonnegative integers. This description is based on labeled sets equipped with m partial orders and n irreflexive symmetric relations such that any two distinct elements are related by exactly one of the relations. Each relation determines an operation and the free algebras encompass those structures generated from the singletons by the operations. These structures are in turn characterized by 3 conditions: the N-condition for each partial order, the P_4-condition for each symmetric relation, and the "triangle condition". (See Theorem 2.14.)

2 The structures

Let m and n denote nonnegative integers with $m+n \geq 1$, and let A denote a set of labels. For an integer $k \geq 0$, let $[k] = \{1, \ldots, k\}$.

DEFINITION 2.1 *An* (m,n)-structure *over* A*, or just* (m,n)*-structure, for short, is a finite set* P *of vertices equipped with* m *irreflexive transitive relations* $<_1, \ldots, <_m$*,* n *irreflexive symmetric relations* $\sim_1, \ldots, \sim_n$*, and a labeling function* $\lambda : P \to A$ *subject to the following condition: For any two distinct vertices* $x, y \in P$*, either there is a unique* $i \in [m]$ *with* $x <_i y$ *or* $y <_i x$*, or else there is a unique* $j \in [n]$ *with* $x \sim_j y$*. A morphism of* (m,n)*-structures is a function which preserves the relations and the labeling. An isomorphism is a bijective morphism.*

Thus, the relations $<_i$ and $\sim_j$ are pairwise disjoint. Moreover, each $(P, <_i)$ is a (strict) partial order, and each $(P, \sim_j)$ is a graph. Below, when needed, we will add the superscript

P on the relations and labeling of an (m,n)-structure P. Note that any morphism $f: A \to B$ between (m,n)-structures is injective, and for any $x, y \in A$ and $i \in [m]$, $j \in [n]$, $x <_i y$ iff $f(x) <_i f(y)$ and $x \sim_j y$ iff $f(x) \sim_j f(y)$.

In order to simplify the treatment, we introduce the notation ρ_i for the relation $<_i \cup <_i^{-1}$, for all $i \in [m]$, and define $\rho_{m+j} = \sim_j$, for all $j \in [n]$. Note that for any (m,n)-structure $(P, <_1, \ldots, <_m, \sim_1, \ldots, \sim_n, \lambda)$, $(P, \rho_1, \ldots, \rho_{m+n}, \lambda)$ is a $(0, m+n)$-structure. In fact, for every $i \in [m]$, $(P, <_1, \ldots, <_i, \rho_{i+1}, \ldots, \rho_{m+n}, \lambda)$ is an $(i, m+n-i)$-structure.

REMARK 2.2 A $(1,0)$-structure may be identified with a labeled total order. A $(0,1)$-structure is a labeled complete graph (or a labeled set). Any $(1,1)$-structure $(P, <, \sim, \lambda)$ may be identified with the labeled partial order $(P, <, \lambda)$, and any $(0,2)$-structure $(P, \sim_1, \sim_2, \lambda)$ with the labeled graph $(P, \sim_1, \lambda)$. However, the correspondence between $(2,0)$-structures and labeled partial orders is not bijective, since in general a labeled partial order may be equipped with a second partial order in several different ways such that the resulting structure becomes a $(2,0)$-structure. More generally, when $n > 0$, every (m,n)-structure $P = (P, <_1, \ldots, <_m, \sim_1, \ldots, \sim_n, \lambda)$ may be identified with the structure $(P, <_1, \ldots, <_m, \sim_1, \ldots, \sim_{n-1}, \lambda)$ having one less relation. (Unless $\sim_n$ is empty, this is not an $(m, n-1)$-structure.)

To define operations on (m,n)-structures, we identify any two isomorphic structures, so that in the definition below we may assume that P and Q are disjoint.

DEFINITION 2.3 *Suppose that* $P = (P, <_1^P, \ldots, <_m^P, \sim_1^P, \ldots, \sim_n^P, \lambda^P)$ *and* $Q = (Q, <_1^Q, \ldots, <_m^Q, \sim_1^Q, \ldots, \sim_n^Q, \lambda^Q)$ *are disjoint* (m,n)*-structures. For each* $i \in [m]$, *we define* $P \odot_i Q$ *to be the* (m,n)*-structure* $(P \cup Q, <_1, \ldots, <_m, \sim_1, \ldots, \sim_n, \lambda)$ *with*

$$\begin{aligned} <_k &= <_k^P \cup <_k^Q, \quad k \neq i, k \in [m] \\ <_i &= <_i^P \cup <_i^Q \cup (P \times Q) \\ \sim_j &= \sim_j^P \cup \sim_j^Q, \quad j \in [n] \\ \lambda &= \lambda^P \cup \lambda^Q. \end{aligned}$$

