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Gröbner Bases of Acyclic Tournament Graphs and Hypergeometric Systems on the Group of Unipotent Matrices

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Abstract

Gelfand, Graev and Postnikov have shown that the number of independent solutions of hypergeometric systems on the group of unipotent matrices is equal to the Catalan number. These hypergeometric systems are related with the vertex-edge incidence matrices of acyclic tournament graphs. In this paper, we show that their results can be obtained by analyzing the Gröbner bases for toric ideals of acyclic tournament graphs. Moreover, we study an open problem given by Gelfand, Graev and Postnikov.

1 Introduction

In recent years, Gröbner bases for toric ideals of graphs have been studied and applied to many combinatorial problems such as triangulations of point configurations, optimization problems in graph theory, enumerations of contingency tables, and so on [3, 4, 11, 13, 14, 15]. On the other hand, Gröbner bases techniques can be applied to partial differential equations via left ideals in the Weyl algebra [12, 16]. Especially, in [16] Gröbner bases techniques have been applied to *GKZ-hypergeometric systems* (or *A-hypergeometric equations*).

Gelfand, Graev and Postnikov [5] studied GKZ-hypergeometric systems called *hypergeometric systems on the group of unipotent matrices.* The hypergeometric systems on the group of unipotent matrices are hypergeometric systems associated with the set of all positive roots A_{n-1}^{+} of the root system A_{n-1}. They showed that the hypergeometric system on the group of unipotent matrices gives a holonomic D-module and the number of linearly independent solutions of the system in a neighborhood of a generic point is equal to the normalized volume of the convex hull of the point set A_{n-1}^{+} and the origin. They also showed that the normalized volume is equal to the Catalan number C_n by giving the triangulations of the convex hull. Since these triangulations are *unimodular* (i.e. the normalized volume of each maximal simplex equals 1), the normalized volume of the convex hull can be calculated by enumerating all maximal simplices.

On the other hand, A_{n-1}^{+} can be viewed as the vertex-edge incidence matrix of acyclic tournament graph with n vertices. The toric ideals of acyclic tournament graphs have been studied in [11]. Sturmfels [17, Chapter 8] showed that for each Gröbner basis for toric ideal of a matrix A, we can define the *regular triangulation* of A. Sturmfels [17, Corollary 8.9.] also showed that for *homogeneous* toric ideals, a regular triangulation becomes unimodular if and only if the initial ideal of the toric ideal is square-free.

In this paper, we apply the result in [11] to the hypergeometric systems on the group of unipotent matrices. The toric ideals of acyclic tournament graphs are not homogeneous for standard positive grading, though those of undirected graphs [3, 4, 13] are all homogeneous.

We can homogenize the toric ideals of acyclic tournament graphs by (i) changing the positive grading or (ii) introducing an extra variable [2, Chapter 8]. We homogenize the toric ideals in [11] by (ii), and apply Sturmfels' result to triangulations of the convex hull of the point set A^+_{n-1} and the origin. For some Gröbner bases, they remain square-free when they are homogenized, so that we can construct unimodular triangulations. These triangulations are same as those constructed by Gelfand, Graev and Postnikov. Thus we give an alternative proof for the result by Gelfand, Graev and Postnikov from the view point of Gröbner bases of non-homogeneous toric ideals. Next we consider the triangulations of $A^+_{n-1} \cup \{\mathbf{0}\}$ ($\mathbf{0}$ is the origin in $\mathbb{R}^n$) which all of maximal simplices contain the origin. Such triangulations are called *local triangulations* [5]. To find all regular local triangulations is an open problem. We enumerate all regular local triangulations for small n using TiGERS [9].

2 Preliminaries

In this section, we give basic definitions of Gröbner bases, toric ideals and regular triangulations. We refer to [2] for introductions of Gröbner bases, and [17] for introductions of toric ideals and regular triangulations.

2.1 Gröbner Bases

Let k be a field and $k[x_1, \ldots, x_n]$ be a polynomial ring. For a non-negative integer vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ ($\mathbb{Z}_{\geq 0}$ means the set of all non-negative integers), we denote $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$.

Definition 2.1 *Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal and $\prec$ be a term order. A finite subset $\mathcal{G} = \{g_1, \ldots, g_s\} \subseteq I$ is a* Gröbner basis *for I with respect to $\prec$ if the* initial ideal $in_\prec(I) := \langle in_\prec(f) \colon f \in I \rangle$ *is generated by $in_\prec(g_1), \ldots, in_\prec(g_s)$. In addition, Gröbner basis $\mathcal{G}$ is* reduced *if $\mathcal{G}$ satisfies the following:*

1. *For any i, the coefficient of $in_\prec(g_i)$ equals 1.*
2. *For any i, any term of g_i is not divisible by $in_\prec(g_j)$ $(i \neq j)$.*

Proposition 2.2 *For an ideal and a term order, the reduced Gröbner basis is defined uniquely.*

Definition 2.3 *Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal. Then the union of all reduced Gröbner bases for I with respect to all term orders is a Gröbner basis for I with respect to any term order. This basis is called the* universal Gröbner basis *for I.*

Although there are infinite term orders, the elements of a universal Gröbner basis are finite.

2.2 Toric Ideals

Fix a subset $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_d\} \subset \mathbb{Z}^n$. Each vector $\mathbf{a}_i$ is identified with a monomial $\mathbf{t}^{\mathbf{a}_i}$ in the Laurent polynomial ring $k[\mathbf{t}^{\pm 1}] := k[t_1, \ldots, t_d, t_1^{-1}, \ldots, t_d^{-1}]$.

