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On a nature of a soliton cellular automaton

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abstract
We give a proof about a nature of "solitons" in a soliton cellular automaton by means of inverse ultra-discretization.

Almost a decade ago, Takahashi and one of the authors (J.S.) proposed a (filter type) cellular automaton (CA) [1]. The CA is 1 (space) +1 (time) dimensional and two valued (0 and 1). The state at time t is an infinite sequence composed of 0's and finite number of 1's. The rule to determine the state at $t+1$ is:
1. Move every 1 only once.
2. Exchange the leftmost 1 with its nearest right 0.
3. Exchange the leftmost 1 among the rest 1's with its nearest right 0.
4. Repeat this procedure until all 1's are moved.

A peculiar feature of the CA is that any state consists only of solitons, interacting in the same manner as KdV solitons (Fig.1).

$$
\begin{array}{l}
t=0 \cdots 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots \\
t=1 \cdots 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots \\
t=2 \cdots 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots \\
t=3 \cdots 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots \\
t=4 \cdots 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0 \cdots \\
t=5 \cdots 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0 \cdots
\end{array}
$$

Fig. 1 An example of time evolution of the CA. Three patterns of 1's (1111, 11 and 1) retain their forms with some phase shifts after collisions.

A block of 1's is regarded as a soliton in the CA. Its velocity is proportional to the number of 1's in it. If the initial state is composed of N solitons arranged in the decreasing order in their size, then, after their interactions, they will be arranged in the opposite order to the initial state.

Recently, a direct connection between the CA and the KdV equation was established by means of a limiting procedure called ultra-discretization [2]. The purpose of the present article is to give a proof of the feature of the CA mentioned above in terms of the inverse process of the ultra-discretization. Precisely speaking, we shall prove the following theorem:

Theorem 1 *Let Q_n^t $(n = 1, 2, ..., N)$ and E_n^t $(n = 0, 1, 2, ..., N)$ be respectively the length (the number of 1's) of the n-th soliton and the number of 0's between n-th soliton and n + 1-th soliton at time t with boundary conditions $E_0^t = +\infty$, $E_N^t = +\infty$. If the following conditions are satisfied at time $t = 0$,*

$$\begin{array}{ll} Q_n^0 \geq Q_{n+1}^0 & (1 \leq n \leq N-1) \\ E_n^0 \geq Q_{n+1}^0 & (1 \leq n \leq N-1), \end{array}$$

then there exists a time T such that $Q_n^t = Q_{N+1-n}^0$ at $t \geq T$.

As an example, in Fig. 1 $Q_1^0 = 4, Q_2^0 = 2, Q_3^0 = 1, E_1^0 = 5$ and $E_2^0 = 2$, which satisfy the assumption of the theorem, and for $t \geq 4$ we see $Q_1^t = 1 = Q_3^0, Q_2^t = 2 = Q_2^0, Q_3^t = Q_1^0$.

Prior to the proof of the theorem, we shall show a relation of the CA to the celebrated Kadomtsev-Petviashvili (KP) hierarchy of nonlinear partial differential equations [3, 4] in terms of ultra-discretization.

We put the evolution rule of the CA in another way. The value of jth cell at time t, $u_j^t (= 0 \mathrm{or} 1)$, is given as

$$u_j^{t+1} = \begin{cases} 1 & \text{if } u_j^t = 0 \text{ and } \sum_{i=-\infty}^{j-1} u_i^t > \sum_{i=-\infty}^{j-1} u_i^{t+1}, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $u_j^t = 0$ for $|j| \gg 1$ is satisfied due to the boundary conditons. This is equivalent to the equation:

$$u_j^{t;1} = \min\left[1 - u_j^t, \sum_{i=-\infty}^{j-1} u_i^t - \sum_{i=-\infty}^{j-1} u_i^{t+1}\right] \tag{2}$$

