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A direct linearization of the KP hierarchy and an initial value problem for tau functions

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Abstract

It is shown that KP tau functions can be characterized by a single, linear integral equation. The measure and contour appearing in the integral are directly related to the representation of a tau function by an element of $gl(\infty)$ (or a suitable completion thereof). Furthermore, it turns out that this measure and contour can be determined from the values a tau function $\tau(x_1, x_2, x_3, \ldots)$ takes in the x_1x_2-plane, thereby providing a method for reconstructing a tau function from an initial profile defined in the x_1x_2-plane.

1 Direct linearization for the KP equation

The original motivation for the work reported here is a rather remarkable direct linearization procedure [1] by which solutions for the Kadomtsev-Petviashvili (KP) equation can be obtained. It is remarkable in the sense that it only requires very little input but yet covers a very broad range of solutions.
The procedure first of all consists of solving the linear integral equation

$$\varphi(k) = e^{\theta(k)} - e^{\theta(k)} \iint_C \mathrm{d}\eta(\lambda, \mu)\ \frac{e^{-\theta(\mu)}}{k-\mu}\ \varphi(\lambda) \tag{1}$$

with $\theta(k) \equiv kx + k^2y + k^3t$, for a contour $\mathcal{C}$ and measure $\eta(\lambda, \mu)$ which are essentialy arbitrary except for the requirement that the homogeneous equation :

$$F(k) = -e^{\theta(k)} \iint_C \mathrm{d}\eta(\lambda, \mu)\ \frac{e^{-\theta(\mu)}}{k-\mu}\ F(\lambda)$$

should have $F(k) \equiv 0$ as its unique solution. We also demand that the measure be suitably well-behaved such that differentiation with respect to the variables x, y and t can be interchanged with the integrations in (1). Under these assumptions it can be shown that the solution $\varphi(k)$ to (1) solves the Lax pair for the KP equation :

$$\psi_y = \psi_{2x} + 2u\psi \tag{2}$$

$$\psi_t = \psi_{3x} + 3u\psi_x + 3(u_x + v)\psi\ , \qquad \text{where} \quad v_x = \frac{1}{2}(u_y - u_{2x}) \tag{3}$$

for a potential u defined in terms of the function $\varphi(k)$ and the same contour $\mathcal{C}$ and measure $\eta(\lambda, \mu)$:

$$u(x, y, t) = \frac{\partial}{\partial x} \iint_C \mathrm{d}\eta(\lambda, \mu)\ \ e^{-\theta(\mu)}\ \varphi(\lambda). \tag{4}$$

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Hence, with this expression for $u(x,y,t)$ one has effectively (re-)constructed a solution to the KP equation

$$4u_{x,t} - 3u_{2y} - u_{4x} - 12(uu_x)_x = 0,$$

from the mere input of a contour C and a measure $\eta(\lambda,\mu)$. The above procedure was actually first introduced for the case of the Korteweg-de Vries equation [2], were the great freedom which is allowed in choosing contours and measures was exploited to construct solutions to the Painlevé II equation, solutions which were not accessible using the usual inverse scattering method based upon the Gel'fand-Levitan equation. As a matter of fact, it can be shown that the modern formulation of the inverse scattering problem for the KP equation(s) (if one distinguishes between the KPI and II cases) as a Riemann-Hilbert problem is actually contained in the above approach (see e.g. [3] and references therein). Furthermore, Nijhoff *et al.* [4] studied so called Bäcklund transformations for the measures used in the integral equation (1) and were able to construct a discrete (lattice) version of the KP equation from the compatibility conditions of those Bäcklund transformations. A result which suggests that the measures involved play an fundamental rôle in the integrability of the KP equation.

In part, the aim of this paper is to explain the above results in the light of the Sato theory [5] for the KP equation (and its associated hierarchy) as it will be shown that equation (1) allows for a natural interpretation in terms of KP tau functions. But most importantly, in doing so it will become clear that the solution of a different type of inverse problem also lies at hand, i.e. : the reconstruction of a tau function from the knowledge of an initial profile which is only defined in the two (lowest weight) coordinates x and y.