Definition 2.4 *Consider the homomorphism*

$$\pi \colon k[x_1, \ldots, x_n] \longrightarrow k[\mathbf{t}^{\pm 1}], \quad x_i \longmapsto \mathbf{t}^{\mathbf{a}_i}.$$

The kernel of π is denoted I_A and called the toric ideal *of A.*

Every vector $\mathbf{u} \in \mathbb{Z}^n$ can be written uniquely as $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$ where $\mathbf{u}^+$ and $\mathbf{u}^-$ are non-negative and have disjoint support.

Lemma 2.5

$$I_A = \langle \mathbf{x}^{\mathbf{u}_i^+} - \mathbf{x}^{\mathbf{u}_i^-} : \mathbf{u}_i \in \mathrm{Ker}(A) \cap \mathbb{Z}^n,\ i = 1, \ldots, s \rangle$$

Furthermore, a toric ideal is generated by finite binomials.

Definition 2.6 *A binomial* $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in I_A$ *is called* circuit *if the support of* $\mathbf{u}$ *is minimal with respect to inclusion in* $\mathrm{Ker}(A)$ *and the coordinates of* $\mathbf{u}$ *are relatively prime. We denote the set of all circuits in* I_A *by* $\mathcal{C}_A$.

Definition 2.7 *A binomial* $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in I_A$ *is called* primitive *if there exists no other binomial* $\mathbf{x}^{\mathbf{v}^+} - \mathbf{x}^{\mathbf{v}^-} \in I_A$ *such that both* $\mathbf{u}^+ - \mathbf{v}^+$ *and* $\mathbf{u}^- - \mathbf{v}^-$ *are non-negative. The set of all primitive binomials in* I_A *is called the* Graver basis *of* A *and written as* Gr_A.

Let $\mathcal{U}_A$ be the universal Gröbner basis of I_A.

Proposition 2.8 ([17, Proposition 4.11.]) *For any matrix* A,

$$\mathcal{C}_A \subseteq \mathcal{U}_A \subseteq Gr_A.$$

2.3 Regular Triangulations

Assume that a subset $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_d\} \subset \mathbb{Z}^n$ is the set of d points in $\mathbb{R}^n$. Let $\mathrm{conv}(A)$ be the convex hull of A.

Definition 2.9 *Let* $q = \dim \mathrm{conv}(A)$. $T = \{T_1, \ldots, T_p\}$ *is a* triangulation *of* $\mathrm{conv}(A)$ *if*

1. $T_i \subseteq A,\ |T_i| = q + 1,\ \dim \mathrm{conv}(T_i) = q.$
2. $\cup_{i=1}^{p} \mathrm{conv}(T_i) = \mathrm{conv}(A).$
3. $\mathrm{conv}(T_i) \cap \mathrm{conv}(T_j) = \mathrm{conv}(T_i \cap T_j) \quad (i \neq j).$

Sturmfels [17, Chapter 8] showed that for each point set A and generic term order $\prec$, we can define a triangulation of A with respect to $\prec$.

Definition 2.10 *Let* $\prec$ *be a generic term order and* $\sqrt{in_\prec(I_A)}$ *a radical ideal of the initial ideal* $in_\prec(I_A)$. *Then we can define the triangulation* $\Delta_\prec(I_A)$ *as follows:*

$$\Delta_\prec(I_A) := \left\{ \mathrm{conv}(F) \colon F \subseteq A,\ \prod_{i \colon \mathbf{a}_i \in F} x_i \notin \sqrt{in_\prec(I_A)} \right\}.$$

We call $\Delta_\prec(I_A)$ *the* regular triangulation *of* A *with respect* $\prec$.

Definition 2.11 *Let* $q = \dim \mathrm{conv}(A)$. *Then we define the* normalized volume *of* $\mathrm{conv}(A)$ *as* $q!$ *times the Euclidean volume of* $\mathrm{conv}(A)$.

Hilbert polynomial $H_A(r)$ of $k[x_1, \ldots, x_n]/I_A$ is the k-dimension of the r-th grade component of $k[x_1, \ldots, x_n]/I_A$ for $r \gg 0$.

Theorem 2.12 ([17, Theorem 4.16.]) *Let $q = \dim \mathrm{conv}(A)$. Then $q!$ times the leading coefficient of the Hilbert polynomial $H_A(r)$ of $k[x_1, \ldots, x_n]/I_A$ is equal to the normalized volume of* $\mathrm{conv}(A)$.

Definition 2.13 *The triangulation $T = \{T_1, \ldots, T_p\}$ of* $\mathrm{conv}(A)$ *is* unimodular *if for any $T_i \in T$, the normalized volume of T_i equals 1. The matrix A is also called* unimodular *if all triangulations of* $\mathrm{conv}(A)$ *are unimodular.*

The unimodularity of a matrix induces good properties as follows.

Proposition 2.14 ([17, Corollary 8.9.]) *Suppose that I_A be a homogeneous toric ideal. Then the initial ideal $in_\prec(I_A)$ is square-free (i.e. $\sqrt{in_\prec(I_A)} = in_\prec(I_A)$) if and only if the corresponding regular triangulation $\Delta_\prec(I_A)$ of* $\mathrm{conv}(A)$ *is unimodular.*

Proposition 2.15 ([17, Proposition 8.11.]) *If A is a unimodular matrix, then $\mathcal{C}_A = \mathcal{U}_A = Gr_A$.*

3 Result of Gelfand, Graev and Postnikov

Gelfand, Graev and Postnikov [5] studied hypergeometric systems on the group of unipotent matrices. They showed that these systems give holonomic D-modules, and calculated the number of independent solutions. In this section, we summarize their results.