We introduce ρ_j^t as $\rho_j^t \equiv \sum_{i=-\infty}^{j} \sum_{s=t}^{+\infty} u_i^s$. Thus we have $u_j^t = \rho_j^t - \rho_j^{t+1} - \rho_{j-1}^t - \rho_{j-1}^{t+1}$. Then Eq. (2) is rewritten as

$$\rho_{j+1}^{t+1} + \rho_j^{t-1} = \max\left[\rho_{j+1}^t + \rho_j^t, \rho_{j+1}^{t-1} + \rho_j^{t+1} - 1\right]. \tag{3}$$

The generating function of the KP hierarchy is given [3, 4]:

$$\mathrm{Res}_{z=\infty} \tau\left(\mathbf{x} - \epsilon(\frac{1}{z})\right) \tau\left(\mathbf{x}' + \epsilon(\frac{1}{z})\right) \exp\left[\xi(\mathbf{x} - \mathbf{x}'; z)\right] = 0, \tag{4}$$

where $\mathbf{x} = (x_1, x_2, x_3, ...)$ and $\mathbf{x}' = (x'_1, x'_2, x'_3, ...)$ are arbitrary two sets of infinite number of independent variables, $\epsilon(\frac{1}{z}) \equiv \left(\frac{1}{z}, \frac{1}{2z^2}, \frac{1}{3z^3}, ...\right)$ and $\xi(\mathbf{x}, z) \equiv \sum_{i=1}^{\infty} x_i z^i$. If we replace $\mathbf{x}$ by $\ell\epsilon\left(\frac{1}{a}\right) + m\epsilon\left(\frac{1}{b}\right) + n\epsilon\left(\frac{1}{c}\right)$ and $\mathbf{x}'$ by $(\ell+1)\epsilon\left(\frac{1}{a}\right) + (m+1)\epsilon\left(\frac{1}{b}\right) + (n+1)\epsilon\left(\frac{1}{c}\right)$, we obtain the discrete KP equation (Hirota-Miwa equation):

$$(a-b)\tau^{n}_{\ell+1,m+1}\tau^{n+1}_{\ell,m} + (b-c)\tau^{\ell}_{m+1,n+1}\tau^{\ell+1}_{m,n} + (c-a)\tau^{m}_{\ell+1,n+1}\tau^{m+1}_{\ell,n} = 0, \tag{5}$$

with $\tau^n_{\ell,m} \equiv \tau\left(\ell\epsilon\left(\frac{1}{a}\right) + m\epsilon\left(\frac{1}{b}\right) + n\epsilon\left(\frac{1}{c}\right)\right)$. We set $a - b = 1$, $b - c = \delta$, $c - a = -1 - \delta$ and impose a reduction conditon $\tau^n_{\ell,m} = \tau^n_{\ell+1,m+1}$. Then, putting $\sigma^t_j \equiv \tau^j_{t,0}$, we obtain

$$(1+\delta)\sigma^{t+1}_j\sigma^{t-1}_{j-1} = \sigma^t_j\sigma^t_{j-1} + \delta\sigma^{t-1}_j\sigma^{t+1}_{j-1}. \tag{6}$$

Equation (3) presents a resemblance to Eq. (6). When we put $\delta = \exp\left[-\frac{1}{\varepsilon}\right]$, from Eq. (6), σ^t_j depends on ε, *i.e.* $\sigma^t_j = \sigma^t_j(\varepsilon)$. If the limit $\lim_{\varepsilon\to+0} \varepsilon \log \sigma^t_j(\varepsilon) \equiv \rho^t_j$ exists, we find ρ^t_j satisfies Eq. (3). Therefore, we find a relation between the CA and the KP hierarchy.

This limiting procedure is called ultra-discretization, through which we can construct CA's from usual continuous equations. It should be noticed that if we have one parameter (ε) family of the solutions to a continuous equation, then we can construct a solution to the corresponding CA as far as its limit exists.

