数理解析研究所講究録
1170 巻 2000 年 23-34

Soliton Equations exhibiting "Pfaffian Solutions"

早大理工　広田良吾 (Ryogo Hirota)
早大理工　岩尾昌央 (Masataka Iwao)
阪大基礎工　辻本諭 (Satoshi Tujimoto)

Soliton equations whose solutions are expressed by pfaffians are briefly discussed. Included are "Discrete-time Toda equation","Modified Toda equation of BKP type", the coupled modified KdV equation and "Coupled modified equation of derivative type" etc. The Bäcklund transformation of the discrete BKP equation in biliner form is described in detail in Appendix.

1 Introduction

Pfaffian is known as a square root of an anti-symmetric determinant of order 2n:

$$\det|a_{jk}|_{1\leq j,k\leq 2n}=[pf(a_1,a_2,\cdots,a_{2n})]^2.$$

We have, for example, for n=1

$$\begin{vmatrix} 0 & a_{12} \\ a_{21} & 0 \end{vmatrix} = pf(a_1,a_2)^2.$$

Hence

$$pf(a_1,a_2)=a_{12}.$$

For n=2,

$$\begin{vmatrix} 0 & a_{12} & \ldots & a_{14} \\ a_{21} & 0 & \ldots & a_{24} \\ \cdots & \cdots & \cdots & \cdots \\ a_{41} & a_{42} & \ldots & 0 \end{vmatrix} = [a_{12}a_{34}-a_{13}a_{24}+a_{14}a_{23}]^2.$$

Hence

$$pf(a_1,a_2,a_3,a_4)=a_{12}a_{34}-a_{13}a_{24}+a_{14}a_{23}.$$

A square root of a determinant of a matrix M of order n which is a sum of a unit matrix E and a product of anti-symmetric matrices A and B is expressed by a two-component pfaffian:

$$\det|E+A\times B|=[pf(a_1,a_2,\cdots,a_n,b_1,b_2,\cdots,b_n)]^2,$$

where

$$pf(a_j, a_k) = a_{jk} = -a_{kj},$$
$$pf(b_j, b_k) = b_{jk} = -b_{kj},$$
$$pf(a_j, b_k) = \delta_{j,k}$$

and where $\delta_{j,k}$ is Kronecker's delta.

We have, for example, for n=3

$$E + A \times B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix} \begin{pmatrix} 0 & b_{12} & b_{13} \\ b_{21} & 0 & b_{23} \\ b_{31} & b_{32} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + a_{12}b_{21} + a_{13}b_{31} & a_{13}b_{32} & a_{12}b_{23} \\ a_{23}b_{31} & 1 + a_{21}b_{12} + a_{23}b_{32} & a_{21}b_{13} \\ a_{32}b_{21} & a_{31}b_{12} & 1 + a_{31}b_{13} + a_{32}b_{23} \end{pmatrix}.$$

Accordingly we find

$$\det|E + A \times B| = [-1 + a_{12}b_{12} + a_{13}b_{13} + a_{23}b_{23}]^2.$$

On the other hand using the expansion rule of the pfaffian

$$pf(c_1, c_2, \cdots, c_{2n}) = \sum_{j=1}^{2n}(-1)^{j-1} pf(c_1, c_j) pf(c_2, c_3, \cdots, \hat{c}_j, \cdots, c_{2n}),$$

we obtain

$$pf(a_1, a_2, a_3, b_1, b_2, b_3) = -1 + a_{12}b_{12} + a_{13}b_{13} + a_{23}b_{23}.$$

Hence

$$\det|E + A \times B| \;=\; pf(a_1, a_2, a_3, b_1, b_2, b_3)^2.$$

The two-component pfaffian plays an important role in expressing soliton-solutions of the coupled modified KdV equations and "Coupled modified equation of derivative type" described below.
We shall show that soliton-solutions of the following equations are expressed by pfaffians.

