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TORIC IDEALS AND ROOT SYSTEMS

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概要

根系 A_{n-1} の正根に付随するGKZ-超幾何方程式系の解空間の次元を計算するために，Gelfand–Graev–Postnikov は根系 A_{n-1} の正根に原点を加えた配置の正則単模三角形分割を具体的に構成し，その配置の正規化体積がカタラン数 $\binom{2n-2}{n-1}/n$ に一致することを示した ([1] 参照)．彼等の結果の本質はその配置のトーリックイデアルの2次 squarefree 単項式から成るイニシャルイデアルを発見した点にある．そのようなイニシャルイデアルがあればその配置の正規化体積，エルハート多項式などがイニシャルイデアルから導かれる有限グラフの独立集合の数え上げで計算できる．Gelfand–Graev–Postnikov の結果に刺激され，[5] では根系 B_n,C_n,D_n,BC_n の各々について，その根系のすべての正根に原点を加えた配置のトーリックイデアルの2次 squarefree 単項式から成るイニシャルイデアルの存在を証明した．

また，根系 A_{n-1} の幾つかの正根に原点を加えた配置は完全単模であるから常に squarefree 単項式から成るイニシャルイデアルを持つ (Stanley) という既知の結果を背景に，[6] では根系 BC_n の幾つかの正根に原点を加えた配置の正規性を議論した．他方，配置に原点を加えるという操作は GKZ-超幾何方程式系を考察する際には自然なことであるが，計算可換代数の観点からみると，根系 B_n, C_n,D_n,BC_n の各々について，その根系のすべての正根から成る配置 (原点は加えない) が squarefree 単項式から成るイニシャルイデアルを持つことが予想される．[6] では根系 A_{n-1} についてこれが成り立つことを示した．

研究集会「D加群のアルゴリズム」では，これら [5]，[6] の内容について講演したが，本稿の内容は，そのうちの [5] に基づいている．

Let $K[\mathbf{t},\mathbf{t}^{-1},s] = K[t_1,t_1^{-1},\ldots,t_n,t_n^{-1},s]$ denote the Laurent polynomial ring over a field K. Let $\mathbf{t}^{\mathbf{a}}s = t_1^{a_1}t_2^{a_2}\cdots t_n^{a_n}s \in K[\mathbf{t},\mathbf{t}^{-1},s]$ if $\mathbf{a} = (a_1,a_2,\ldots,a_n) \in \mathbb{Z}^n$. We associate given a finite set $\{\mathbf{a}_1,\mathbf{a}_2,\ldots,\mathbf{a}_N\} \subset \mathbb{Z}^n$ with the affine semigroup ring R $(\subset K[\mathbf{t},\mathbf{t}^{-1},s])$ generated by the monomials $\mathbf{t}^{\mathbf{a}_1}s,\mathbf{t}^{\mathbf{a}_2}s,\ldots,\mathbf{t}^{\mathbf{a}_N}s$. Let $A = K[x_1,x_2,\ldots,x_N]$ denote the polynomial ring over K and write I $(\subset A)$ for the kernel of the surjective homomorphism $\pi : A \to R$ defined by setting $\pi(x_i) = \mathbf{t}^{\mathbf{a}_i}s$ for all i. The ideal I, called the *toric ideal* of R, is generated by binomials. We are interested in the questions when the toric ideal of an affine semigroup ring is generated by quadratic binomials as well as when the toric ideal of an affine semigroup ring possesses a quadratic initial ideal. Consult, e.g., [3] and [4].

Let $\Phi \subset \mathbb{Z}^n$ be one of the root systems $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$ and $\mathbf{BC}_n$ ([2, pp. 64 – 65]) and write R_Φ for the affine semigroup ring associated with the finite set consisting of all positive roots of Φ together with the origin of $\mathbb{R}^n$. The purpose of the present paper is to show the existence of a squarefree quadratic initial ideal of the toric ideal I_Φ of R_Φ. In particular, the convex polytope which is the convex hull of the positive roots of Φ together with the origin of $\mathbb{R}^n$ possesses a regular unimodular triangulation and, in addition, the affine semigroup ring R_Φ is Koszul. We refer the reader to [1] for related results on the root system $\mathbf{A}_{n-1}$.