2 Tau functions and infinite dimensional algebras

First, let us introduce some notation and definitions regarding tau functions and especially concerning the algebraic formulation of the Sato-theory [6, 7] for the KP hierarchy.

For an algebra of (charged) *free fermion* creation and annihilation operators, satisfying the anti-commutation relations :

$$[\psi_i,\psi_j^*]_+ \equiv \psi_i\psi_j^* + \psi_j^*\psi_i = \delta_{i+j,0}\ , \qquad [\psi_i,\psi_j]_+ = [\psi_i^*,\psi_j^*]_+ = 0, \qquad \text{for } i,j \in \mathbb{Z}+\frac{1}{2},$$

one has the standard *Fock representation* defining the Fock space $\mathcal{F}$ as well as its dual $\mathcal{F}^*$, with cyclic vectors $|\mathrm{vac}\rangle \in \mathcal{F}$ and $\langle\mathrm{vac}| \in \mathcal{F}^*$:

$$\begin{aligned} |\mathrm{vac}\rangle &= \psi_{1/2}^*\psi_{3/2}^*\psi_{5/2}^*\cdots|0\rangle \\ \langle\mathrm{vac}| &= \langle 0|\cdots\psi_{-5/2}\psi_{-3/2}\psi_{-1/2}\ , \end{aligned}$$

themselves defined in terms of "fake" vacuum states $|0\rangle$ and $\langle 0|$, respectively annihilated by all operators ψ_j or ψ_j^* ($j \in \mathbb{Z}+1/2$). A usual, expectation values for the states in the Fock space are defined using the pairing $\mathcal{F}^*\times\mathcal{F}\to\mathbb{C}$ with $\langle\mathrm{vac}|1|\mathrm{vac}\rangle = 1$; these can be calculated using Wick's theorem.

Let us also introduce (formal) fermion operators

$$\psi(k) \equiv \sum_{j\in\mathbf{Z}+1/2} \psi_j\ k^{-j-1/2} \qquad \psi^*(k) \equiv \sum_{j\in\mathbf{Z}+1/2} \psi_j^*\ k^{-j-1/2},$$

for which "time" evolutions with respect to (infinitely many) time variables $\mathbf{x} \equiv (x_1,x_2,x_3,\ldots)$ can be introduced using a Hamiltonian :

$$H(\mathbf{x}) \equiv \sum_{n=1}^{\infty} x_n\ H_n \qquad \text{with} \quad H_n \equiv \sum_{j\in\mathbf{Z}+1/2} :\psi_{-j}\psi_{j+n}^*: \qquad \forall n\geq 1 \tag{5}$$

(the symbol : : denotes normal ordering as in : $\psi_i\psi_j^*$: $=$ $\psi_i\psi_j^* - \langle\mathrm{vac}|\psi_i\psi_j^*|\mathrm{vac}\rangle$). Note also that $H_n|\mathrm{vac}\rangle = 0$ ($\forall n\geq 1$) and hence that $e^{H(\mathbf{x})}|\mathrm{vac}\rangle = 0$.

In terms of the above Hamiltonian, the evolutions of $\psi(k)$ and $\psi^*(k)$ w.r.t. the coordinates $\mathbf{x}$ are defined as

$$\psi(k;\mathbf{x}) \equiv e^{H(\mathbf{x})}\ \psi(k)\ e^{-H(\mathbf{x})} = \psi(k)\ e^{\xi(\mathbf{x},k)} \tag{6}$$

$$\psi^*(k;\mathbf{x}) \equiv e^{H(\mathbf{x})}\ \psi^*(k)\ e^{-H(\mathbf{x})} = \psi^*(k)\ e^{-\xi(\mathbf{x},k)}, \tag{7}$$

for phase functions $\xi(\mathbf{x},k) = \sum_{n=1}^{\infty} x_n k^n$.