Now we shall prove the theorem 1. The idea of the proof is to use the inverse process of the ultra-discretization. It is illustrated as:

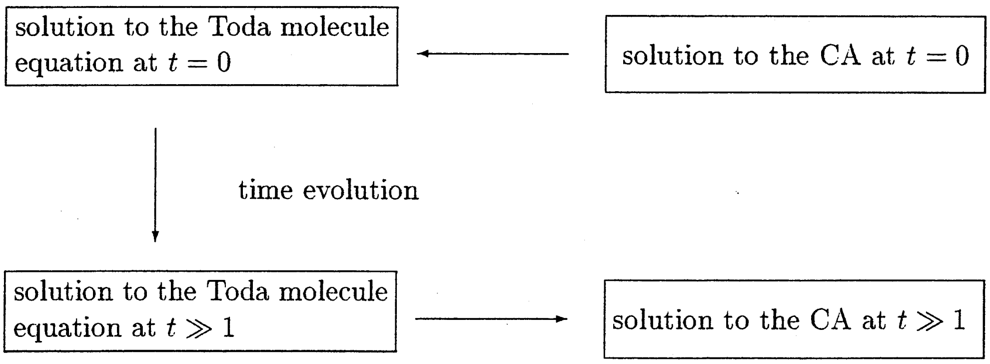

Fig.2 Schetch of the proof of theorem 1

We consider a system of N solitons. It is easy to see that the length of solitons Q_n^t and the distance E_n^t $(n = 1, 2, \cdots, N)$ satisfy

$$\begin{aligned} Q_n^{t+1} &= \min\left[\sum_{j=1}^{n} Q_j^t - \sum_{j=1}^{n-1} Q_j^{t+1},\ E_n^t\right] & \text{(7a)} \\ E_n^{t+1} &= Q_{n+1}^t + E_n^t - Q_n^{t+1}. & \text{(7b)} \end{aligned}$$

These equations are the ultra-discretization of the Toda molecule equation [5]:

$$\begin{aligned} I_n^{t+1} &= I_n^t + V_n^t - V_{n-1}^{t+1} & \text{(8a)} \\ V_n^{t+1} &= \frac{I_{n+1}^t V_n^t}{I_n^{t+1}}. & \text{(8b)} \end{aligned}$$

where $n = 1, 2, \cdots, N$, $V_0^t = V_N^t = 0$. In fact, taking into account of the boundary conditions, we obtain

$$I_n^{t+1} = \frac{\prod_{j=1}^{n} I_j^t}{\prod_{j=1}^{n-1} I_j^{t+1}} + V_n^t.$$

Thus, when we put

$$\begin{aligned} I_n^t = I_n^t(\varepsilon) &\equiv \exp[-\frac{1}{\varepsilon}\tilde{Q}_n^t(\varepsilon)], \qquad Q_n^{*t} = \lim_{\varepsilon\to+0} \tilde{Q}_n^t(\varepsilon) \\ V_n^t = V_n^t(\varepsilon) &\equiv \exp[-\frac{1}{\varepsilon}\tilde{E}_n^t(\varepsilon)], \qquad E_n^{*t} = \lim_{\varepsilon\to+0} \tilde{E}_n^t(\varepsilon), \end{aligned}$$

Q_n^{*t} and E_n^{*t} satisfies Eqs (8).

Next propositon is trivial, but fundamental.

Proposition 1 *Suppose that one parameter family of solutions $\tilde{Q}_n^t(\varepsilon)$ and $\tilde{E}_n^t(\varepsilon)$ satisfy Eq. (8) for $0 < \varepsilon < \exists C$, and that the limits Q_n^{*t} and E_n^{*t} exist. Then, if Q_n^{*0} and E_n^{*0} coincide with the initial values Q_n^0 and E_n^0 of Eqs. (7), Q_n^{*t} and E_n^{*t} coincide with Q_n^t and E_n^t for any $t \geq 0$.*

To prove the theorem 1, we need three Lemmas.