For each $j \in [m]$, *we let* $P \otimes_j Q$ *to be the* (m,n)*-structure* $(P \cup Q, <_1, \ldots, <_m, \sim_1, \ldots, \sim_n, \lambda)$ *with*

$$\begin{aligned} <_i &= <_i^P \cup <_i^Q, \quad i \in [m] \\ \sim_k &= \sim_k^P \cup \sim_k^Q, \quad k \neq j, k \in [n] \\ \sim_j &= \sim_j^P \cup \sim_j^Q \cup (P \times Q) \cup (Q \times P) \\ \lambda &= \lambda^P \cup \lambda^Q. \end{aligned}$$

Moreover, we define 1 *to be the empty structure.*

It is clear that all of the operations are associative and that the operations $\otimes_j$ are commutative. Moreover, 1 is a unit for all operations.

REMARK 2.4 Identifying $(1,1)$-structures with the corresponding labeled partial orders, $\odot_1$ corresponds to the series product of labeled partial orders and $\otimes_1$ corresponds to parallel product or disjoint union, cf. [VTL82]. In Definition 2.10 we will denote these two operations by $\cdot$ and $\otimes$. For $(0,2)$-structures, i.e., labeled graphs, $\otimes_1$ is the operation of taking the disjoint union and connecting each vertex in any of the two graphs with all vertices of the other. Operation $\otimes_2$ is just disjoint union. In Definition 2.12, the operations $\otimes_1$ and $\otimes_2$ will be denoted as $\cdot$ and $\otimes$.

DEFINITION 2.5 *An* (m,n)-semigroup *is a set M equipped with m associative product operations $\odot_i$, $i \in [m]$, n associative and commutative product operations $\otimes_j$, $j \in [n]$. An* (m,n)-monoid *is an (m,n)-semigroup equipped with a constant 1 which is a unit for all operations. A morphism of (m,n)-semigroups is a function that preserves the operations. Morphisms of (m,n)-monoids also preserve the unit.*

Thus, (m,n)-structures over A form an (m,n)-monoid denoted $\mathbf{S}_A$. Note that a $(1,0)$-semigroup is just a semigroup and a $(0,1)$-semigroup is a commutative semigroup. $(1,1)$-monoids are called bimonoids in [BE96], double monoids in [Gra81] and dioids in [BC88]. A $(2,0)$-semigroup is termed a binoid in [Has99].

DEFINITION 2.6 *An (m,n)-structure is* reducible *if it can be generated from the singletons corresponding to the letters in A by the $m+n$ product operations, i.e., when it can be reduced to a singleton by decomposition with respect to the operations.*

Thus, the reducible (m,n)-structures form the least subalgebra of $\mathbf{S}_A$ containing the singletons. We let $\mathbf{RS}_A$ denote this (m,n)-semigroup.

REMARK 2.7 When $m+n=1$, every nonempty (m,n)-structure is reducible. The reducible $(1,1)$-structures over A correspond to the series-parallel A-labeled posets, and the reducible $(0,2)$-structures to the A-labeled cographs defined below.

Call an (m,n)-structure P *decomposable with respect to* $\odot_i$, where $i \in [m]$, if there exist nonempty structures Q and R with $P = Q \odot_i R$. Similarly, call P *decomposable with respect to* $\otimes_j$, where $j \in [n]$, if there exist nonempty structures Q and R with $P = Q \otimes_j R$. Otherwise call P *indecomposable (with respect to $\odot_i$ or $\otimes_j$, respectively)*. It is clear that each reducible structure P is either a singleton, or there is a unique i that P has a nontrivial decomposition

$$P \;=\; P_1 \odot_i \ldots \odot_i P_k, \tag{1}$$

where the P_t are reducible and indecomposable with respect to $\odot_i$, or else there is a unique j such that P has a nontrivial decomposition

$$P \;=\; P_1 \otimes_j \ldots \otimes_j P_k, \tag{2}$$

where the P_t are reducible and indecomposable with respect to $\otimes_j$. Using this fact, we immediately obtain

THEOREM 2.8 $\mathbf{RS}_A$ *is freely generated by the set* A *in the variety of all* (m,n)*-semigroups.*