The connection between this algebro-field theoretic construction and integrable (partial) differential equations is provided by the so called *fermion - boson correspondence* , stating that the (fermionic) operators $\psi(k)$ and $\psi^*(k)$ possess a realization on a bosonic Fock space. In particular, the fermion-boson correspondence allows one to define a tau function as the orbit of the cyclic vector $|\mathrm{vac}\rangle$ under the action of the Lie group associated with the infinite dimensional Lie algebra :

$$gl(\infty): \quad \{ \sum_{i,j\in\mathbf{Z}+1/2} a_{ij} : \psi_i\psi_j^* : + a_0 \mid \exists\, R \text{ s.t. } a_{ij}=0\ \forall\ |i-j|>R, \text{ with } a_{ij}, a_0 \in \mathbb{C} \} ;$$

the corresponding Lie group is denoted by $GL(\infty)$ and in the following the symbol $\mathbf{g}$ will be used to denote an element of $GL(\infty)$. A tau function is then defined by the pairing :

$$\tau \equiv \langle \mathrm{vac}|e^{H(\mathbf{x})}\mathbf{g}|\mathrm{vac}\rangle = \langle \mathrm{vac}|\mathbf{g}(\mathbf{x})|\mathrm{vac}\rangle \tag{8}$$

where $\mathbf{g}(\mathbf{x}) \equiv e^{H(\mathbf{x})}\mathbf{g}e^{-H(\mathbf{x})}$. In fact, it will prove necessary to extend the Lie group $GL(\infty)$ to what one might call its formal completion $\overline{GL}(\infty)$, containing elements generated by products of formal operators such as $\psi(\lambda)\psi^*(\mu)$ – which are *not* contained in $gl(\infty)$ – provided that such elements still give rise to a well defined pairing (8).

The fundamental property of the tau functions is of course that they satisfy the bilinear identity :

$$\operatorname*{Res}_k \left[\tau(\mathbf{x}-\varepsilon[k])\ \tau(\mathbf{x}'+\varepsilon[k])\ e^{\xi(\mathbf{x}-\mathbf{x}',k)}\right] = 0, \quad \forall \mathbf{x}, \mathbf{x}' \tag{9}$$

where the symbol $\varepsilon[k]$ stand for the infinite sequence of shifts $(\frac{1}{k}, \frac{1}{2k^2}, \frac{1}{3k^3}, \cdots)$. In this identity, the operation $\mathrm{Res}_k\,[E(k)]$ denotes the residue for (formal) expansions of $E(k)$ at $k=\infty$. As is well known, the bilinear identity (9) can be used as a generator for the equations in the KP hierarchy expressed in terms of the Hirota D-operators [8] (i.e. expressed in terms of tau functions instead of the field $u = \partial_{x_1}^2 \log\tau$). For example, the KP equation takes the following form :

$$\left[4D_{x_1}D_{x_3} - D_{x_1}^4 - 3D_{x_2}^2\right]\ \tau\cdot\tau = 0.$$

To complete the desccription of the KP theory, one also defines wave functions $\Psi_\lambda(\mathbf{x})$ and adjoint wave functions $\Psi_\lambda^*(\mathbf{x})$

$$\Psi_\lambda(\mathbf{x}) \equiv \frac{\langle \mathrm{vac}|\psi_{1/2}^*\ \psi(\mathbf{x},\lambda)\mathbf{g}(\mathbf{x})|\mathrm{vac}\rangle}{\langle \mathrm{vac}|\mathbf{g}(\mathbf{x})|\mathrm{vac}\rangle} = \frac{\tau(\mathbf{x}-\varepsilon[\lambda])}{\tau(\mathbf{x})}\ e^{\xi(\mathbf{x},\lambda)}$$

$$\Psi_\lambda^*(\mathbf{x}) \equiv \frac{\langle \mathrm{vac}|\psi_{1/2}\ \psi^*(\mathbf{x},\lambda)\mathbf{g}(\mathbf{x})|\mathrm{vac}\rangle}{\langle \mathrm{vac}|\mathbf{g}(\mathbf{x})|\mathrm{vac}\rangle} = \frac{\tau(\mathbf{x}+\varepsilon[\lambda])}{\tau(\mathbf{x})}\ e^{-\xi(\mathbf{x},\lambda)}$$