(i) Discrete-time Toda equation

$$W_n^{m+1} - W_n^{m-1} = \log \frac{\delta^2[\exp(W_{n+1}^m - V_n^m) + \exp(V_n^m - W_n^m)] + (1 - 2\delta^2)}{\delta^2[\exp(W_n^m - V_{n-1}^m) + \exp(V_{n-1}^m - W_{n-1}^m)] + (1 - 2\delta^2)},$$
$$V_n^{m+1} - V_n^m = W_{n+1}^m - W_n^m.$$

(ii) Modified Toda equation of BKP type

$$\frac{d}{dt}\log\frac{\beta+V_n}{\beta-\alpha(I_n+I_{n-1})}=I_n-I_{n+1},$$
$$\frac{d}{dt}I_n=V_{n-1}-V_n.$$

(iii) Coupled modified KdV equation

$$\frac{\partial}{\partial t}v_j+6[\sum_{1\le j<k\le N}c_{j,k}v_jv_k]\frac{\partial v_j}{\partial x}+\frac{\partial^3 v_j}{\partial x^3}=0,\quad j=1,2,\ldots,N.$$

(iv) Coupled modified KdV equations of "derivative type"

$$\frac{\partial}{\partial t}v_j+6[\sum_{1\le j<k\le N}c_{j,k}D_xv_j\cdot v_k]\frac{\partial v_j}{\partial x}+\frac{\partial^3 v_j}{\partial x^3}=0,\quad j=1,2,\ldots,N.$$

We have the discrete KP equation

$$[z_1\exp(D_1)+z_2\exp(D_2)+z_3\exp(D_3)]f\cdot f=0$$

where D_1,D_2,D_3 and z_1,z_2,z_3 are bilinear operators and constants,respectively. Soliton solutions to the KP equation are known to be expressed by determinants [1]. While the discrete BKP equation is expressed by

$$[z_1\exp(D_1)+z_2\exp(D_2)+z_3\exp(D_3)+z_4\exp(D_4)]f\cdot f=0$$

where D_1,D_2,D_3,D_4 and z_1,z_2,z_3,z_4 are bilinear operators and constants, respectively, satisfying the relations

$$\begin{aligned} D_1+D_2+D_3+D_4 &= 0,\\ z_1+z_2+z_3+z_4 &= 0. \end{aligned}$$

Solutions to the discrete BKP equation are well parametrized by parameters a,b and c which are the intervals of the coordinates l,m,n [2]. We rewrite the discrete BKP equation using these parameters:

$$\begin{aligned}&(a+b)(a+c)(b-c)\tau(l+1,m,n)\tau(l,m+1,n+1)\\&+(b+c)(b+a)(c-a)\tau(l,m+1,n)\tau(l+1,m,n+1)\\&+(c+a)(c+b)(a-b)\tau(l,m,n+1)\tau(l+1,m+1,n)\\&+(a-b)(b-c)(c-a)\tau(l,m,n)\tau(l+1,m+1,n+1)=0.\end{aligned}$$

Solutions are expressed by the pfaffian [2],[3]:

$$\tau(l,m,n)=pf(1,2,3,\cdots,2N),$$

where the elements of the pfaffian are

$$\begin{aligned}&pf(j,k)\equiv c_{j,k}\\&+\sum_{s=-\infty}^{n-1}[\phi_j(l,m,s+1)\phi_k(l,m,s)-\phi_j(l,m,s)\phi_k(l,m,s+1)]\end{aligned}$$

and $\phi_j(l,m,n)$ are a linear combination of "exponential functions in discrete space"

$$(\frac{1-ap_\mu}{1+ap_\mu})^l(\frac{1-bp_\mu}{1+bp_\mu})^m(\frac{1-cp_\mu}{1+cp_\mu})^n,$$

namely

$$\phi_j(l,m,n)=\sum_\mu \alpha_{j\mu}(\frac{1-ap_\mu}{1+ap_\mu})^l(\frac{1-bp_\mu}{1+bp_\mu})^m(\frac{1-cp_\mu}{1+cp_\mu})^n.$$

2 Soliton Equations Generated by the Bilinear BKP Equation

The followings are well-known soliton equations generated by the discrete BKP equation.