Proof. We need to show that for any (m,n)-semigroup M and function $h : A \to M$ there is a unique morphism $h^\sharp : \mathbf{RS}_A \to M$ extending h. (We identify any letter $a \in A$ with the corresponding singleton structure labeled a.) Since each $P \in \mathbf{RS}_A$ can be generated from the singletons by the operations, it follows that $h^\sharp$ is unique. As for the existence, define $h^\sharp(P)$, for $P \in \mathbf{RS}_A$, in the following way: If $P = a$, for some $a \in A$, define $h^\sharp(P) = h(a)$. Suppose that P has two or more elements. Then either there is a unique $i \in [m]$ such that P has a decomposition (1), or else there is a unique $j \in [m]$ such that P has a decomposition (2). In the first case, we define $h^\sharp(P) = h^\sharp(P_1) \odot_i \ldots \odot_i h^\sharp(P_k)$. In the second, we let $h^\sharp(P) = h^\sharp(P_1) \otimes_j \ldots \otimes_j h^\sharp(P_k)$. It is a routine matter to verify that $h^\sharp$ preserves the operations. □

Let $\mathbf{RS}^1_A$ denote the (m,n)-monoid $\mathbf{RS}_A \cup \{1\}$.

COROLLARY 2.9 $\mathbf{RS}^1_A$ *is freely generated by the set* A *in the variety of all* (m,n)*-monoids.*

DEFINITION 2.10 *We call an* A*-labeled poset* $P = (P, <, \lambda)$ series-parallel *if* P *can be generated from the singletons by the operations* $\cdot$ *and* $\otimes$ *of series and parallel product. Moreover, we say that* P*, or the underlying poset of* P *satisfies the* N-condition *if there exist no distinct vertices* x, y, u, v *in* P *such that the only order relations between these vertices are* $x < u$, $y < u$ *and* $y < v$.

THEOREM 2.11 Grabowski, Valdes et al [Gra81, VTL82] *An* A*-labeled poset is series-parallel iff it satisfies the* N*-condition.*

DEFINITION 2.12 *An* A*-labeled graph* $P = (P, \sim, \lambda)$ *is a* cograph *if it can be generated from the the singletons by the two product operations* $\cdot$ *and* $\otimes$ *defined on labeled graphs (cf. Remark 2.4). Moreover, we say that* P *satisfies the* P_4-condition *if* P *does not contain an induced subgraph which is a path of length* 4*, i.e.,* P *does not have distinct vertices* x, y, u, v *such that the only relations between these vertices are* $x \sim y$, $y \sim u$, $u \sim v$.

It is well-known that P is a cograph iff P can be generated from the singletons by the operations of disjoint union and complementation. Note that the P_4-condition is equivalent to the requirement that any two vertices x and y are either disconnected, or there is a path of length 1 or 2 between x and y, i.e., the distance between x and y is at most 2.

THEOREM 2.13 Corneil et al. [CLB81] *An* A*-labeled graph is a cograph iff it satisfies the* P_4*-condition.*

Using the above results we now prove:

THEOREM 2.14 *A nonempty* (m,n)*-structure* $P = (P, <_1, \ldots, <_m, \sim_1, \ldots, \sim_n)$ *is reducible iff the following conditions hold.*

1. *For each* $i \in [m]$*, the poset* $(P, <_i)$ *satisfies the* N*-condition.*
2. *For each* $j \in [n]$*, the graph* $(P, \sim_j)$ *satisfies the* P_4*-condition.*
3. *There exist no distinct vertices* x, y, z *and distinct integers* $i, j, k \in [m+n]$ *with* $x\rho_i y$*,* $y\rho_j z$ *and* $z\rho_k x$*. (*Triangle condition.*)*

Proof. The necessity of the conditions follows by a straightforward induction argument on the number of vertices of P. As for the sufficiency, we may assume that $m+n > 1$, since if $m+n=1$ then every (m,n)-structure is reducible. Our claim is clear if P has a single vertex. Suppose now that P has two or more vertices. We prove that one of the graphs (P, ρ_i), $i \in [m+n]$ is connected. Indeed, if this condition does not hold, then let Q denote a maximal subset of P such that for some i, (Q, ρ_i) is connected. By assumption, there is a vertex $v \in Q - P$. Let $u_0 \in Q$, and for $n = 1, 2, \ldots$ let U_n denote the set of vertices $u_n \in Q$ at distance n from u_0, i.e., such that the minimal path between u_0 and u_n in (Q, ρ_i) is of length n. Let $u_0 \rho_j v$, say. Since $(Q \cup \{v\}, \rho_i)$ is not connected, we have $i \neq j$. It follows now by a straightforward induction argument using the triangle condition that $u_n \rho_j v$ for all $u_n \in U_n$. But $\cup_{n \geq 0} U_n = Q$, so that $u\rho_j v$, for all $u \in Q$. But then $(Q \cup \{v\}, \rho_j)$ is connected, contradicting our assumption on Q.