3.1 Hypergeometric Systems on the Group of Unipotent Matrices

Let $\mathbf{e}_i$ be the i-th standard basis in $\mathbb{R}^n$. We denote

$$A_{n-1}^+ := \{\mathbf{a}_{ij} := \mathbf{e}_i - \mathbf{e}_j : 1 \leq i < j \leq n\} \subset \mathbb{R}^n$$

and $\widetilde{A}_{n-1}^+ = A_{n-1}^+ \cup \{\mathbf{0}\} \subset \mathbb{R}^n$ where $\mathbf{0}$ is the origin. A_{n-1}^+ is the set of all positive roots of the root system $A_{n-1} := \{\mathbf{e}_i - \mathbf{e}_j : 1 \leq i, j \leq n,\ i \neq j\} \subset \mathbb{R}^n$. Let $\mathrm{conv}(A_{n-1}^+)$ be the convex hull of all points in A_{n-1}^+ and $\mathrm{conv}(\widetilde{A}_{n-1}^+)$ the convex hull of all points in $\widetilde{A}_{n-1}^+$.

Definition 3.1 *The* hypergeometric system on the group of unipotent matrices *is the following system of differential equation with coordinates z_{ij}, $1 \leq i < j \leq n$:*

$$-\sum_{i=1}^{j-1} z_{ij} \frac{\partial f}{\partial z_{ij}} + \sum_{k=j+1}^{n} z_{jk} \frac{\partial f}{\partial z_{jk}} = \alpha_j f, \qquad j = 1, 2, \ldots, n \tag{1}$$

$$\frac{\partial f}{\partial z_{ik}} = \frac{\partial^2 f}{\partial z_{ij} \partial z_{jk}}, \qquad 0 \leq i < j < k \leq n \tag{2}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ such that $\sum_{j=1}^n \alpha_j = 0$.

Theorem 3.2 ([5, Theorem 2.3.])

(i) *The hypergeometric system (1), (2) gives a holonomic D-module. The number of linearly independent solutions of this system in a neighborhood of a generic point is equal to the normalized volume of* $\mathrm{conv}(\widetilde{A}_{n-1}^+)$.

(ii) *The normalized volume of* $\mathrm{conv}(\widetilde{A}^+_{n-1})$ *is equal to the Catalan number*

$$C_{n-1} = \frac{1}{n}\binom{2(n-1)}{n-1}.$$

Definition 3.3 *Let* T *be a tree on the set* $\{1, 2, \ldots, n\}$.

- T *is* admissible *if there are no* $1 \leq i < j < k \leq n$ *such that both* (i, j) *and* (j, k) *are edges of* T.
- T *is* standard *if* T *is admissible and there are no* $1 \leq i < j < k < l \leq n$ *such that both* (i, k) *and* (j, l) *are edges of* T.
- T *is* anti-standard *if* T *is admissible and there are no* $1 \leq i < j < k < l \leq n$ *such that both* (i, l) *and* (j, k) *are edges of* T.

Theorem 3.4 ([5, Theorem 6.3. and Theorem 6.6.]) *Let*

$$T_{ST} = \left\{ \mathrm{conv}\left(\bigcup_{(i,j)\in ST} \mathbf{a}_{ij} \cup \{\mathbf{0}\} \right) : ST \text{ is standard tree on } \{1, \ldots, n\} \right\}$$

$$T_{AT} = \left\{ \mathrm{conv}\left(\bigcup_{(i,j)\in AT} \mathbf{a}_{ij} \cup \{\mathbf{0}\} \right) : AT \text{ is anti-standard tree on } \{1, \ldots, n\} \right\}.$$

Both T_{ST} *and* T_{AT} *give local unimodular triangulations of* $\widetilde{A}^+_{n-1}$.

Theorem 3.5 ([5, Theorem 6.4. and Corollary 6.7.]) *The number of standard trees (resp. anti-standard trees) on the set* $\{1, 2, \ldots, n\}$ *is equal to the Catalan number*

$$C_{n-1} = \frac{1}{n}\binom{2(n-1)}{n-1}.$$

Theorem 3.2 follows from Theorem 3.4 and Theorem 3.5.

4 Gröbner Bases for Acyclic Tournament Graphs

In this section, we summarize our results in [11].

Let D_n be an acyclic tournament graph with n vertices which have labels $1, 2, \ldots, n$ such that each edge (i, j) $(i < j)$ is directed from i to j. Let $m = \binom{n}{2}$ be the number of edges in D_n. We associate each edge (i, j) with a variable x_{ij} in the polynomial ring $k[\mathbf{x}] := k[x_{ij} : 1 \leq i < j \leq n]$. Since we can consider A^+_{n-1} as the vertex-edge incidence matrix of D_n, we can also associate each edge (i, j) with a point $\mathbf{a}_{ij} \in A^+_{n-1}$. In this section, we analyze the toric ideal $I_{A^+_{n-1}}$.

4.1 Toric Ideals of A_{n-1}^+

A *walk* of D_n is the sequence of vertices $(v_1, v_2, \dots, v_p)$ such that (v_i, v_{i+1}) or (v_{i+1}, v_i) is an arc of D_n for each $1 \leq i < n$. A *cycle* is a walk $(v_1, v_2, \dots, v_p, v_1)$. A *circuit* is a cycle $(v_1, v_2, \dots, v_p, v_1)$ such that $v_i \neq v_j$ for any $i \neq j$.