To begin with, we discuss the toric ideal of the root system $\mathbf{BC}_n$. The affine semigroup ring associated with (the finite set consisting of the origin of $\mathbb{R}^n$ together with the positive roots of) the root system $\mathbf{BC}_n$ is the subalgebra $R_{\mathbf{BC}_n}$ of $K[\mathbf{t}, \mathbf{t}^{-1}, s]$ generated by the monomial s together with the monomials $t_i t_j s$ with $1 \leq i \leq j \leq n$, $t_i t_j^{-1} s$ with $1 \leq i < j \leq n$, and $t_i s$ with $1 \leq i \leq n$. Let $A_{\mathbf{BC}_n}$ denote the polynomial rings

$$A_{\mathbf{BC}_n} = K[\{x\} \cup \{y_i\}_{1 \leq i \leq n} \cup \{e_{i,j}\}_{1 \leq i \leq j \leq n} \cup \{f_{i,j}\}_{1 \leq i < j \leq n}]$$

over K and write $\pi : A_{\mathbf{BC}_n} \to R_{\mathbf{BC}_n}$ for the surjective homomorphism defined by setting $\pi(x) = s$, $\pi(y_i) = t_i s$, $\pi(e_{i,j}) = t_i t_j s$ and $\pi(f_{i,j}) = t_i t_j^{-1} s$. Let $I_{\mathbf{BC}_n}$ denote the kernel of π and call $I_{\mathbf{BC}_n}$ the toric ideal of $R_{\mathbf{BC}_n}$.

We fix the reverse lexicographic monomial order $<_{rev}$ on the polynomial ring $A_{\mathbf{BC}_n}$ induced by the ordering of the variables

$$\begin{aligned} y_1 &< y_2 < \cdots < y_n < x < f_{1,n} < f_{1,n-1} < \cdots < f_{1,2} < f_{2,n} < \cdots < f_{n-1,n} \\ &< e_{1,n} < e_{1,n-1} < \cdots < e_{1,2} < e_{1,1} < e_{2,n} < \cdots < e_{n-1,n} < e_{n-1,n-1} < e_{n,n}. \end{aligned}$$

To simplify the notation below, we understand $e_{j,i} = e_{i,j}$ if $i < j$. First of all, the quadratic binomials

(1) $e_{i,j}e_{k,\ell} - e_{i,\ell}e_{j,k}$, $\quad i \leq j < k \leq \ell$;
(2) $e_{i,k}e_{j,\ell} - e_{i,\ell}e_{j,k}$, $\quad i < j \leq k < \ell$;
(3) $f_{i,k}f_{j,\ell} - f_{i,\ell}f_{j,k}$, $\quad i < j < k < \ell$;
(4) $f_{i,j}f_{j,k} - xf_{i,k}$, $\quad i < j < k$;
(5) $f_{j,k}e_{i,\ell} - f_{i,k}e_{j,\ell}$, $\quad i < j < k$;
(6) $f_{i,j}e_{j,k} - y_iy_k$, $\quad i < j$;
(7) $y_je_{i,k} - y_ie_{j,k}$, $\quad i < j$;
(8) $y_jf_{i,k} - y_if_{j,k}$, $\quad i < j < k$;
(9) $y_jf_{i,j} - y_ix$, $\quad i < j$;
(10) $xe_{i,j} - y_iy_j$, $\quad i \leq j$,

belong to $I_{\mathbf{BC}_n}$ and their initial monomials

(1') $e_{i,j}e_{k,\ell}$, $\quad i \leq j < k \leq \ell$;
(2') $e_{i,k}e_{j,\ell}$, $\quad i < j \leq k < \ell$;
(3') $f_{i,k}f_{j,\ell}$, $\quad i < j < k < \ell$;
(4') $f_{i,j}f_{j,k}$, $\quad i < j < k$;
(5') $f_{j,k}e_{i,\ell}$, $\quad i < j < k$;

(6') $f_{i,j}e_{j,k}$, $\quad i < j$;
(7') $y_j e_{i,k}$, $\quad i < j$;
(8') $y_j f_{i,k}$, $\quad i < j < k$;
(9') $y_j f_{i,j}$, $\quad i < j$;
(10') $x e_{i,j}$, $\quad i \leq j$,

belong to $in_{<_{rev}}(I_{\mathbf{BC}_n})$.