which solve the KP linear problem

$$p_n(-\tilde\partial)\Psi_\lambda(\mathbf{x}) = \Psi_\lambda(\mathbf{x})\ p_{n-1}(-\tilde\partial)[\log\tau(\mathbf{x})]_{x_1}\ , \qquad \forall n\ge 2$$

and adjoint linear problem

$$p_n(\tilde\partial)\Psi_\lambda^*(\mathbf{x}) = \Psi_\lambda^*(\mathbf{x})\ p_{n-1}(\tilde\partial)[\log\tau(\mathbf{x})]_{x_1}\ , \qquad \forall n\ge 2.$$

Here the operators p_n are defined as the Schur polynomials $\sum_{n=0}^{\infty} p_n(-\tilde\partial)\lambda^n = \exp\left[\sum_{n=1}^{\infty} -\frac{1}{n}\frac{\partial}{\partial x_n}\lambda^n\right]$. Note that the KP Lax pair (2) and (3) is of course obtained at $n=2$ and 3, upon identifying $x=x_1, y=x_2$ and $t=x_3$ and consequently $u=\partial_x^2\log\tau$.

In this context it is important to mention that a general solution to the KP linear problem can always be expressed in the form [9] :

$$\Phi(\mathbf{x}) = \int_C \frac{\mathrm{d}\lambda}{2\pi i}\ h(\lambda)\Psi_\lambda(\mathbf{x}) \tag{10}$$

for a density function

$$h(\lambda) = \frac{1}{\lambda}\ \Phi(\mathbf{x}'+\varepsilon[\lambda])\Psi^*(\mathbf{x}') \qquad \forall \mathbf{x}',$$

which can be taken at any value of $\mathbf{x}'$. The integral expression (10) should in fact be understood as a $\mathrm{Res}_\lambda[\cdots]$ operation similar to that in the bilinear identity (9), i.e. for formal expansions of both $\Psi_\lambda(\mathbf{x})$ and $h(\lambda)$ near $\lambda = \infty$. The contour C is chosen such that it always includes the singularities induced by the exponential factor $\exp\xi(\mathbf{x},\lambda)$ present in the wave functions $\Psi_\lambda(\mathbf{x})$ and in particular it should include the essential singularity at $\lambda = \infty$. Excluded however are all singularities present in the density $h(\lambda)$ as well as those which might arise from the tau functions making up the wave function (i.e. arising from the shift $\tau(\mathbf{x} - \varepsilon[\lambda])$). See ref. [9] for an explanation of the rationale behind these restrictions as well as for a similar discussion concerning the adjoint wave functions.

3 Direct linearization for the KP hierarchy

We can now state the main result connecting the tau function approach to the KP hierarchy with the direct linearization procedure described in Sec. 1. Here and in what follows only the principal results will be stated. The relevant proofs, requiring careful consideration as well as considerable detail, will appear in a forthcoming paper [10].

It can be shown that all KP tau functions satisfy the integral equation

$$\tau(\mathbf{x} - \varepsilon[k]) = \tau(\mathbf{x}) - \int_{C_\lambda} \frac{d\lambda}{2\pi i} \int_{C_\mu^*} \frac{d\mu}{2\pi i} h(\lambda,\mu) \, \frac{e^{-\xi(\mathbf{x},\mu)+\xi(\mathbf{x},\lambda)}}{k-\mu} \tau(\mathbf{x} - \varepsilon[\lambda]) \tag{11}$$

with a density $h(\lambda,\mu)$ which is given by the following expression :

$$h(\lambda,\mu) = \frac{\tau(\mathbf{x}' - \varepsilon[\mu] + \varepsilon[\lambda])}{(\mu-\lambda)\tau(\mathbf{x}')} e^{\xi(\mathbf{x}',\mu)-\xi(\mathbf{x}',\lambda)} - \frac{1}{\mu-\lambda} \qquad \forall\, \mathbf{x}'; \tag{12}$$

the combined contours $\mathcal{C}_\lambda$ and $\mathcal{C}_\mu$ are subject to requirements similar to those for formula (10).