Lemma 1 *If τ_n^t satisfies*

$$\tau_n^{t+1}\tau_n^{t-1} = \left(\tau_n^t\right)^2 + \tau_{n+1}^{t-1}\tau_{n-1}^{t+1}, \quad \tau_{-1}^t = \tau_{N+1}^t = 0,$$

then

$$I_n^t = \frac{\tau_{n-1}^t \tau_n^{t+1}}{\tau_n^t \tau_{n-1}^{t+1}}, \quad V_n^t = \frac{\tau_{n+1}^t \tau_{n-1}^{t+1}}{\tau_n^t \tau_n^{t+1}},$$

satisfy Eqs. (8).

The proof of this lemma is simply done by substitution.

Lemma 2 $\forall c_j,\ \forall p_j \in \mathbf{C}\ \ (j=1,2,\cdots,N)$,

$$\begin{aligned}\tau_n^t &= \det(\mathbf{A}_n(t)\mathbf{B}_n)\\ &= \sum_{1\le i_1<i_2<\cdots<i_n\le N}\left(\prod_{1\le k<\ell\le n}(p_{i_k}-p_{i_\ell})^2\prod_{s=1}^{n} c_{i_s}p_{i_s}^t\right)\end{aligned}$$

gives a solution to the bilinear equation in Lem. 1. Here

$$\mathbf{A}_n(t)=\begin{pmatrix} c_1p_1^t & c_2p_2^t & \cdots & c_Np_N^t\\ c_1p_1^{t+1} & c_2p_2^{t+1} & \cdots & c_Np_N^{t+1}\\ \vdots & \vdots & \ddots & \vdots\\ c_1p_1^{t+n-1} & c_2p_2^{t+n-1} & \cdots & c_Np_N^{t+n-1}\end{pmatrix}\qquad \mathbf{B}_n=\begin{pmatrix}1 & p_1 & p_1^2 & \cdots & p_1^{n-1}\\ 1 & p_2 & p_2^2 & \cdots & p_2^{n-1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & p_N & p_N^2 & \cdots & p_N^{n-1}\end{pmatrix}$$

Proof. The Jacobi identity for an arbitrary $(n+1)\times(n+1)$ matrix $\mathbf{M}$ is given as

$$|\mathbf{M}||\mathbf{M}_{1,n+1}^{1,n+1}| = |\mathbf{M}_1^1||\mathbf{M}_{n+1}^{n+1}| - |\mathbf{M}_1^{n+1}||\mathbf{M}_{n+1}^1|,$$

where $\mathbf{M}_i^j$ denotes a minor of order n obtained from $\mathbf{M}$ by striking out raw i and column j, and $\mathbf{M}_{i_1,i_2}^{j_1,j_2}$ denotes a minor of order $n-1$ obtained by striking out raws i_1, i_2 and columns j_1, j_2. When we take $\mathbf{M}=(\mathbf{A}_{n+1}(t-1)\mathbf{B}_{n+1})$, the Jacobi identity turns into the bilinear equation in Lem. 1 with $\tau_{-1}^t=0$. Another boundary condition $\tau_{N+1}^t=0$ comes from the fact that the rank of $(\mathbf{A}_{N+1}(t-1)\mathbf{B}_{N+1})$ is N. Furthermore, since

$$(\mathbf{A}_n(t)\mathbf{B}_n)=\begin{pmatrix}\sum_{i=1}^N c_ip_i^t & \sum_{i=1}^N c_ip_i^{t+1} & \cdots & \sum_{i=1}^N c_ip_i^{t+n-1}\\ \sum_{i=1}^N c_ip_i^{t+1} & \sum_{i=1}^N c_ip_i^{t+2} & \cdots & \sum_{i=1}^N c_ip_i^{t+n}\\ \vdots & \vdots & \ddots & \vdots\\ \sum_{i=1}^N c_ip_i^{t+n-1} & \sum_{i=1}^N c_ip_i^{t+n} & \cdots & \sum_{i=1}^N c_ip_i^{t+2n-1}\end{pmatrix},$$

