COMMUTING DIFFERENTIAL OPERATORS OF TYPE B_2

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1. INTRODUCTION

1.1. Several integral systems are accidentally related to root systems. Olshanetsky-Perelomov ([OP1], [OP2]) considered integrable n-particle models in dimension one arising from root systems. The systems of differential operators satisfied by zonal spherical functions give such integrable systems and these were generalized by Sekiguchi and Heckman-Opdam ([Sj], [HO]).

In [OOS] we announce a classification of integrable systems invariant under simple classical Weyl groups. The precise discussion has already been given by [OS] and [O] except for the case of type B_2. As is shown in [OS], the classification problem for type B_2 is reduced to a functional differential equation (2.1).

In §2 we give a complete list of solutions of this functional equation. Some solutions have already been obtained, after [OP2], by Inozemtsev [IM], [I] (See also [P]). The main result of §2 is Theorem 2.9, which is stated in §1.3 in a different form.

In §3 we examine the reducibility of the system obtained in §2. We note that if the system coincides with the system satisfied by zonal spherical functions of a semisimple Lie group, the reducibility is related to degenerate series representations.

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1.2. Now we give a quick review of the results in [OS, §6] concerning with type B_2. Let $W(B_2)$ be the Weyl group of type B_2, which is identified with the group of coordinate transformations of (x_1, x_2) generated by $(x_1, x_2) \mapsto (x_2, x_1)$ and $(x_1, x_2) \mapsto (x_1, -x_2)$. Consider $W(B_2)$-invariant differential operators

$$(1.0) \quad \begin{cases} P_1 = \partial_1^2 + \partial_2^2 + R(x), \\ P_2 = \partial_1^2 \partial_2^2 + \text{lower order terms} \end{cases}$$

which satisfies $[P_1, P_2] = 0$ and ${}^t P_2 = P_2$. Here we denote $\partial_1 = \frac{\partial}{\partial x_1}$ and $\partial_2 = \frac{\partial}{\partial x_2}$ for simplicity and the map ^{t} is the anti-automorphism of the algebra of differential operators such that ${}^t a(x) = a(x)$ for functions $a(x)$ and ${}^t \partial_i = -\partial_i$ for $i = 1$ and 2. We assume that the coefficients of differential operators are extended to

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holomorphic functions on a Zariski open subset of an open connected neighborhood of the origin of the complexification $\mathbb{C}^2$ of $\mathbb{R}^2$.

The operators are proved to be expressed by even functions u and v of one variable as follows ([OS, Proposition 6.3]):

(1.1)
$$\begin{cases} P_1 &= \partial_1^2 + \partial_2^2 + u(x_1 + x_2) + u(x_1 - x_2) + v(x_1) + v(x_2), \\ P_2 &= \left(\partial_1\partial_2 + \dfrac{u(x_1 + x_2) - u(x_1 - x_2)}{2}\right)^2 + v(x_2)\partial_1^2 + v(x_1)\partial_2^2 \\ &\quad + v(x_1)v(x_2) + T(x_1, x_2), \end{cases}$$

where T is determined by the following equations up to a constant.
(1.2)
$$\begin{cases} 2\partial_2 T = v'(x_1)\big(u(x_1 + x_2) - u(x_1 - x_2)\big) + 2v(x_1)\big(u'(x_1 + x_2) - u'(x_1 - x_2)\big), \\ 2\partial_1 T = v'(x_2)\big(u(x_1 + x_2) - u(x_1 - x_2)\big) + 2v(x_2)\big(u'(x_1 + x_2) + u'(x_1 - x_2)\big). \end{cases}$$

As the compatibility condition for the existence of the solution T of the equation (1.2), we have an equation

$$(1.4) \quad \partial_2 \Big(v'(x_2)\big(u(x_1 + x_2) - u(x_1 - x_2)\big) + 2v(x_2)\big(u'(x_1 + x_2) + u'(x_1 - x_2)\big)\Big)$$
$$= \partial_1 \Big(v'(x_1)\big(u(x_1 + x_2) - u(x_1 - x_2)\big) + 2v(x_1)\big(u'(x_1 + x_2) - u'(x_1 - x_2)\big)\Big),$$

which have been posed in [OS, Proposition 6.3] (cf. [P, §2.2.C]).

Conversely for any solution (u, v) of (1.4) and the pair (P_1, P_2) of the operators which are given by (1.1) with

$$(1.3) \qquad T = \frac{1}{2}\Big(\partial_1^2 - \partial_2^2\Big)\Big(V(x_1)\big(U(x_1 + x_2) + U(x_1 - x_2)\big) - G(x_1)\Big)$$

under the notation in Remark 2.1 and Lemma 2.2, we have $[P_1, P_2] = 0$

1.3. We give a complete list of solutions of the functional equation (1.4). Remind that the Schrödinger operator P_1 is explicitly expressed as in (1.1) using u and v.

1) (Trivial case) $u = $ constant, $v = $ an arbitrary even function,

1^d) $u = $ an arbitrary even function, $v = $ constant.

Let ω_1 and ω_2 denote the primitive half periods of the Weierstrass elliptic function $\wp(t)$ and put $\omega_3 = -\omega_1 - \omega_2$ and $\omega_4 = 0$.