Definition 4.1 *Let C be a circuit of D_n. If we fix a direction of C, we can partition the edges of C into two sets C^+ and C^- such that C^+ is the set of forward edges and C^- is the set of backward edges. Then the vector $\mathbf{c} = (c_{ij})_{1 \leq i < j \leq n} \in \mathbb{R}^m$ defined by*

$$c_{ij} = \begin{cases} 1 & \text{if } (i,j) \in C^+ \\ -1 & \text{if } (i,j) \in C^- \\ 0 & \text{if } (i,j) \notin C \end{cases}$$

is called the incidence vector *of C.*

Lemma 4.2 ([1]) *A binomial $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in I_{A_{n-1}^+}$ is a circuit if and only if $\mathbf{u}$ is the incidence vector of a circuit of D_n.*

By Proposition 2.15, $\mathcal{C}_{A_{n-1}^+} = \mathcal{U}_{A_{n-1}^+} = Gr_{A_{n-1}^+}$ since the incidence matrix A_{n-1}^+ is unimodular.

Corollary 4.3 *The universal Gröbner basis $\mathcal{U}_{A_{n-1}^+}$ is the set of binomials which correspond to all of the circuits of D_n.*

Corollary 4.4 *The number of elements in $\mathcal{U}_{A_{n-1}^+}$ is of exponential order with respect to n.*

Since $x_{12}x_{23} - x_{13} \in I_{A_{n-1}^+}$, $I_{A_{n-1}^+}$ is not homogeneous for the standard grading $\deg(x_{ij}) = 1$ $(^\forall i, j)$.

Corollary 4.5 *$I_{A_{n-1}^+}$ is not unimodular for the grading $\deg(x_{ij}) = 1$ $(^\forall i, j)$.*

But we can change the positive grading such that $I_{A_{n-1}^+}$ is homogeneous.

Theorem 4.6 *If we set a positive grading as*

$$\deg(x_{ij}) = j - i, \quad 1 \leq i < j \leq n, \tag{3}$$

then $I_{A_{n-1}^+}$ is a homogeneous ideal.

(**Proof**) It suffices to show that any elements in the universal Gröbner basis $\mathcal{U}_{A_{n-1}^+}$ are homogeneous with respect to the positive grading (3).

Let $C = i_1, i_2, \dots, i_s, i_1$ be a circuit in D_n. Let $C^+ := \{k\colon i_k < i_{k+1}\}$ and $C^- := \{k\colon i_k > i_{k+1}\}$ (we set $i_{s+1} = i_1$). The binomial f_C corresponding to C is

$$f_C = \prod_{k \in C^+} x_{i_k i_{k+1}} - \prod_{k \in C^-} x_{i_{k+1} i_k}.$$

Then, since $C^+ \cap C^- = \emptyset$,

$$\begin{aligned} \deg\left(\prod_{k \in C^+} x_{i_k i_{k+1}}\right) - \deg\left(\prod_{k \in C^-} x_{i_{k+1} i_k}\right) &= \sum_{k \in C^+} (i_{k+1} - i_k) - \sum_{k \in C^-} (i_k - i_{k+1}) \\ &= \sum_{k=1}^{s} (i_{k+1} - i_k) \\ &= 0 \end{aligned}$$

Thus f_C is homogeneous. ∎

4.2 Some Reduced Gröbner Bases of $I_{A_{n-1}^+}$

In this section, we show that the elements in reduced Gröbner bases with respect to some specific term orders can be given in terms of graphs. As a corollary, we can show that there exist term orders for which reduced Gröbner bases remain in polynomial order.

Remark 4.7 *In this section, we line under the initial term of each polynomial.*

Theorem 4.8 *There exists a term order on $k[\mathbf{x}]$ for which the reduced Gröbner basis for $I_{A_{n-1}^+}$ is*

$$\{\underline{x_{ij}x_{jk}} - x_{ik} \colon 1 \leq i < j < k \leq n\} \cup \{\underline{x_{ik}x_{jl}} - x_{il}x_{jk} \colon 1 \leq i < j < k < l \leq n\}. \tag{4}$$

Let $g_{ijk} := x_{ij}x_{jk} - x_{ik}$ and $g_{ijkl} := x_{ik}x_{jl} - x_{il}x_{jk}$. Then the set $\{g_{ijk} \colon 1 \leq i < j < k \leq n\}$ corresponds to all of the circuits of length three in D_n, and $\{g_{ijkl} \colon 1 \leq i < j < k < l \leq n\}$ corresponds to some of the circuits of length four (Figure 1).

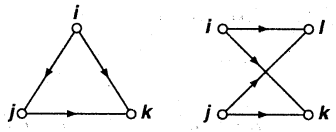

Figure 1: The circuit corresponding to g_{ijk} and the circuit corresponding to g_{ijkl}.

(**Proof**) Let $\prec$ be a purely lexicographic term order induced by the following variable ordering:

$$x_{ij} \succ x_{kl} \Longleftrightarrow i < k \text{ or } (i = k \text{ and } j < l).$$

We show that (4) is the reduced Gröbner basis for $I_{A_{n-1}^+}$ with respect to $\prec$.

For any circuit of length three defined by three vertices i, j, k $(i < j < k)$, the associated binomial equals $\underline{x_{ij}x_{jk}} - x_{ik}$, which is g_{ijk}.

The circuits defined by four vertices $i < j < k < l$ are $C_1 := i, j, k, l, i$, $C_2 := i, j, l, k, i$, $C_3 := i, k, j, l, i$ and their opposites. The binomial which corresponds to C_1 or its opposite is $\underline{x_{ij}x_{jk}x_{kl}} - x_{il}$, whose initial term is divisible by $in_{\prec}(g_{ijk})$. Similarly, the initial term of binomial which corresponds to C_2 or its opposite is divisible by $in_{\prec}(g_{ijl})$. The binomial which corresponds to C_3 or its opposite is g_{ijkl}.