(a) Sawada-Kotera equation

$$D_x(D_t+D_x^5)f\cdot f=0,$$

$$u=2\frac{\partial^2}{\partial x^2}\log f,$$

$$u_t+15(u^3+uu_{xx})_x+u_{xxxxx}=0.$$

(b) Model equation of shallow-water wave

$$D_x(D_t-D_tD_x^2+D_x)f\dot{f}=0,$$

$$u=2\frac{\partial^2}{\partial x^2}\log f,$$

$$u_t-u_{xxt}-3uu_t+3u_x\int_x^\infty u_t dx'=0.$$

Here we add two more examples.
(1) Discrete-time Toda equation.
Let

$$\begin{aligned}
D_1 &= \frac{3}{2}\delta D_t, \qquad z_1=1,\\
D_2 &= -\frac{1}{2}\delta D_t, \qquad z_2=-1+2\delta^2,\\
D_3 &= D_n-\frac{1}{2}\delta D_t, \quad z_3=-\delta^2,\\
D_4 &= -D_n-\frac{1}{2}\delta D_t, \quad z_4=-\delta^2.
\end{aligned}$$

Then

$$\begin{aligned}
&[e^{\delta D_t+\frac{1}{2}\delta D_t}+e^{-\delta D_t+\frac{1}{2}\delta D_t}-2e^{-\frac{1}{2}\delta D_t}\\
&\quad -\delta^2(e^{D_n-\frac{1}{2}\delta D_t}+e^{-D_n-\frac{1}{2}\delta D_t}-2e^{-\frac{1}{2}\delta D_t})]f\cdot f=0,
\end{aligned}$$

which is rewritten as

$$\cosh \frac{1}{2}\delta D_t[\sinh^2 \frac{1}{2}\delta D_t - \delta^2 \sinh^2 \frac{1}{2}D_n]f \cdot f = 0. \qquad (1)$$

The bilinear form indicates that 1-soliton solution is the same as that of the discrete-time Toda equation

$$[\sinh^2 \frac{1}{2}\delta D_t - \delta^2 \sinh^2 \frac{1}{2}D_n]f \cdot f = 0.$$

The bilinear equation(1) is transformed into the nonlinear difference-difference equation in the ordinary form

$$W_n^{m+1} - W_n^{m-1} = \log \frac{\delta^2[\exp(W_{n+1}^m + V_n^m) + \exp(V_n^m - W_n^m)] + (1-2\delta^2)}{\delta^2[\exp(W_n^m + V_{n-1}^m) + \exp(V_{n-1}^m - W_{n-1}^m)] + (1-2\delta^2)},$$
$$V_n^{m+1} - V_n^m = W_{n+1}^m - W_n^m,$$

through a series of dependent variable transformations

$$f_n^m = e^{\phi_n^m},$$
$$\Delta_m \phi_n^m = T_n^m,$$
$$\Delta_n \phi_n^m = S_n^m,$$
$$\Delta_m S_n^m = W_n^m,$$
$$\Delta_n S_n^m = V_n^m,$$

where Δ is the forward difference operator defined by

$$\Delta_m f_n^m = \delta^{-1}[f_n^{m+1} - f_n^m],$$
$$\Delta_n f_n^m = f_{n+1}^m - f_n^m.$$

(2) Modified Toda equation of BKP type.
Let

$$\begin{aligned} D_1 &= \frac{1}{2}(\delta D_x + \epsilon D_y), & z_1 &= 1, \\ D_2 &= \frac{1}{2}(\delta D_x - \epsilon D_y), & z_2 &= -1 + \beta\delta\epsilon, \\ D_3 &= -D_n - \frac{1}{2}(\delta D_x + \epsilon D_y), & z_3 &= -\alpha\epsilon - \beta\delta\epsilon, \\ D_4 &= D_n - \frac{1}{2}(\delta D_x - \epsilon D_y), & z_4 &= \alpha\epsilon. \end{aligned}$$

Then

$$\begin{aligned} &\{e^{\frac{1}{2}(\delta D_x + \epsilon D_y)} - e^{\frac{1}{2}(\delta D_x - \epsilon D_y)} \\ &+\alpha\epsilon[e^{D_n - \frac{1}{2}(\delta D_x - \epsilon D_y)} - e^{-D_n - \frac{1}{2}(\delta D_x + \epsilon D_y)}] \\ &+\beta\delta\epsilon[e^{\frac{1}{2}(\delta D_x - \epsilon D_y)} - e^{-D_n - \frac{1}{2}(\delta D_x + \epsilon D_y)}]\} f \cdot f = 0, \end{aligned}$$

which is rewritten as

$$\begin{aligned}
&\frac{2}{\delta\epsilon}\sinh\frac{1}{2}\delta D_x \sinh\frac{1}{2}\epsilon D_y f_n\cdot f_n\\
&-\alpha\frac{1}{\delta}e^{\epsilon D_y}\sinh\frac{1}{2}\delta D_x f_{n+1}\cdot f_{n-1}\\
&-\beta[e^{\frac{1}{2}(\delta D_x+\epsilon D_y)}f_{n+1}\cdot f_{n-1}-e^{\frac{1}{2}(\delta D_x-\epsilon D_y)}f_n\cdot f_n]=0.
\end{aligned}$$