we have

$$\begin{aligned}\tau_n^t &= \sum_{i_1=1}^N\sum_{i_2=1}^N\cdots\sum_{i_n=1}^N p_{i_2}p_{i_3}^2\cdots p_{i_n}^{n-1}\begin{vmatrix} c_{i_1}p_{i_1}^t & c_{i_2}p_{i_2}^t & \cdots & c_{i_n}p_{i_n}^t\\ c_{i_1}p_{i_1}^{t+1} & c_{i_2}p_{i_2}^{t+1} & \cdots & c_{i_n}p_{i_n}^{t+1}\\ \vdots & \vdots & \ddots & \vdots\\ c_{i_1}p_{i_1}^{t+n-1} & c_{i_2}p_{i_2}^{t+n-1} & \cdots & c_{i_n}p_{i_n}^{t+n-1}\end{vmatrix}\\ &= \sum_{1\le i_1<i_2<\cdots<i_n\le N}\sum_{\sigma}\mathrm{sgn}(\sigma)p_{i_\sigma(2)}p_{i_\sigma(3)}^2\cdots p_{i_\sigma(n)}^{n-1}\left(\prod_{s=1}^n c_{i_s}p_{i_s}^t\right)\begin{vmatrix}1 & 1 & \cdots & 1\\ p_{i_1} & p_{i_2} & \cdots & p_{i_n}\\ \vdots & \vdots & \ddots & \vdots\\ p_{i_1}^{n-1} & p_{i_2}^{n-1} & \cdots & p_{i_n}^{n-1}\end{vmatrix}\end{aligned}$$

$$= \sum_{1\le i_1<i_2<\cdots<i_n\le N} \left(\prod_{s=1}^{n} c_{i_s} p_{i_s}^t\right) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ p_{i_1} & p_{i_2} & \cdots & p_{i_n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{i_1}^{n-1} & p_{i_2}^{n-1} & \cdots & p_{i_n}^{n-1} \end{vmatrix}^2$$

$$= \sum_{1\le i_1<i_2<\cdots<i_n\le N} \left(\prod_{1\le k<\ell\le n} (p_{i_k}-p_{i_\ell})^2 \prod_{s=1}^{n} c_{i_s} p_{i_s}^t\right).$$

∎

Lemma 3 *If nonnegative integers* P_j, γ_j $(j = 1, 2, \cdots, N,\ \gamma_1 = 0)$ *satisfy the inequalities:*

$$P_1 \ge P_2 \ge \cdots \ge P_N \ge 1$$
$$\gamma_{j+1} \ge \gamma_j + jP_j - (j-1)P_{j+1} \quad (j = 1, 2, \cdots, N),$$

then, for $\forall n$, $\{\ell_j\}_{j=1}^n \subseteq (1, 2, 3, \cdots, N)$, *we have*

$$\sum_{j=1}^{n} (jP_j + \gamma_j) \le \sum_{j=1}^{n} \left(jP_{\ell_j} + \gamma_{\ell_j}\right).$$

Proof. To prove the lemma, it is sufficient to show

$$kP_{k+\ell-1} + \gamma_{k+\ell-1} < kP_{k+\ell} + \gamma_{k+\ell} \qquad \cdots(\#)$$

for $\forall k, (1 \le k \le N-1)$ and for $\forall \ell, (1 \le \ell)$. However,

$$\begin{aligned} & kP_{k+\ell} + \gamma_{k+\ell} - (kP_{k+\ell-1} + \gamma_{k+\ell-1}) \\ & \ge k(P_{k+\ell} - P_{k+\ell-1}) + (k+\ell-1)P_{k+\ell-1} - (k+\ell-2)P_{k+\ell} \\ & = (\ell-1)(P_{k+\ell-1} - P_{k+\ell}) + P_{k+\ell} \\ & \ge P_{k+\ell} > 0. \end{aligned}$$

Thus the inequality (#) holds. ∎

Now we shall prove theorem 1.