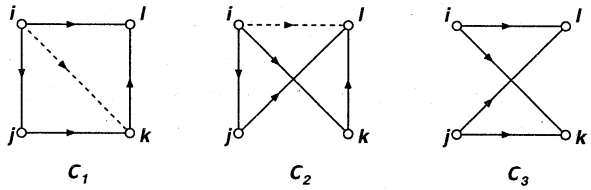

Figure 2: The circuits C_1, C_2, C_3.

Let C be a circuit of length more than five. Let i_1 be the vertex whose label is minimum in C, and $C := i_1, i_2, \ldots, i_s, i_1$. Without loss of generality, we set $i_2 < i_s$. Let f_C be the binomial

corresponding to C, then $in_{\prec}(f_C)$ is product of all variables whose associated edges have same direction with (i_1, i_2) on C. We show that $in_{\prec}(f_C)$ is divisible by initial term of some g_{ijk} or g_{ijkl}, which implies that (4) is Gröbner basis of $I_{A^+_{n-1}}$ with respect to *prec*.

If $i_2 < i_3$, then (i_1, i_2) and (i_2, i_3) have same direction on C. Thus the variables $x_{i_1 i_2}$ and $x_{i_2 i_3}$ appear in $in_{\prec}(f_C)$, and $in_{\prec}(f_C)$ is divisible by $in_{\prec}(g_{i_1 i_2 i_3})$ (Figure 3 left).

If $i_2 > i_3$, then since $i_3 < i_2 < i_s$, there exists k $(3 \leq k < s)$ such that $i_1 < i_k < i_2 < i_{k+1}$. Then the variables $x_{i_1 i_2}$ and $x_{i_k i_{k+1}}$ appear in $in_{\prec}(f_C)$, and $in_{\prec}(f_C)$ is divisible by $in_{\prec}(g_{i_1 i_k i_2 i_{k+1}})$ (Figure 3 right).

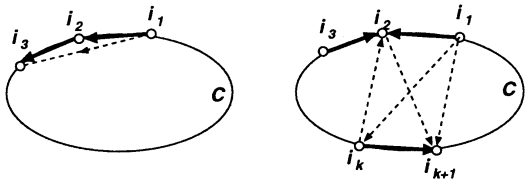

Figure 3: $x_{i_1 i_2}$ and $x_{i_2 i_3}$ (left) or $x_{i_1 i_2}$ and $x_{i_k i_{k+1}}$ (right) appear in $in_{\prec}(f_C)$.

Any term of g_{ijk} is not divisible by the initial term of any other binomials g_{ijk} or g_{ijkl}, and so as g_{ijkl}. This implies that (4) is reduced. ∎

Theorem 4.9 *There exists a term order on* $k[\mathbf{x}]$ *for which the reduced Gröbner basis for* $I_{A^+_{n-1}}$ *is*

$$\{\underline{x_{ij}x_{jk}} - x_{ik} \colon 1 \leq i < j < k \leq n\} \cup \{\underline{x_{il}x_{jk}} - x_{ik}x_{jl} \colon 1 \leq i < j < k < l \leq n\}. \tag{5}$$

Let $g_{ijk} := x_{ij}x_{jk} - x_{ik}$ and $g_{ijkl} := x_{il}x_{jk} - x_{ik}x_{jl}$. Then the set $\{g_{ijk} \colon 1 \leq i < j < k \leq n\}$ corresponds to all of the circuits of length three in D_n, and $\{g_{ijkl} \colon 1 \leq i < j < k < l \leq n\}$ corresponds to the circuits of length four same as in Figure 1, but the directions are opposite. (**Proof**) Let $\prec$ be a purely lexicographic term order induced by the following variable ordering:

$$x_{ij} \succ x_{kl} \Longleftrightarrow j - i < l - k \text{ or } (j - i = l - k \text{ and } i < k).$$

We show that (5) is the reduced Gröbner basis for $I_{A^+_{n-1}}$ with respect to $\prec$.

For any circuit of length three defined by three vertices i, j, k $(i < j < k)$, the associated binomial equals $\underline{x_{ij}x_{jk}} - x_{ik}$, which is g_{ijk}.

The circuits defined by four vertices $i < j < k < l$ are $C_1 := i, j, k, l, i$, $C_2 := i, j, l, k, i$, $C_3 := i, k, j, l, i$ and their opposites. The binomial which corresponds to C_1 or its opposite is $\underline{x_{ij}x_{jk}x_{kl}} - x_{il}$, whose initial term is divisible by $in_{\prec}(g_{ijk})$. The binomial which corresponds to C_2 or its opposite is $x_{ij}x_{jl} - x_{ik}x_{kl}$. If its initial term is $x_{ij}x_{jl}$, it is divisible by $in_{\prec}(g_{ijl})$. If initial term is $x_{ik}x_{kl}$, it is divisible by $in_{\prec}(g_{ikl})$. The binomial which corresponds to C_3 or its opposite is g_{ijkl}.

Let C be a circuit of length more than five. Let (i_1, i_2) $(i_1 < i_2)$ be a edge which the difference of labels is minimum in C, and $C := i_1, i_2, \ldots, i_s, i_1$. Let f_C be the binomial corresponding to C, then $in_{\prec}(f_C)$ is product of all variables whose associated edges have same direction with (i_1, i_2) on C.