In the limit of small δ, ϵ we have

$$D_xD_yf_n\cdot f_n=\alpha D_xf_{n+1}\cdot f_{n-1}+\beta(f_{n+1}f_{n-1}-f_n^2).$$

The bilinear equation is transformed into the ordinary form

$$\begin{aligned}
&\frac{\partial}{\partial x}\log\frac{\beta+V_n}{\beta-\alpha(I_n+I_{n+1})}=I_n-I_{n+1},\\
&\frac{\partial}{\partial y}I_n=V_{n-1}-V_n,
\end{aligned}$$

through the transformation

$$\begin{aligned}
f_n&=e^{S_n},\\
V_n&=\frac{\partial^2}{\partial x\partial y}S_n,\\
I_n&=\frac{\partial}{\partial x}(S_{n-1}-S_n),
\end{aligned}$$

which we call "modified Toda equation of BKP type". The bilinear form of the modified Toda equation of BKP type

$$D_xD_yf_n\cdot f_n=\alpha D_xf_{n+1}\cdot f_{n-1}+\beta(f_{n+1}f_{n-1}-f_n^2),$$

suggests that a new equation may be obtained by replacing the term

$$f_{n+1}f_{n-1}-f_n^2$$

by

$$D_xf_{n+1}\cdot f_{n-1}.$$

We shall replace the term g_jg_k by $D_xg_j\cdot g_k$ in order to obtain a coupled modified KdV equation of "derivative type".
The well-known modified KdV equation

$$\frac{\partial}{\partial t}v+6v^2\frac{\partial}{\partial x}v+\frac{\partial^3}{\partial x^3}v=0,$$

is transformed into the bilinear form

$$\begin{aligned}
&(D_t+D_x^3)f\cdot g=0,\\
&D_x^2f\cdot f=2g^2,
\end{aligned}$$

through the dependent variable transformation

$$v = \frac{g}{f}.$$

A coupled modified KdV equation is obtained by considering the following coupled form of the modified KdV equations [4].

$$\begin{cases} (D_t + D_x^3) f \cdot g_j = 0, & \text{for } j = 1, 2, \cdots N, \\ Dx^2 f \cdot f = 2 \sum_{1 \le j < k \le N} c_{j,k} g_j g_k, & \end{cases} \tag{2}$$

which is transformed into the ordinary form

$$\frac{\partial}{\partial t} v_j + 6[\sum_{1 \le j < k \le N} c_{j,k} v_j v_k] \frac{\partial v_j}{\partial x} + \frac{\partial^3 v_j}{\partial x^3} = 0,$$

through the transformation

$$v_j = \frac{g_j}{f}.$$

The multisoliton solution to eq.(2) is expressed by "two-component pfaffians" [4]:

$$f = pf(a_1, a_2, \cdots, a_L, b_1, b_2, \cdots, b_L),$$
$$g_j = pf(d_0, a_1, a_2, \cdots, a_L, b_1, b_2, \cdots, b_L, c_j), \text{ for } j = 1, 2, \cdots, N,$$

where the elements of pfaffians are defined as follows

$$pf(\delta_n, a_\nu) \equiv \frac{\partial^n}{\partial x^n} \exp(\eta_\nu), \text{ for } n = 1, 2, 3, \ldots,$$
$$pf(a_\mu, a_\nu) \equiv \frac{p_\mu - p_\nu}{p_\mu + p_\nu} \exp(\eta_\mu + \eta_\nu),$$
$$pf(a_\mu, b_\nu) \equiv \delta_{\mu,\nu},$$
$$pf(b_\mu, b_\nu) \equiv -\frac{c_j k}{p_\mu^2 - p_\nu^2} \exp(\eta_\mu + \eta_\nu), \quad \begin{Bmatrix} b_\mu \in B_j \\ b_\nu \in B_k \end{Bmatrix},$$
$$pf(b_\mu, c_j) \equiv \begin{cases} 1, & \text{if } b_\mu \in B_j, \\ 0, & \text{if } b_\nu \notin B_k \end{cases}$$
$$pf(otherwise) \equiv 0.$$