Theorem 1. *The initial ideal $in_{<_{rev}}(I_{\mathbf{BC}_n})$ of the toric ideal $I_{\mathbf{BC}_n}$ with respect to the reverse lexicographic monomial order $<_{rev}$ is generated by the quadratic monomials (1') – (10') listed above.*

Proof. Let $\mathcal{G}$ denote the set of standard monomials of $R_{\mathbf{BC}_n}$ with respect to the ideal generated by the quadratic monomials (1') – (10') listed above. Thus a monomial

$$u = s^{\alpha}(t_{k_1}s)\cdots(t_{k_r}s)(t_{a_1}t_{b_1}s)\cdots(t_{a_p}t_{b_p}s)(t_{i_1}t_{j_1}^{-1}s)\cdots(t_{i_q}t_{j_q}^{-1}s),$$

of $R_{\mathbf{BC}_n}$, where

$$y_{k_1} \leq_{rev} \cdots \leq_{rev} y_{k_r} \leq_{rev} f_{i_1,j_1} \leq_{rev} \cdots \leq_{rev} f_{i_q,j_q} \leq_{rev} e_{a_1,b_1} \leq_{rev} \cdots \leq_{rev} e_{a_p,b_p},$$

belongs to $\mathcal{G}$ if and only if the following conditions are satisfied:

(BC–1) $a_1 \leq a_2 \leq \cdots \leq a_p \leq b_p \leq \cdots \leq b_2 \leq b_1$;
(BC–2) If $\xi < \eta$ then either $i_\xi \leq i_\eta < j_\eta \leq j_\xi$ or $i_\xi < j_\xi < i_\eta < j_\eta$;
(BC–3) $i_q \leq a_1$;
(BC–4) $k_1 \leq \cdots \leq k_r \leq a_1$;
(BC–5) $i_\eta < k_\xi \leq j_\eta$ for no ξ and no η;
(BC–6) $\{k_1, \ldots, k_r, a_1, \ldots, a_p, b_1, \ldots, b_p\} \cap \{j_1, \ldots, j_q\} = \emptyset$;
(BC–7) If $\alpha \neq 0$, then $p = 0$.

To obtain the required result, what we must prove is that if the monomial u above and

$$u' = s^{\alpha'}(t_{k'_1}s)\cdots(t_{k'_{r'}}s)(t_{a'_1}t_{b'_1}s)\cdots(t_{a'_{p'}}t_{b'_{p'}}s)(t_{i'_1}t_{j'_1}^{-1}s)\cdots(t_{i'_{q'}}t_{j'_{q'}}^{-1}s)$$

belong to $\mathcal{G}$ and if $u = u'$ in $R_{\mathbf{BC}_n}$, then

$\alpha = \alpha', r = r', p = p', q = q'$,
$k_1 = k'_1, \ldots, k_r = k'_r$,
$a_1 = a'_1, \ldots, a_p = a'_p, b_1 = b'_1, \ldots, b_p = b'_p$,
$i_1 = i'_1, \ldots, i_q = i'_q, j_1 = j'_1, \ldots, j_q = j'_q$.

First, one has $q = q'$, $\alpha + r + p = \alpha' + r' + p'$ and $r + 2p = r' + 2p'$. Hence, if $\alpha = \alpha' = 0$, then $p = p'$ and $r = r'$. If $\alpha \geq \alpha' > 0$, then $p = p' = 0$ by (BC–7); thus $r = r'$ and $\alpha = \alpha'$. If $\alpha = 0$ and $\alpha' > 0$, then $r + p = \alpha' + r'$ and $r + 2p = r'$. Thus $\alpha' + p = 0$, a contradiction.

Second, in case $\alpha = \alpha' = 0$ and $q = q' > 0$, we prove $i_q = i'_q$ and $j_q = j'_q$. Let $i'_q < i_q$. Then $j'_q \leq j_q$ by (BC–2). Thus by (BC–5) there is no k'_ℓ with $i'_q < k'_\ell \leq j_q$ ($= j'_\eta$ for some η). Hence there is no k'_ℓ with $k'_\ell = i_q$. Note, in particular, that $i'_q = i_q$ if $p = p' = 0$. Thus either $a'_\xi = i_q$ or $b'_\xi = i_q$ for some ξ. Hence by (BC–2), (BC–3), (BC–4) and (BC–5) one has $k'_r \leq i'_\eta \leq i'_q \leq a'_1 \leq i_q < j_q = j'_\eta$. Since $i'_\mu \leq i'_q\,(< i_q)$ for all μ, the total number of variables t_δ with $\delta \geq i_q$ appearing in u' is at most $2p$. Since $i_q \leq a_1$, the total number of variables t_δ with $\delta \geq i_q$ appearing in u is at least $2p+1$. This contradicts $u = u'$ in $R_{\mathbf{BC}_n}$. Hence $i'_q = i_q$. Suppose $i_q = i'_q < j'_q < j_q$. If t_δ^{-1} appears in u, then either $\delta \geq j_q$ or $\delta < i_q$. Thus $t_{j'_q}^{-1}$ never appears in u', a contradiction. Hence $j'_q = j_q$. Thus one has $i_q = i'_q$ and $j_q = j'_q$, as desired. It follows by induction (on q) that $i_1 = i'_1, \ldots, i_q = i'_q, j_1 = j'_1, \ldots, j_q = j'_q$. If $\alpha = \alpha' = 0$ and $q = q' = 0$, then (BC–1), (BC–4) together with $p = p'$, $r = r'$ guarantee that $k_1 = k'_1, \ldots, k_r = k'_r$ and $a_1 = a'_1, \ldots, a_p = a'_p, b_1 = b'_1, \ldots, b_p = b'_p$.