If $i_2 < i_3$, then the variables $x_{i_1 i_2}$ and $x_{i_2 i_3}$ appear in $in_{\prec}(f_C)$, and $in_{\prec}(f_C)$ is divisible by $in_{\prec}(g_{i_1 i_2 i_3})$. Similarly, if $i_s < i_1$, then $in_{\prec}(f_C)$ is divisible by $in_{\prec}(g_{i_s i_1 i_2})$.

Let $i_3 < i_2$ and $i_1 < i_n$. Then $i_3 < i_1 < i_2 < i_n$ by the definition of i_1 and i_2. If there exists some p such that $i_p < i_{p+1} < i_{p+2}$, then $in_\prec(f_C)$ is divisible by $in_\prec(g_{i_p i_{p+1} i_{p+2}})$. We show that when there is not such p, there exists some q $(3 \leq q \leq n-1)$ such that $i_q < i_1 < i_2 < i_{q+1}$ (Figure 4 left)).

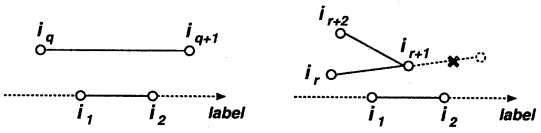

Figure 4: $i_q < i_1 < i_2 < i_{q+1}$ $(^\exists q)$ (left).If $i_r < i_1 < i_{r+1} < i_2$, it must be $i_{r+2} < i_1$ (right).

Let $i_r < i_1 < i_{r+1} < i_2$ (Figure 4 right). Then $i_{r+2} < i_{r+1}$, and $i_{r+2} < i_1$ by the definition of i_1 and i_2. Thus there must be some q $(3 \leq q \leq n-1)$ such that $i_q < i_1 < i_2 < i_{q+1}$ since $i_3 < i_1 < i_2 < i_s$.

Then $in_\prec(f_C)$ is divisible by $in_\prec(g_{i_q i_1 i_2 i_{q+1}})$.

Any term of g_{ijk} is not divisible by the initial term of any other binomials g_{ijk} or g_{ijkl}, and so as g_{ijkl}. This implies that (5) is reduced. ∎

Theorem 4.10 *There exists a term order on $k[\mathbf{x}]$ for which the reduced Gröbner basis for $I_{A^+_{n-1}}$ is*

$$\{\underline{x_{ij}} - x_{i,i+1}x_{i+1,i+2}\cdots x_{j-1,j} \colon 1 \leq i < j-1 < n\}. \tag{6}$$

Let $g_{ij} := x_{ij} - x_{i,i+1}x_{i+1,i+2}\cdots x_{j-1,j}$. Then the set $\{g_{ij} \colon 1 \leq i < j-1 < n\}$ corresponds to all of the fundamental circuits of D_n for the spanning tree $T := \{(i, i+1) \colon 1 \leq i < n\}$.

(**Proof**) Let $\prec$ be a purely lexicographic term order induced by the following variable ordering:

$$x_{ij} \succ x_{kl} \Longleftrightarrow i < k \text{ or } (i = k \text{ and } j > l).$$

We show that (6) is the reduced Gröbner basis for $I_{A^+_{n-1}}$ with respect to $\prec$.

Let C be a circuit which is not a fundamental circuit for T. Let i_1 be the vertex whose label is minimum in C, and $C := i_1, i_2, \ldots, i_s, i_1$. Without loss of generality, we set $i_2 < i_s$. Then the variable $x_{i_1 i_s}$ appears in the initial term of associated binomial f_C. Thus $in_\prec(f_C)$ is divisible by $in_\prec(g_{i_1 i_s})$.

The initial term of g_{ij} corresponds to an edge which is not contained in T, and other term corresponds to several edges which are contained in T. Thus any term of g_{ij} is not divisible by the initial term of other binomial in (6), which implies that (6) is reduced. ∎

Remark 4.11 *Since Gröbner basis of $I_{A^+_{n-1}}$ is a basis of $I_{A^+_{n-1}}$, the number of elements in Gröbner basis of $I_{A^+_{n-1}}$ is more than the number of elements in the basis for $I_{A^+_{n-1}}$. $I_{A^+_{n-1}}$ corresponds to the cycle space of D_n. Thus the number of elements in reduced Gröbner basis for $I_{A^+_{n-1}}$ equals the dimension of the cycle space, which is $\binom{n}{2} - (n-1)$, and the reduced Gröbner basis in Theorem 4.10 is the example achieving this bound.*

5 Regular Triangulations of $\widetilde{A}^+_{n-1}$

In this section, we study the regular triangulations of $\widetilde{A}^+_{n-1}$. To analyze the triangulation from the view point of toric ideals, we consider the $(n+1) \times (m+1)$ matrix

$$A'^+_{n-1} := \begin{pmatrix} {}^t\mathbf{1} & 1 \\ A^+_{n-1} & \mathbf{0} \end{pmatrix} \subset \mathbb{R}^{n+1}$$

where ${}^t\mathbf{1}$ is a row vector whose components are all 1, and $\mathbf{0}$ is a column zero vector. Then the toric ideal of A'^+_{n-1} is homogeneous.

Remark 5.1

1. *The triangulation* Δ *of* $\mathrm{conv}(A'^+_{n-1})$ *can be associated with the triangulation of* $\mathrm{conv}(\widetilde{A}^+_{n-1})$ *by projecting* Δ *to the hyperplane* $x_{n+1} = 0$ *in* $\mathbb{R}^{n+1}$.
2. *If* $\mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v} \in I_{A^+_{n-1}}$ *and* $\deg(\mathbf{x}^\mathbf{u}) - \deg(\mathbf{x}^\mathbf{v}) = k$, *then* $\mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{u} x_0^k \in I_{A'^+_{n-1}}$. *Conversely, If* $\mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v} x_0^k \in I_{A'^+_{n-1}}$, *then* $\mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v} \in I_{A^+_{n-1}}$.