A coupled modified KdV equation of "derivative type" is obtained by replacing the coupling terms of product form

$$c_{jk} g_j g_k, \quad (c_{jk} = c_{kj})$$

by coupling terms of derivative type

$$c_{jk} D_x g_j \cdot g_k, \quad (c_{jk} = -c_{kj}).$$

We obtain

$$\begin{cases} (D_t + D_x^3) f \cdot g_j = 0, & \text{for } j = 1, 2, \cdots N, \\ D_x^2 f \cdot f = 2 \sum_{1 \le j < k \le N} c_{j,k} D_x g_j \cdot g_k, \end{cases} \tag{3}$$

which is transformed into the ordinary form

$$\frac{\partial}{\partial t} v_j + 6[\sum_{1 \le j < k \le N} c_{j,k} (D_x v_j \cdot v_k)] \frac{\partial v_j}{\partial x} + \frac{\partial^3 v_j}{\partial x^3} = 0.$$

through the transformation

$$v_j = \frac{g_j}{f}.$$

The multisoliton solution to eq.(3) is expressed by the same form as that of the coupled modified KdV equation [5]:

$$f = pf(a_1, a_2, \cdots, a_L, b_1, b_2, \cdots, b_L),$$
$$g_j = pf(d_0, a_1, a_2, \cdots, a_L, b_1, b_2, \cdots, b_L, c_j), \text{ for } j = 1, 2, \cdots, N,$$

where the elements of pfaffians are the same as those of the modified KdV equation except the element

$$pf(b_\mu, b_\nu) \equiv -\frac{c_{jk}}{p_\mu^2 - p_\nu^2} \exp(\eta_\mu + \eta_\nu), \quad \left\{ \begin{array}{l} b_\mu \in B_j \\ b_\nu \in B_k \end{array} \right\}, \quad (c_{jk} = c_{kj}),$$

is replaced by

$$pf(b_\mu, b_\nu) \equiv -\frac{c_{jk}}{p_\mu + p_\nu} \exp(\eta_\mu + \eta_\nu), \quad \left\{ \begin{array}{l} b_\mu \in B_j \\ b_\nu \in B_k \end{array} \right\}, \quad (c_{jk} = -c_{kj}).$$

In conclusion we have shown that soliton-solutions of the following equations are expressed by pfaffians.

(i) Discrete-time Toda equation

$$W_n^{m+1} - W_n^{m-1} = \log \frac{\delta^2[\exp(W_{n+1}^m - V_n^m) + \exp(V_n^m - W_n^m)] + (1 - 2\delta^2)}{\delta^2[\exp(W_n^m - V_{n-1}^m) + \exp(V_{n-1}^m - W_{n-1}^m)] + (1 - 2\delta^2)},$$
$$V_n^{m+1} - V_n^m = W_{n+1}^m - W_n^m.$$

(ii) Modified Toda equation of BKP type

$$\frac{d}{dt} \log \frac{\beta + V_n}{\beta - \alpha(I_n + I_{n-1})} = I_n - I_{n+1},$$
$$\frac{d}{dt} I_n = V_{n-1} - V_n.$$

(iii) Coupled modified KdV equation

$$\frac{\partial}{\partial t} v_j + 6[\sum_{1 \le j < k \le N} c_{j,k} v_j v_k] \frac{\partial v_j}{\partial x} + \frac{\partial^3 v_j}{\partial x^3} = 0, \quad j = 1, 2, \ldots, N.$$

(iv) Coupled modified equations of "derivative type"

$$\frac{\partial}{\partial t} v_j + 6[\sum_{1 \le j < k \le N} c_{j,k} \underline{D_x v_j \cdot v_k}] \frac{\partial v_j}{\partial x} + \frac{\partial^3 v_j}{\partial x^3} = 0, \quad j = 1, 2, \ldots, N.$$