As it is written in expression (12) the density $h(\lambda,\mu)$ depends explicitly on an arbitrary set of coordinates $\mathbf{x}'$; one can however show that the combined integrals in the equation (11) are invariant for changes in $\mathbf{x}'$. Hence, all KP tau functions satisfy a *linear* integral equation with a measure which can be calculated from the tau function itself (by using formula (12) at any convenient choice of coordinates $\mathbf{x}'$). The converse also holds : every function $\tau(\mathbf{x})$ which satisfies equation (11) for a density (12) *is* a KP tau function. A corollary of equation (11) – used in the proof of the converse statement – is that all wave functions satisfy the integral equation :

$$\Psi_k(\mathbf{x}) = e^{\xi(\mathbf{x},k)} - e^{\xi(\mathbf{x},k)} \int_{C_\lambda} \frac{d\lambda}{2\pi i} \int_{C_\mu^*} \frac{d\mu}{2\pi i} h(\lambda,\mu) \, \frac{e^{-\xi(\mathbf{x},\mu)}}{k-\mu} \, \Psi_\lambda(\mathbf{x}).$$

This integral equation is of course nothing but equation (1) for a measure defined in terms of the density $h(\lambda,\mu)$ introduced above and extended to evolutions involving the entire KP hierarchy (i.e. $x_{n\geq 1}$). Moreover, expanding equation (11) in terms of λ^{-1}, we find expressions for the logaritmic derivatives of the tau function in terms of the wave functions, as in the formulae

$$(\log\tau)_x = \int_{C_\lambda} \frac{d\lambda}{2\pi i} \int_{C_\mu^*} \frac{d\mu}{2\pi i} h(\lambda,\mu) \; e^{-\xi(\mathbf{x},\mu)} \Psi_\lambda(\mathbf{x})$$

$$(\log\tau)_y = (\log\tau)_{2x} + (\log\tau)_x^2 + 2\int_{C_\lambda} \frac{d\lambda}{2\pi i} \int_{C_\mu^*} \frac{d\mu}{2\pi i} \; \mu\, h(\lambda,\mu) \; e^{-\xi(\mathbf{x},\mu)} \Psi_\lambda(\mathbf{x})$$

$\vdots$ *etc.* ,

the first one of which coincides with the reconstruction formula (4) for the potential u presented in Sec. 1. This then provides a direct connection between the direct linearization described in that section and the Sato theory. It does however not explain why the reconstruction formula (4) can yield a solution to the KP hierarchy for almost *any* contour or measure. Furthermore, up to this point there is no major advantage in adopting the tau function reformulation of the method. For if one would like to reconstruct a tau function from its corresponding density and contour, one is still required to solve an integral equation, after which one has to reconstruct the tau function from relations such as the logarithmic derivatives given above. It is here that the algebraic approach introduced in the previous section will come into play.

4 Initial value problems for tau functions

Quite generally, we can consider tau functions $\tau(\mathbf{x}) = \langle \text{vac}| e^{H(\mathbf{x})}\ e^X|\text{vac}\rangle$ by starting from generators X which are defined as a general "superpositions" of products of formal operators $\psi(k)$ and $\psi^*(k)$:

$$X = c + \iint_{\mathcal{C}} \mathrm{d}\eta(\lambda,\mu)\ \ \psi(\lambda)\psi^*(\mu) \qquad\qquad (c \in \mathbb{C})$$

as long as both measure and contour are such that $e^X \in \overline{GL}(\infty)$, i.e. as long as the expectation value defining the tau function makes sense. For such general tau functions it can be shown that if the generator X is separable in the sense that there exists a sequence of pairs of operators ϕ_j and ϕ_j^* $(\forall j \in \mathcal{J})$

$$\phi_j \equiv \int_{\mathcal{C}_\lambda} \frac{\mathrm{d}\lambda}{2\pi i} h_j(\lambda)\psi(\lambda)\ , \qquad \phi_j^* \equiv \int_{\mathcal{C}_\mu} \frac{\mathrm{d}\mu}{2\pi i} h_j^*(\mu)\psi^*(\mu)$$