In the rest of this section, we consider the toric ideal of A'^+_{n-1}. We associate the point $\mathbf{a}_{ij} \in \widetilde{A}^+_{n-1}$ with the point $\mathbf{a}'_{ij} := \binom{1}{\mathbf{a}_{ij}} \in A'^+_{n-1}$ and the variable x_{ij} in the polynomial ring $k[\mathbf{x}, x_0] := k[x_{12}, \ldots, x_{1n}, x_{23}, \ldots, x_{n-1,n}, x_0]$, and the point $\mathbf{0} \in \widetilde{A}^+_{n-1}$ with $\mathbf{a}'_0 := \binom{1}{\mathbf{0}} \in A'^+_{n-1}$ and the variable x_0 in $k[\mathbf{x}, x_0]$.

For the case of $n \geq 4$, Proposition 2.15 does not hold since A'^+_{n-1} is not unimodular.

Claim 5.2 *If* $n = 3$, $\mathcal{C}_{A'^+_2} = \mathcal{U}_{A'^+_2}$. *If* $n \geq 4$, *then* $\mathcal{C}_{A'^+_{n-1}}$ *is a proper subset of* $\mathcal{U}_{A'^+_{n-1}}$.

(**Proof**) If $n = 3$, Proposition 2.15 holds since A'^+_2 is unimodular.

Let $n \geq 4$. Then $x_{13}^2 x_{34} - x_{12}x_{14}x_{23} \notin \mathcal{C}_{A'^+_{n-1}}$ but $x_{13}^2 x_{34} - x_{12}x_{14}x_{23} \in \mathcal{U}_{A'^+_{n-1}}$. ∎

Thus analyzing reduced Gröbner bases for $I_{A'^+_{n-1}}$ is much difficult.

Theorem 4.8 (or Theorem 4.9) shows that $I_{A'^+_{n-1}}$ is generated by $\{x_{ij}x_{jk} - x_{ik}x_0 \colon 1 \leq i < j < k \leq n\} \cup \{x_{ik}x_{jl} - x_{il}x_{jk} \colon 1 \leq i < j < k < l \leq n\}$. Thus in these case, we can extend the term order $\prec$ in Theorem 4.8 (resp. Theorem 4.9) to the term order $\prec'$ on $k[\mathbf{x}, x_0]$ such that $in_\prec(I_{A^+_{n-1}}) = in_{\prec'}(I_{A'^+_{n-1}})$.

Corollary 5.3

(**i**) *There exists a term order on* $k[\mathbf{x}, \mathbf{0}]$ *for which the initial ideal of* $I_{A'^+_{n-1}}$ *is* $\langle \{x_{ij}x_{jk} \colon 1 \leq i < j < k \leq n\} \cup \{x_{ik}x_{jl} \colon 1 \leq i < j < k < l \leq n\} \rangle$.

(**ii**) *There exists a term order on* $k[\mathbf{x}, \mathbf{0}]$ *for which the initial ideal of* $I_{A'^+_{n-1}}$ *is* $\langle \{x_{ij}x_{jk} \colon 1 \leq i < j < k \leq n\} \cup \{x_{il}x_{jk} \colon 1 \leq i < j < k < l \leq n\} \rangle$.

(**Proof**) (i). Let $\prec'$ be a purely lexicographic term order induced by the following variable ordering:

$$x_{ij} \succ x_{kl} \Longleftrightarrow i < k \text{ or } (i = k \text{ and } j < l), \quad \text{and } x_{ij} \succ x_0 \text{ for any } 1 \leq i < j \leq n.$$

Since $I_{A'^+_{n-1}}$ is generated by

$$\{\underline{x_{ij}x_{kl}} - x_{il}x_0 \colon 1 \leq i < j < k \leq n\} \cup \{\underline{x_{ik}x_{jl}} - x_{il}x_{jk} \colon 1 \leq i < j < k < l \leq n\} \qquad (7)$$

and initial terms of binomials in (7) are same as those in (4), the reduced Gröbner basis for $I_{A'^{+}_{n-1}}$ with respect to $\prec$ is (7). Thus the initial ideal of $I_{A'^{+}_{n-1}}$ is $\langle \{x_{ij}x_{jk} \colon 1 \leq i < j < k \leq n\} \cup \{x_{ik}x_{jl} \colon 1 \leq i < j < k < l \leq n\} \rangle$.

The proof of (ii) is similar to that of (i). ∎

Thus we get two regular unimodular triangulations Δ_1, Δ_2 of A'^{+}_{n-1} by applying Definition 2.10. The normalized volume of $\mathrm{conv}(A'^{+}_{n-1})$ can be obtained by calculating the Hilbert polynomial of $k[\mathbf{x}, x_0]/I_{A'^{+}_{n-1}}$.

As a matter of fact, $F \subset A'^{+}_{n-1}$ is the face of Δ_1 (resp. Δ_2) if and only if $\{(i,j) \colon \mathbf{a}_{ij} \in F\}$ is standard tree (resp. anti-standard tree). Thus we obtain our main theorem using the result by Gelfand, Graev and Postnikov that the number of linearly independent solutions of the system (1), (2) in a neighborhood of a generic point is equal to the normalized volume of $\mathrm{conv}(\widetilde{A}^{+}_{n-1})$.