A Bäcklund Transformation of the Discrete BKP Equation

In this appendix we describe in detail how to obtain the Bäcklund transformation of the discrete BKP equation in bilinear form.
The Bäcklund transformation which relates a solution f of the discrete BKP euqation

$$[z_1 \exp(D_1) + z_2 \exp(D_2) + z_3 \exp(D_3) + z_4 \exp(D_4)] f \cdot f = 0$$

with another solution f' of the same equation

$$[z_1 \exp(D_1) + z_2 \exp(D_2) + z_3 \exp(D_3) + z_4 \exp(D_4)] f' \cdot f' = 0$$

is obtained by considering the following quantity P

$$\begin{aligned} P = &\{[z_1 \exp(D_1) + z_2 \exp(D_2) + z_3 \exp(D_3) + z_4 \exp(D_4)] f \cdot f\} \\ &\qquad\qquad \times \{[z_3 \exp(D_3) + z_4 \exp(D_4)] f' \cdot f'\} \\ &-\{[z_3 \exp(D_3) + z_4 \exp(D_4)] f \cdot f\} \\ &\qquad \times \{[z_1 \exp(D_1) + z_2 \exp(D_2) + z_3 \exp(D_3) + z_4 \exp(D_4)] f' \cdot f'\} \\ = &\, 0 \end{aligned}$$

and using the exchange formula

$$(e^{D_1} f \cdot f)(e^{D_3} f' \cdot f') = e^{\frac{1}{2}(D_1+D_3)} [e^{\frac{1}{2}(D_1-D_3)} f \cdot f'] \cdot [e^{-\frac{1}{2}(D_1-D_3)} f \cdot f']. \tag{4}$$

In fact we find

$$\begin{aligned} &(e^{D_1} f \cdot f)(e^{D_3} f' \cdot f') - (e^{D_1} f' \cdot f')(e^{D_3} f \cdot f) \\ &= e^{\frac{1}{2}(D_1+D_3)} \{[e^{\frac{1}{2}(D_1-D_3)} f \cdot f'] \cdot [e^{-\frac{1}{2}(D_1-D_3)} f \cdot f'] - [e^{-\frac{1}{2}(D_1-D_3)} f \cdot f'] \cdot [e^{\frac{1}{2}(D_1-D_3)} f\} \\ &= 2 \sinh \frac{1}{2}(D_1 + D_3) [e^{\frac{1}{2}(D_1-D_3)} f \cdot f'] \cdot [e^{-\frac{1}{2}(D_1-D_3)} f \cdot f']. \end{aligned} \tag{5}$$

Using eq(5) and noting that $D_1 + D_2 + D_3 + D_4 = 0$ we rewrite P as

$$\begin{aligned} P = &\, 2 z_1 z_3 \sinh \frac{1}{2}(D_1 + D_3) [e^{\frac{1}{2}(D_1-D_3)} f \cdot f'] \cdot [e^{-\frac{1}{2}(D_1-D_3)} f \cdot f'] \\ &-2 z_2 z_4 \sinh \frac{1}{2}(D_1 + D_3) [e^{\frac{1}{2}(D_2-D_4)} f \cdot f'] \cdot [e^{-\frac{1}{2}(D_2-D_4)} f \cdot f'] \\ &-2 z_1 z_4 \sinh \frac{1}{2}(D_2 + D_3) [e^{\frac{1}{2}(D_1-D_4)} f \cdot f'] \cdot [e^{-\frac{1}{2}(D_1-D_4)} f \cdot f'] \\ &+2 z_2 z_3 \sinh \frac{1}{2}(D_2 + D_3) [e^{\frac{1}{2}(D_2-D_3)} f \cdot f'] \cdot [e^{-\frac{1}{2}(D_2-D_3)} f \cdot f'] \\ = &\, 2 \sinh \frac{1}{2}(D_1 + D_3) \\ &\quad \{z_1 z_3 [e^{\frac{1}{2}(D_1-D_3)} f \cdot f'] \cdot [e^{-\frac{1}{2}(D_1-D_3)} f \cdot f'] \\ &\qquad - z_2 z_4 [e^{\frac{1}{2}(D_2-D_4)} f \cdot f'] \cdot [e^{-\frac{1}{2}(D_2-D_4)} f \cdot f']\} \\ &+ 2 \sinh \frac{1}{2}(D_2 + D_3) \\ &\quad \{z_2 z_3 [e^{\frac{1}{2}(D_2-D_3)} f \cdot f'] \cdot [e^{-\frac{1}{2}(D_2-D_3)} f \cdot f'] \\ &\qquad - z_1 z_4 [e^{\frac{1}{2}(D_1-D_4)} f \cdot f'] \cdot [e^{-\frac{1}{2}(D_1-D_4)} f \cdot f']\}. \end{aligned}$$