Proof of Th. 1 For the initial state Q_n^0, E_n^0 $(n = 1, 2, ..., N-1)$, which satisfy the conditions in the statement of the theorem, we set $Q_n^{*0} = Q_n^0$, $E_n^{*0} = E_n^0$. We also define the nonnegative integers P_n, γ_n $(n = 1, 2, ..., N,\ \gamma_1 = 0)$ by

$$P_n = Q_n^{*0}, \quad n(P_{n+1} - P_n) + (\gamma_{n+1} - \gamma_n) = E_n^{*0}.$$

Then $P_n,\ \gamma_n$ satisfy the assumption in Lem. 3 and, hence, they satisfy the inequalities in Lem. 3.

Then we choose

$$c_j = \exp\left[-\frac{1}{\varepsilon}\gamma_j\right],\ p_j = \alpha_j \exp\left[-\frac{1}{\varepsilon}P_j\right],$$

where, $\alpha_{j+1} - 1 > \alpha_j \geq 1,\ \alpha_j \sim O(1)$. (As an example $\alpha_j = j$) Then

$$\exists \rho_n^t \equiv \lim_{\varepsilon \to +0} -\varepsilon \log \tau_n^t \tag{9a}$$

$$= \min_{1 \leq \ell_1 < \ell_2 < \cdots < \ell_n \leq N} \left(\sum_{j=1}^{n} (t+j-1) P_{\ell_j} + \gamma_{\ell_j} \right) \tag{9b}$$

From Lem. 3, we find

$$\begin{aligned} \rho_n^1 &= \min\left(\textstyle\sum_{j=1}^{n} jP_{\ell_j} + \gamma_{\ell_j}\right) &&= \sum_{j=1}^{n} jP_j + \gamma_j \\ \rho_n^0 &= \min\left(\textstyle\sum_{j=1}^{n} (j-1)P_{\ell_j} + \gamma_{\ell_j}\right) &&= \sum_{j=1}^{n} (j-1)P_j + \gamma_j. \end{aligned}$$

From Lem. 1, in the limit $\varepsilon \to +0$, we have

$$\begin{aligned} Q_n^{*t} &= \rho_{n-1}^t + \rho_n^{t+1} - \rho_n^t - \rho_{n-1}^{t+1}, \\ E_n^{*t} &= \rho_{n+1}^t + \rho_{n-1}^{t+1} - \rho_n^t - \rho_n^{t+1}. \end{aligned}$$

On the other hand, we find from (9) that there exists a time T such that for $t \geq T$

$$\rho_n^t = \sum_{j=1}^{n} (t+j-1) P_{N-n+j} + \gamma_{N-n+j}.$$

Substituting this expression into Q_n^{*t}, we have $Q_n^{*t} = P_{N-n+1} = Q_{N-n+1}^0 \quad \cdots (\%)$. Since $Q_n^{*0} = Q_n^0,\ E_n^{*0} = E_n^0$, from Prop. 1, these Q_n^{*t}, E_n^{*t} give the solution to the CA equation (8) with the given initial conditions. Thus (%) means $Q_n^t = Q_{N-n+1}^0$. This completes the proof. ∎

We have proved that the "solitons" in the CA behave exactly like KdV solitons. We utilized the inverse process of ultra-discretization for the proof. Since all the variables of a CA are discrete, it is suitable for numerical analysis. In fact, the soliton-like feature of the CA is easily found by numerical calculations. However, it is fairly difficult to prove such features of a CA for we can not directly apply analytical methods to it because of its discrete nature. We believe, as was demonstrated in this article, that ultra-discretization (or inverse ultra-discretization) offers an effective tool for the analysis of CA's.

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