(or for that matter, sequences of densities $h_j(\lambda)$ and $h_j^*(\mu)$ for j in some index set $\mathcal{J}$, possibly infinite) such that X can be expressed in the form :

$$X = c + \sum_{j\in\mathcal{J}} \phi_j\phi_j^*\ ,$$

then the resulting tau function $\tau(\mathbf{x}) = \langle \text{vac}| e^{H(\mathbf{x})}\ e^X|\text{vac}\rangle$ will satisfy equation (11) for the density

$$h(\lambda,\mu) \equiv \sum_{j\in\mathcal{J}} h_j(\lambda)h_j^*(\mu). \tag{13}$$

Here again, the integral expressions defining the operators ϕ_j and ϕ_j^* should in fact be understood as $\text{Res}_\lambda[\cdots]$ operations, i.e. for formal expansions of both $\Psi_\lambda(\mathbf{x})$ and $h(\lambda)$ around $\lambda = \infty$. The contours are subject to the same constraints as before.

Obviously, this now provides a way to reconstruct an element of $gl(\infty)$ (or rather in an "extension" of $gl(\infty)$ in the sense explained above) for any given tau function. It suffices to calculate the density (12) for a particularly convenient value of the coordinates $\mathbf{x}'$ (such as to facilitate the resulting expression ; most often this will simply be $\mathbf{x}' = \mathbf{0} \equiv (0,0,\ldots)$ if this is not a zero of the tau function) and subsequently decompose it as in expression (13). From the resulting densities $h_j(\lambda)$ and $h_j^*(\mu)$ one can then calculate the operators ϕ_j and ϕ_j^* and hence the generator X. The constant c appearing in this generator can be determined afterwards, by normalizing the resulting expectation value such that it coincides with $\tau(\mathbf{x})$. Note that the decomposition (13) is only subject to the requirement that, ultimately, e^X has to be part of $\overline{GL}(\infty)$. Which also offers an explanation for the remarkable freedom one has in choosing measures and contours in the integral equation (11).

As a simple example illustrating this method, let us calculate the densitiy $h(\lambda,\mu)$ (at $\mathbf{x} = \mathbf{0}$) for the trivial tau function $e^{\xi(\mathbf{x},p)}$:

$$\tau(\mathbf{x}) \equiv e^{\xi(\mathbf{x},p)} \qquad \overset{(12)}{\Longrightarrow} \qquad h^0(\lambda,\mu) = \frac{1}{\mu-\lambda}\left(\frac{1-p/\mu}{1-p/\lambda} - 1\right) = \frac{p}{\mu}\,\frac{1}{\lambda-p}$$

which gives rise to the generator

$$X \equiv (\sum_{j\leq -1/2} \psi_j\ p^{1/2-j})\psi^*_{-1/2}\ ,$$

as can be seen from the following epressions for the operators ϕ_j and ϕ_j^* :

$$\phi = \int_{\mathcal{C}_\lambda} \frac{\mathrm{d}\lambda}{2\pi i}\frac{\psi(\lambda)}{\lambda-p} = \sum_{j\leq -1/2} \psi_j\ p^{-j-1/2}\ , \qquad \phi^* = \int_{\mathcal{C}_\mu^*} \frac{\mathrm{d}\mu}{2\pi i}\frac{1}{\mu}\ \psi^*(\mu) = \psi^*_{-1/2}$$

(for contours $\mathcal{C}_\lambda$ and $\mathcal{C}_\mu$ around $\lambda = \infty$, excluding the point $\lambda = p$). It is easily verified that this generator does produce the correct tau function $\tau(\mathbf{x}) = \langle \text{vac}| e^{H(\mathbf{x})}\ e^X|\text{vac}\rangle \equiv e^{\xi(\mathbf{x},p)}$ (i.e. $c = 0$).