Finally, when $\alpha = \alpha' > 0$, since $p = p' = 0$, in the discussion above we already know $i'_q = i_q$ and, in addition, $j'_q = j_q$. Moreover, if $\alpha = \alpha' > 0$, $p = p' = 0$ and $q = q' = 0$, then obviously $k_1 = k'_1, \ldots, k_r = k'_r$, as required. □

We now turn to the study of the toric ideal of the root system $\mathbf{B}_n$. With the same notation as in the discussion of $in_{<_{rev}}(I_{\mathbf{BC}_n})$, just note that none of $t_1^2 s, \ldots, t_n^2 s$ appears in $R_{\mathbf{B}_n}$ and that none of $e_{1,1}, \ldots, e_{n,n}$ appears in $A_{\mathbf{B}_n}$.

Theorem 2. *The initial ideal $in_{<_{rev}}(I_{\mathbf{B}_n})$ of the toric ideal $I_{\mathbf{B}_n}$ with respect to the reverse lexicographic monomial order $<_{rev}$ is generated by the quadratic monomials listed below:*

(1") $e_{i,j}e_{k,\ell}$, $i < j < k < \ell$;
(2") $e_{i,k}e_{j,\ell}$, $i < j < k < \ell$;
(3") $f_{i,k}f_{j,\ell}$, $i < j < k < \ell$;
(4") $f_{i,j}f_{j,k}$, $i < j < k$;
(5") $f_{j,k}e_{i,\ell}$, $i < j < k$, $i \neq \ell$, $j \neq \ell$;
(6") $f_{i,j}e_{j,k}$, $i < j$, $j \neq k$;
(7") $y_j e_{i,k}$, $i < j$, $i \neq k$, $j \neq k$;
(8") $y_j f_{i,k}$, $i < j < k$;
(9") $y_j f_{i,j}$, $i < j$;
(10") $x e_{i,j}$, $i < j$.

Proof. Since our work is to modify the proof of Theorem 1, only a brief sketch will be given below. With the same notation as in the proof of Theorem 1, a monomial u belongs to $\mathcal{G}$ if and only if the following conditions are satisfied:

(B–1) Either $a_1 \leq a_2 \leq \cdots \leq a_p < b_p \leq \cdots \leq b_2 \leq b_1$ or
$a_1 \leq a_2 \leq \cdots \leq a_{p_1} < b_{p_1} = \cdots = b_2 = b_1$
$= a_{p_1+1} = a_{p_1+2} = \cdots = a_p < b_p \leq b_{p-1} \leq \cdots \leq b_{p_1+1}$;

(B–2) If $\xi < \eta$ then either $i_\xi \leq i_\eta < j_\eta \leq j_\xi$ or $i_\xi < j_\xi < i_\eta < j_\eta$;

(B–3) Either $i_q \leq a_1$ or
$i_{q_1} \leq a_1 \leq a_2 \leq \cdots \leq a_{p_1} < i_{q_1+1} = i_{q_1+2} = \cdots = i_q = b_{p_1} = \cdots = b_2 = b_1$
$= a_{p_1+1} = a_{p_1+2} = \cdots = a_p < b_p \leq b_{p-1} \leq \cdots \leq b_{p_1+1}$;

(B–4) Either $k_1 \leq \cdots \leq k_r \leq a_1$ or
$k_1 \leq \cdots \leq k_{r_1} \leq a_1 \leq a_2 \leq \cdots \leq a_{p_1} < k_{r_1+1} = k_{r_1+2} = \cdots = k_r$

$= b_{p_1} = \cdots = b_2 = b_1 = a_{p_1+1} = a_{p_1+2} = \cdots = a_p < b_p \leq b_{p-1} \leq \cdots \leq b_{p_1+1}$;
(B–5) $i_\eta < k_\xi \leq j_\eta$ for no ξ and no η;
(B–6) $\{k_1, \ldots, k_r, a_1, \ldots, a_p, b_1, \ldots, b_p\} \cap \{j_1, \ldots, j_q\} = \emptyset$;
(B–7) If $\alpha \neq 0$, then $p = 0$.