Here we conjecture that the following equations constitute the Bäcklund transformation of the discrete BKP equation

$$[a_1 \exp(\frac{1}{2}(D_1 - D_3)) + a_2 \exp(-\frac{1}{2}(D_1 - D_3))$$
$$+a_3 \exp(\frac{1}{2}(D_2 - D_4)) + a_4 \exp(-\frac{1}{2}(D_2 - D_4))]f \cdot f' = 0, \qquad (6)$$
$$[b_1 \exp(\frac{1}{2}(D_2 - D_3)) + b_2 \exp(-\frac{1}{2}(D_2 - D_3))$$
$$+b_3 \exp(\frac{1}{2}(D_1 - D_4)) + b_4 \exp(-\frac{1}{2}(D_1 - D_4))]f \cdot f' = 0, \qquad (7)$$

where new parameters $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ and b_4 are to be determined.
Using eq.(6) we have

$$\sinh \frac{1}{2}(D_1 + D_3)$$
$$\{[a_1 \exp(\frac{1}{2}(D_1 - D_3)) + a_4 \exp(-\frac{1}{2}(D_2 - D_4)]f \cdot f'\}$$
$$\cdot\{[a_2 \exp(-\frac{1}{2}(D_1 - D_3)) + a_3 \exp(\frac{1}{2}(D_2 - D_4)]f \cdot f'\} = 0$$

which gives a relation

$$\sinh \frac{1}{2}(D_1 + D_3)$$
$$\{a_1 a_2 [\exp(\frac{1}{2}(D_1 - D_3))f \cdot f'] \cdot [\exp(-\frac{1}{2}(D_1 - D_3))f \cdot f']$$
$$-a_3 a_4 [\exp(\frac{1}{2}(D_2 - D_4))f \cdot f'] \cdot [\exp(-\frac{1}{2}(D_2 - D_4))f \cdot f']\}$$
$$= \sinh \frac{1}{2}(D_1 + D_3)$$
$$\{a_1 a_3 [\exp(\frac{1}{2}(D_1 - D_3))f \cdot f'] \cdot [\exp(\frac{1}{2}(D_2 - D_4))f \cdot f']$$
$$-a_2 a_4 [\exp(-\frac{1}{2}(D_1 - D_3))f \cdot f'] \cdot [\exp(-\frac{1}{2}(D_2 - D_4))f \cdot f']\}. \qquad (8)$$

We chose the following relations between the parameters

$$\frac{z_1 z_3}{a_1 a_2} = \frac{z_2 z_4}{a_3 a_4} = k_1, \qquad (9)$$
$$\frac{z_2 z_3}{b_1 b_2} = \frac{z_1 z_4}{b_3 b_4} = k_2. \qquad (10)$$

Then we find

$$P = 2k_1 \sinh \frac{1}{2}(D_1 + D_3)$$
$$\{a_1 a_2 [e^{\frac{1}{2}(D_1 - D_3)} f \cdot f'] \cdot [e^{-\frac{1}{2}(D_1 - D_3)} f \cdot f'] - a_3 a_4 [e^{\frac{1}{2}(D_2 - D_4)} f \cdot f'] \cdot [e^{-\frac{1}{2}(D_2 - D_4)} f \cdot f']\}$$
$$+2k_2 \sinh \frac{1}{2}(D_2 + D_3)$$
$$\{b_1 b_2 [e^{\frac{1}{2}(D_2 - D_3)} f \cdot f'] \cdot [e^{-\frac{1}{2}(D_2 - D_3)} f \cdot f'] - b_3 b_4 [e^{\frac{1}{2}(D_1 - D_4)} f \cdot f'] \cdot [e^{-\frac{1}{2}(D_1 - D_4)} f \cdot f']\},$$

which is, by using eq.(8),rewritten as

$$P = 2k_1 \sinh\frac{1}{2}(D_1+D_3)$$
$$\{a_1a_3[e^{\frac{1}{2}(D_1-D_3)}f\cdot f']\cdot[e^{\frac{1}{2}(D_2-D_4)}f\cdot f'] - a_2a_4[e^{-\frac{1}{2}(D_1-D_3)}f\cdot f']\cdot[e^{-\frac{1}{2}(D_2-D_4)}f\cdot f']\}$$
$$+2k_2 \sinh\frac{1}{2}(D_2+D_3)$$
$$\{b_1b_3[e^{\frac{1}{2}(D_2-D_3)}f\cdot f']\cdot[e^{\frac{1}{2}(D_1-D_4)}f\cdot f'] - b_2b_4[e^{-\frac{1}{2}(D_2-D_3)}f\cdot f']\cdot[e^{-\frac{1}{2}(D_1-D_4)}f\cdot f']\}.$$