Reconstruction methods for tau functions (or rather for their generators) were already described by Takasaki [11] and Takebe [12, 13]. Whereas these approaches provide deep insight in the algebraic nature of the reconstruction problem, the present approach has the apparent advantage of allowing for a reconstruction of the tau function from "partial" data : it turns out that the density $h(\lambda,\mu)$ can be calculated even in the case where only the x_1x_2-dependence of the tau function is known ! This then opens the door to solving genuine initial value problems (in the sense of inverse scattering) directly on the level of the tau functions. An earlier description of an inverse scattering technique for tau functions was given by Oishi [14]. There however extensive use was made of the standard inverse scattering approach (i.e. of the Gel'fand-Levitan equation) and the method therefore essentially amounted to (and suffered from the same restrictions as) the traditional inverse scattering.

Here instead we propose the following reconstruction procedure. For a given initial profile $\tau^0(x=x_1, y=x_2)$ one first has to solve the system of (linear) equations :

$$\begin{aligned} &[D_y - D_x^2 - 2\mu\, D_x] S_\mu \cdot \tau^0 = 0 \\ &[D_y + D_x^2 - 2\lambda\, D_x] S_\mu^\lambda \cdot S_\mu = 0 \end{aligned} \tag{14}$$

under the restriction that the solutions S_μ and S_μ^λ should allow for the following asymptotic expansions in the parameters λ and μ :

$$S_\mu = \tau^0 - \frac{1}{\mu}\,\tau_x^0 - \frac{1}{2\mu^2}\,(\tau_y^0 - \tau_{2x}^0) + \sum_{n=3}^{\infty} s_n\, \mu^{-n}$$

$$S_\mu^\lambda = S_\mu + \frac{1}{\lambda}\,(S_\mu)_x + \frac{1}{2\lambda^2}\,[(S_\mu)_y + (S_\mu)_{2x}] + \sum_{n=3}^{\infty} S_n\, \lambda^{-n}\,.$$

(where, in principle, the functions s_n and S_n $(n \geq 3)$ can be found recursively from (14)). The function S_μ^λ also has to satisfy the additional requirement :

$$\lim_{\lambda -> \mu} S_\mu^\lambda = \tau^0(x,y)\,.$$

If such solutions S_μ and S_μ^λ exist, it can be shown ① that the function S_μ^λ is uniquely determined for a given S_μ and ② that it allows one to calculate a density

$$h^0(\lambda,\mu) = \frac{S_\mu^\lambda\, e^{\xi_\mu^0 - \xi_\lambda^0}}{(\mu-\lambda)\tau^0(x,y)} - \frac{1}{\mu-\lambda} \qquad (\xi_k^0 \equiv kx + k^2 y)$$

determining – up to a normalization constant – a tau function $\tau(\mathbf{x})$ which coincides with $\tau^0(x,y)$ in the plane of the initial data : $\tau^0(x,y) = \tau(\mathbf{x})|_{x_{n\geq 3}=0}$.

Of course such a tau function can not be unique, as can be seen from the simple fact that multiplication of a tau function by an exponential factor $(\exp \sum_{n=3}^{\infty} a_n x_n)$ $(a_n \in \mathbb{C})$ yields another tau function correspondsing to the same initial data $\tau(\mathbf{x})|_{x_{n\geq 3}=0}$. It can however be shown that, under the above requirements on S_μ and S_μ^λ, two tau functions corresponding to the same initial data can only differ by such trivial gauges. Note that this gauge freedom results in a certain ambiguity in determining the function S_μ, a particular choice however always producing a single S_μ^λ and ultimately only manifesting itself in a trivial exponential factor which does not complicate or change the reconstructed tau function in any real way. On the other hand, the question of which initial profiles $\tau^0(x,y)$ actually allow for such a reconstruction is a far more difficult one and will be discussed elsewhere [10].