Theorem 5.4 *The number of linearly independent solutions of the system* (1), (2) *in a neighborhood of a generic point is equal to the the Catalan number*

$$C_{n-1} = \frac{1}{n}\binom{2(n-1)}{n-1}.$$

For the case of Theorem 4.10, we cannot extend the term order $\prec$ in Theorem 4.10 to the term order $\prec'$ on $k[\mathbf{x}, x_0]$ such that $in_{\prec}(I_{A^{+}_{n-1}}) = in_{\prec'}(I_{A'^{+}_{n-1}})$.

Example 5.5 *Let $n = 4$ and $\prec$ be a purely lexicographic term order induced by*

$$x_{14} \succ x_{13} \succ x_{12} \succ x_{24} \succ x_{23} \succ x_{34}.$$

Then

$$in_{\prec}(I_{A^{+}_{n-1}}) = \langle x_{13},\ x_{14},\ x_{24} \rangle.$$

Let $\prec'$ be a purely lexicographic term order induced by

$$x_{14} \succ x_{13} \succ x_{12} \succ x_{24} \succ x_{23} \succ x_{34} \succ x_0.$$

Then

$$in_{\prec'}(I_{A'^{+}_{n-1}}) = \langle x_{13}x_{34},\ x_{13}x_0,\ x_{14}x_{23},\ x_{14}x_0,\ x_{24}x_0 \rangle.$$

Question 5.6 *How can the universal Gröbner basis $\mathcal{U}_{A'^{+}_{n-1}}$ be characterized in terms of graphs?*

6 Open Problems by Gelfand, Graev and Postnikov

In [5], Gelfand, Graev and Postnikov gave the following open problems:

1. Find all regular local triangulations of $\widetilde{A}^{+}_{n-1}$.
2. For $I, J \subset \{1, \ldots, n\}$ such that $I \cap J = \emptyset$, let

   $$A_{IJ} := \mathrm{conv}(\{\mathbf{a}_{ij} \colon 1 \leq i < j \leq n,\ i \in I,\ j \in J\} \cup \{\mathbf{0}\}).$$

   How can the triangulations of A_{IJ} be described?
3. Find analogues of all results in [5] for other root systems.

Ohsugi and Hibi [14, 15] studied the third problem. In this section, we study the first problem.

Remark 6.1 *The special case for the problem 2 is the case* $I = \{1, \ldots, k\}$, $J = \{k+1, \ldots, n\}$ *for some* k. *In this case,* A_{IJ} *is related to the hypergeometric system called* hypergeometric system on the grassmannian. *This system is connected with triangulations of the product of two simplices* $\Delta^k \times \Delta^{n-k}$. *For more details, we refer to [6, 7, 8].*

For the first problem, we calculated all regular local triangulations, regular triangulations, all regular unimodular triangulations, all regular unimodular local triangulations for small n using TiGERS [9, 10] (Table 1).

n	# reg	# reg+local	# reg+uni	# reg+uni+local
3	2	1	2	1
4	18	2	15	2
5	3515	18	1301	18

Table 1: The number of regular triangulations (reg), regular local triangulations (reg+local), regular unimodular triangulations (reg+uni) and regular unimodular local triangulations (reg+uni+local).

Remark 6.2

1. *The problem to find all regular triangulations of* $\widetilde{A}^+_{n-1}$ *is equivalent to the problem to find all initial ideals of* $I_{A'^+_{n-1}}$.
2. *The problem to find all regular local triangulations of* $\widetilde{A}^+_{n-1}$ *is equivalent to the problem to find all initial ideals of* $I_{A'^+_{n-1}}$ *none of whose generators contains* x_0.
3. *The problem to find all regular unimodular triangulations of* $\widetilde{A}^+_{n-1}$ *is equivalent to the problem to find all square-free initial ideals of* $I_{A'^+_{n-1}}$.
4. *The problem to find all regular unimodular local triangulations of* $\widetilde{A}^+_{n-1}$ *is equivalent to the problem to find all square-free initial ideals of* $I_{A'^+_{n-1}}$ *none of whose generators contains* x_0.

Question 6.3 *Is any regular local triangulation of* $\widetilde{A}^+_{n-1}$ *regular?*

For the first problem, we are interested in the bound of the number of regular local triangulations of $\widetilde{A}^+_{n-1}$. The number of regular triangulations are interesting problem in computational geometry.

Question 6.4 *Can the number of regular triangulations, regular local triangulations, regular unimodular triangulations and regular unimodular local triangulations be bounded with respect to* n*?*

As the relation of the complexity of the algorithm of minimum cost flow problem using Gröbner bases [11], we are also interested in the number of elements in reduced Gröbner bases of $I_{A^+_{n-1}}$ and $I_{A'^+_{n-1}}$. The lower bound for $I_{A^+_{n-1}}$ is achieved in Theorem 4.10.

Question 6.5 *Are the number of elements in reduced Gröbner bases of* $I_{A^+_{n-1}}$ *and* $I_{A'^+_{n-1}}$ *of polynomial order with respect to* n*?*

7 Conclusions

In this paper, we showed that the number of linearly independent solutions of the hypergeometric systems on the group of unipotent matrices can be calculated using Gröbner bases for toric ideals of acyclic tournament graphs. We also study an open problem by Gelfand, Graev and Postnikov [5]. To homogenize the toric ideal $I_{A_{n-1}^+}$, we add a column associated with the origin and consider the space $\mathbb{R}^{n+1}$ by adding a row whose components are all 1. But the unimodularity of $I_{A_{n-1}^+}$ is broken by these operations. Characterizations of the universal Gröbner basis $\mathcal{U}_{A'^+_{n-1}}$, and the number of triangulations of $\widetilde{A}^+_{n-1}$ are future works.

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