Now, suppose that u and u' belong to $\mathcal{G}$ with $u = u'$ in $R_{\mathbf{B}_n}$. Then one has $\alpha = \alpha', r = r', p = p'$ and $q = q'$. In case $\alpha = \alpha' = 0$ and $q = q' > 0$, we prove $i_q = i'_q$ and $j_q = j'_q$. Let $i'_q < i_q$. Then $i'_\eta \leq i'_q < i_q < j_q = j'_\eta$. Hence there is no k'_μ with $i_q \leq k'_\mu < j_q$. Thus $a'_1 \leq i_q$. First, if $a_1 < i_q$, then by (B–3) for each ξ either $a_\xi = i_q$ or $b_\xi = i_q$. Thus the total number of the variable t_{i_q} appearing in u is at least $p+1$; while the total number of variable t_{i_q} appearing in u' is at most p since $k_\mu = i_q$ for no μ. Second, let $i_q \leq a_1$. If $k'_r < i_q$, then the total number of variables t_ξ with $\xi \geq i_q$ appearing in u (resp. u') is at least $2p+1$ (resp. at most $2p$). Let $(i'_\eta <) i_q \leq k'_r$. Then $(j'_\eta =) j_q < k'_r$. In addition, if $k'_\mu < k'_r$, then $k'_\mu \leq i'_\eta$ since $k'_\mu \leq a'_1 < j'_\eta$. Hence the total number of variables t_ξ with $k'_r \neq \xi \geq i_q$ appearing in u' is at most p. Since either $i_q \leq a_\eta \neq k'_r$ or $i_q \leq b_\eta \neq k'_r$ for each η, the total number of variables t_ξ with $k'_r \neq \xi \geq i_q$ appearing in u is at least $p+1$. This complete the proof of $i_q = i'_q$. Hence $j_q = j'_q$ by the same reason as in the case of $\mathbf{BC}_n$. Let $\alpha = \alpha' = 0$ and $q = q' = 0$. If $k_1 \leq a_1$ and $k'_1 \leq a'_1$, then $k_1 = k'_1$. If $a_1 < k_1$, then by (B–4) the total number of the variable t_{k_1} appearing in u is $r + p$. Hence $k'_1 = k_1$. Let $\alpha = \alpha' = 0$, $r = r' = 0$ and $q = q' = 0$. If t_ξ^p divides u for no ξ, then $a_1 \leq \cdots \leq a_p < b_p \leq \cdots \leq b_1$. If, for some ℓ, t_ℓ^p divides u, then either $a_\xi = \ell < b_\xi$ or $a_\xi < \ell = b_\xi$ for each ξ. Hence $a_\eta = a'_\eta$ and $b_\eta = b'_\eta$ for all η. The final step of the proof is completely analogous to that of the proof given for $in_{<_{rev}}(I_{\mathbf{BC}_n})$. $\square$

The study of the initial ideal $in_{<_{rev}}(I_{\mathbf{C}_n})$ (resp. $in_{<_{rev}}(I_{\mathbf{D}_n})$) of the root system $\mathbf{C}_n$ (resp. $\mathbf{D}_n$) is much easier than that of $\mathbf{BC}_n$ (resp. $\mathbf{B}_n$); only ignoring the variables $y_1, y_2, \ldots, y_n$ in the polynomial ring $A_{\mathbf{BC}_n}$ (resp. $A_{\mathbf{B}_n}$) and ignoring $t_1 s, t_2 s, \ldots, t_n s$ in the affine semigroup ring $R_{\mathbf{BC}_n}$ (resp. $R_{\mathbf{B}_n}$).

Theorem 3. *The initial ideal $in_{<_{rev}}(I_{\mathbf{C}_n})$ of the toric ideal $I_{\mathbf{C}_n}$ with respect to the reverse lexicographic monomial order $<_{rev}$ is generated by the quadratic monomials* (1') – (6') *listed above.*

Theorem 4. *The initial ideal $in_{<_{rev}}(I_{\mathbf{D}_n})$ of the toric ideal $I_{\mathbf{D}_n}$ with respect to the reverse lexicographic monomial order $<_{rev}$ is generated by the quadratic monomials* (1") – (6") *listed above.*

We conclude the present paper with a remark that the role of the origin of $\mathbb{R}^n$, i.e., the variable x of the polynomial ring is essential in our discussions. In fact, the toric ideal of the affine semigroup ring associated with the set of positive roots of each of the root systems $\mathbf{A}_{n-1}$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$ and $\mathbf{BC}_n$ with $n \geq 6$ is *not* generated by quadratic binomials.

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