On the other hand we have the operator identy

$$e^{\frac{1}{2}(D_1+D_3)}[e^{\frac{1}{2}(D_1-D_3)}f\cdot f']\cdot[e^{\frac{1}{2}(D_2-D_4)}f\cdot f']$$
$$= e^{\frac{1}{2}(D_2+D_3)}[e^{\frac{1}{2}(D_2-D_3)}f\cdot f']\cdot[e^{\frac{1}{2}(D_1-D_4)}f\cdot f']. \tag{11}$$

Using the identiy(11) we obtain

$$P = 2\sinh\frac{1}{2}(D_1+D_3)$$
$$\{(k_1a_1a_3+k_2b_1b_3)[e^{\frac{1}{2}(D_1-D_3)}f\cdot f']\cdot[e^{\frac{1}{2}(D_2-D_4)}f\cdot f']$$
$$-(k_1a_2a_4+k_2b_2b_4)[e^{-\frac{1}{2}(D_1-D_3)}f\cdot f']\cdot[e^{-\frac{1}{2}(D_2-D_4)}f\cdot f']\},$$

which vanishes if

$$k_1a_1a_3 + k_2b_1b_3 = 0, \tag{12}$$
$$k_1a_2a_4 + k_2b_2b_4 = 0.$$

Substituting eqs.(9) and (10) into the above relations we find that they are satisfied if

$$z_1a_3b_2 + z_2a_2b_3 = 0. \tag{13}$$

Hence the Bäcklund transformation of the discrete BKP equation are given by

$$[a_1\exp(\frac{1}{2}(D_1-D_3)) + a_2\exp(-\frac{1}{2}(D_1-D_3))$$
$$+a_3\exp(\frac{1}{2}(D_2-D_4)) + a_4\exp(-\frac{1}{2}(D_2-D_4))]f\cdot f' = 0, \tag{14}$$
$$[b_1\exp(\frac{1}{2}(D_2-D_3)) + b_2\exp(-\frac{1}{2}(D_2-D_3))$$
$$+b_3\exp(\frac{1}{2}(D_1-D_4)) + b_4\exp(-\frac{1}{2}(D_1-D_4))]f\cdot f' = 0, \tag{15}$$

provided that parameters satisfy the relations

$$\begin{aligned} z_1z_3a_3a_4 - z_2z_4a_1a_2 &= 0, \\ z_2z_3b_3b_4 - z_1z_4b_1b_2 &= 0, \\ z_2a_2b_3 + z_1a_3b_2 &= 0. \end{aligned}$$

Finally we note that equations (14) and (15) exhibit soliton solutions provided that the parameters satisfy the conditions

$$\begin{aligned} a_1+a_2+a_3+a_4 &= 0, \\ b_1+b_2+b_3+b_4 &= 0. \end{aligned}$$

References

[1] Y.Ohta, R.Hirota, S.Tujimoto and T.Imai,"Casorati and Discrete Gram Type Determinant Representations of Solutions to the Discrete KP Hierarchy in Bilinear Form", J.Phy.Soc.Jpn. **62** (1993) 1872.

[2] T. Miwa, Proc.Jpn.Acad.**58A**(1982) 9.

[3] Satoshi Tsujimoto and Ryogo Hirota,"Pfaffian Representation of Solutions to the Discrete BKP Hierarchy in Bilinear Form", J.Phy.Soc.Jpn. **65** (1996) 2797.

[4] Masataka Iwao and Ryogo Hirota,"Soliton Solution of a Coupled Modified KdV Equation,J.Phy.Soc.Jpn. **66** (1997) 577.

[5] Masataka Iwao and Ryogo Hirota,"Soliton Solution of a Coupled Derivative Modified KdV Equation, to appear in J.Phy.Soc.Jpn.