Thus, at least one of the (P, ρ_i) is connected. If $i \leq m$, then $(P, <_i)$ is a partial order satisfying the N-condition. Thus, by Theorem 2.11, P has nonempty disjoint subsets R, S with

$$(P, <_i) \quad = \quad (Q, <_i) \cdot (R, <_i),$$

so that $Q \cup R = P$. It is now clear that the (m,n)-structure P satisfies

$$P \quad = \quad (Q, <_1, \ldots, <_m, \sim_1, \ldots, \sim_n, \lambda) \odot_i (R, <_1, \ldots, <_m, \sim_1, \ldots, \sim_n, \lambda).$$

Since the (m,n)-structures Q and R also satisfy the conditions of the Theorem and have fewer vertices than P, it follows from the induction hypothesis that Q and R are reducible, so that P is also reducible. If $i > m$, then $(P, \sim_j)$ is a graph satisfying the P_4-condition, where $j = i - m$. Thus, by Theorem 2.13, $(P, \sim_j)$ has a nontrivial decomposition

$$(P, \sim_j) \quad = \quad (Q, \sim_j) \otimes (R, \sim_j).$$

Using the induction hypothesis it follows as before that the (m,n)-structure P is reducible. □

REMARK 2.15 Suppose that $P = (P, <_1, \ldots, <_m, \sim_1, \ldots, \sim_n)$ is an (m,n)-structure. If $n > 0$ then P is reducible iff P satisfies the triangle condition, the first condition of Theorem 2.14 for all $i \in [m]$, and the second condition for all $i \in [n-1]$. Similarly, if $m > 0$, then P is reducible iff P satisfies the triangle condition, the first condition for all $i \in [m-1]$, and the second for all $i \in [n]$.

REMARK 2.16 When $m = 1$ and $n = 2$, the conditions involved in Theorem 2.14 may be replaced by the following X-condition [BC88]: For all $i, j = 1, 2, 3$ with $i \neq j$ and for all vertices v_0, v_1, v_2 and v_3, if $v_0 \rho_i v_1$, $v_1 \rho_i v_3$ but $v_0 \rho_i v_2$ and $v_1 \rho_i v_3$ do not hold, and if $v_0 \rho_j v_3$, then $v_0 \rho_j v_2$, $v_1 \rho_j v_2$ and $v_1 \rho_j v_3$.

Acknowledgment The author would like to thank Zoltán Németh for a useful comment.

References

[BE96] S. L. Bloom and Z. Ésik. Free shuffle algebras in language varieties. *Theoretical Computer Science*, 163(1996), 55–98.

[BC88] G. Boudol and I. Castellani. Concurrency and atomicity. *Theoret. Comput. Sci.*, 59(1988), 25–34.

[CLB81] D. G. Corneil, H. Lerchs and L. S. Burlinham. Complement reducible graphs. *Discr. Appl. Math.*, 3(1981), 163–174.

[CPB85] D. G. Corneil, Y. Perl and L. K. Stewart. A linear recognition algorithm for cographs. *SIAM J. Comput.*, 14(1985), 926–934.

[Gis88] J. L. Gischer. The equational theory of pomsets. The equational theory of pomsets. *Theoret. Comput. Sci.*, 61(1988), 199–224.

[Gra81] J. Grabowski. On partial languages. *Fund. Inform.*, 4:427–498, 1981.

[Has99] K. Hashiguchi, S. Ichihara and S. Jimbo. Formal languages over free binoids. proc. *Int. Workshop on Descriptional Complexity of Automata, Grammars and Related Structures*, Magdeburg, 1999, 39–50.

[Kus00] D. Kuske. Infinite series-parallel posets: logic and languages. proc. *ICALP 2000*, to appear.

[LW98] K. Lodaya and P. Weil. Kleene iteration for parallelism, proc. *FST & TCS 98*, LNCS 1530, Springer-Verlag, 1998, 355–366.

[LWa] K. Lodaya and P. Weil. Series-parallel languages and the bounded-width property. *Theoret. Comput. Sci.*, to appear.

[LWb] K. Lodaya and P. Weil. Rationality in algebras with series operation. To appear.

[VTL82] J. Valdes, R. E. Tarjan, and E. L. Lawler. The recognition of series-parallel digraphs. *SIAM Journal of Computing*, 11(2):298–313, 1982.

[Win86] G. Winskel. Event structures. proc. *Advances in Petri Nets 86*, LNCS 255, Springer-Verlag, 1987, 325–392.

[ZJ94] J. Zwiers and W. Janssen. Partial order based design of concurrent systems. in: *A Decade of Concurrency*, LNCS 803, 1994, 622–684.