Let us instead give some simple examples in order to illustrate the procedure :

① starting from the initial profile for a 1-soliton solution : $\tau^0(x,y) = c + \dfrac{e^{\xi_p^0 - \xi_q^0}}{p-q}$ $(\xi_k^0 \equiv kx + k^2 y$, $c,p,q \in \mathbb{C})$, the following solutions to the linear system (14) can be easily found :

$$S_\mu = c + \frac{\mu - p}{\mu - q}\,\frac{e^{\xi_p^0 - \xi_q^0}}{p-q}\,, \qquad S_\mu^\lambda = c + \frac{(\mu-p)(\lambda-q)}{(\mu-q)(\lambda-p)}\,\frac{e^{\xi_p^0 - \xi_q^0}}{p-q}.$$

At $x = y = 0$, the function S_μ^λ gives rise to the density :

$$h^0(\lambda,\mu) = \frac{1}{c+\frac{1}{p-q}} \frac{1}{(\lambda-p)(\mu-q)} \tag{15}$$

which is already of the form (13) and which gives rise the pair of operators :

$$\phi = \sum_{j<0} \psi_j \, p^{-j-1/2} \qquad \text{and} \qquad \phi^* = \sum_{j<0} \psi_j^* \, q^{-j-1/2}.$$

From the expectation value

$$\begin{aligned} \langle \mathrm{vac}|e^{H(\mathbf{x})} \, e^{\phi\phi^*}|\mathrm{vac}\rangle &= 1 + \langle \mathrm{vac}|e^{H(\mathbf{x})}\phi\phi^*|\mathrm{vac}\rangle \\ &= \frac{1}{c+\frac{1}{p-q}}(c + \frac{e^{\xi_p^0-\xi_q^0}}{p-q}) \end{aligned}$$

it is easily seen that one recovers the correct tau function (i.e. one that corresponds to τ^0 in the plane of the initial data) by normalizing with $c + \frac{1}{p-q}$. Note that the operators ϕ and ϕ^* are not the usual ones found in the literature when the generator for the KP 1-soliton solution is discussed. It can however be shown (by deforming the contours $\mathcal{C}_\lambda$ and $\mathcal{C}_\mu$) that the densities $h(\lambda) = \frac{1}{\lambda-p}$ and $h^*(\mu) = \frac{1}{\mu-p}$ can actually be replaced by delta functions $\delta(p/\lambda)$ and $\delta(q/\mu)$ respectively (the delta function being defined as : $\delta(p/\lambda) \equiv \frac{1}{\lambda}\sum_{n\in\mathbb{Z}} (p/\lambda)^n)$ for an appropriate normalization of the resulting tau function. From such delta functions one then obtains operators $\phi = \psi(p)$, $\phi^* = \psi^*(q)$ and the well known generator for the 1-soliton solution : $X = \dfrac{1}{c}\psi(p)\psi^*(q)$.

② another simple example is provided by the initial profile $\tau^0 = c+x+ky$ which actually falls outside the class of functions addressed by traditional inverse scattering methods, as it corresponds to a potential $u = \partial_x^2 \log\tau$ which is singular in x and y and has polynomial asymptotic behaviour elsewhere. In the present approach however, one immediately finds the following solutions for the linear system (14) :

$$S_\mu = c - \frac{1}{\mu-k/2} + x + ky \ , \qquad S_\mu^\lambda = c - \frac{1}{\mu-k/2} + \frac{1}{\lambda-k/2} + x + ky$$

and hence the corresponding density and resulting polynomial tau function :

$$h^0(\lambda,\mu) = \frac{1}{c}\frac{1}{\lambda-\frac{k}{2}}\frac{1}{\mu-\frac{k}{2}} \qquad \Longrightarrow \qquad \tau(\mathbf{x}) = c + x + ky + \sum_{n=3}^{\infty} \frac{n}{2^{n-1}} k^{n-1} \, x_n.$$

Note that this density can be obtained by a coalescence $p = q = k/2$ of the poles in the 1-soliton density (15) (for a suitable redefinition of the constant c). For the case of N-soliton solutions, such a coalescense leads to tau functions which will produce the lump solutions [15] for the KP I equation. An extension of this idea gives rise the so called multipole solutions recently described by Ablowitz *et al.* [16], a case which can also be easily handled in the present approach.

Finally, adaptations of this method to the case of the discrete KP equation or to reductions of the KP hierarchy are the subject of ongoing and future research.

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