2) (Elliptic case) For $\omega_1, \omega_2 < \infty$
$$\begin{cases} u(t) &= C_6\wp(t) + C_7 E, \\ v(t) &= \sum_{i=1}^{4} C_i \wp(t + \omega_i) + C_5, \end{cases}$$

$2^d)$

$$\begin{cases} u(t) = \sum_{i=1}^{4} C_i \wp(t + \omega_i) + C_5, \\ v(t) = C_6 \wp(2t) + C_7. \end{cases}$$

$2)'$ (Trigonometric case)

$$\begin{cases} u(t) = C_6 \sinh^{-2} \lambda t + C_7, \\ v(t) = C_1 \sinh^{-2} \lambda t + C_2 \sinh^{-2} 2\lambda t + C_3 \sinh^2 \lambda t + C_4 \sinh^2 2\lambda t + C_5, \end{cases}$$

$2^d)'$

$$\begin{cases} u(t) = C_1 \sinh^{-2} \lambda t + C_2 \sinh^{-2} 2\lambda t + C_3 \sinh^2 \lambda t + C_4 \sinh^2 2\lambda t + C_5 \\ v(t) = C_6 \sinh^{-2} 2\lambda t + C_7. \end{cases}$$

$2)''$ (Rational case)

$$\begin{cases} u(t) = C_6 t^{-2} + C_7, \\ v(t) = C_1 t^{-2} + C_2 + C_3 t^2 + C_4 t^4 + C_5 t^6, \end{cases}$$

$2^d)''$

$$\begin{cases} u(t) = C_1 t^{-2} + C_2 + C_3 t^2 + C_4 t^4 + C_5 t^6, \\ v(t) = C_6 t^{-2} + C_7. \end{cases}$$

$3)$ (Elliptic case) For $\omega_1, \omega_2 < \infty$

$$\begin{cases} u(t) = C_1 \left(\wp(\tfrac{t}{2} + \omega_1) + \wp(\tfrac{t}{2} + \omega_2) \right) + C_2 \wp(t) + C_3, \\ v(t) = C_4 \wp(t) + C_5 \wp(t + \omega_3) + C_6. \end{cases}$$

$3)'$ (Trigonometric case)

$$\begin{cases} u(t) = C_1 \sinh^{-2} \tfrac{\lambda}{2} t + C_2 \sinh^{-2} \lambda t + C_3, \\ v(t) = C_4 \sinh^{-2} \lambda t + C_5 \sinh^2 \lambda t + C_6, \end{cases}$$

$3^d)'$

$$\begin{cases} u(t) = C_4 \sinh^{-2} \lambda t + C_5 \sinh^2 \lambda t + C_6, \\ v(t) = C_1 \sinh^{-2} \lambda t + C_2 \sinh^{-2} 2\lambda t + C_3. \end{cases}$$

$3)''$ (Rational case)

$$\begin{cases} u(t) = C_1 t^{-2} + C_2 + C_3 t^2, \\ v(t) = C_4 t^{-2} + C_5 + C_6 t^2. \end{cases}$$

1.4. Although we deal with the commuting differential operators of type B_2 with the Weyl group symmetry in the main body of this paper, we will give a brief summary of the related works.

The commuting differential operators of type A have been studied very well. The commuting differential operators of type A with the Weyl group invariant condition are classified in [OS]. This work is generalized to the commuting differential operators of type A_2 without Weyl group invariant condition

$$\begin{cases} \Delta_1 = \partial_1 + \partial_2 + \partial_3, \\ \Delta_2 = \partial_1 \partial_2 + \partial_2 \partial_3 + \partial_3 \partial_1 + R(x), \\ \Delta_3 = \partial_1 \partial_2 \partial_3 + \text{lower order terms}. \end{cases}$$

To classify the potential function $R(x)$, we may assume that ${}^t\Delta_3 = -\Delta_3$. Then there exist one-variable functions $u_1 = u_1(x_2 - x_3)$, $u_2 = u_2(x_3 - x_1)$ and $u_3 = u_3(x_1 - x_2)$ such that $R(x) = -u_1 - u_2 - u_3$, and

$$(1.6) \qquad \begin{vmatrix} 1 & 1 & 1 \\ u_1(x) & u_2(y) & u_3(z) \\ u_1'(x) & u_2'(y) & u_3'(z) \end{vmatrix} = 0 \qquad \text{for } x + y + z = 0.$$

For the Weyl group invariant case, we have $u_1(z) = u_2(z) = u_3(z)$ and the proof of this fact is given in Proposition 4.2 (with $m = 3$) of [OS], which is valid for the general case with no change. For the Weyl group invariant case, the functional differential equation (1.6) is solved in [WW] and the solution is a Weierstrass elliptic function $\wp$. The corresponding potential $R(x)$ is of Calogero-Moser type. For the general case, the equation (1.6) is solved in [BP] and [BB]. Besides the $\wp$ solutions, we also have solutions expressed by exponential functions. The corresponding potential is known as of type periodic/non-periodic Toda, which can be regarded as a degenerating limit of a Weyl group invariant potential [vD].

For type B_2, the classification of the commuting differential operators (1.0) without the Weyl group symmetry has not been done yet. It is known that the similar functional differential equation (see (2.4')) is related to such operators. The following results are obtained in [Oc]:

(i) We have the expression of the (non Weyl group invariant) operators P_1 and P_2 by using four functions $u_1 = u_1(x_1 + x_2)$, $u_2 = u_2(x_1 - x_2)$, $v_1 = v_1(x_1)$ and $v_2 = v_2(x_2)$ with one-variable. Actually, if we replace $u(x_1 + x_2)$ by $u_1(x_1 + x_2)$, $u(x_1 - x_2)$ by $u_2(x_1 - x_2)$, and so on, the formula (1.1) is also valid for non-invariant operators. These functions satisfy the functional differential equation like (1.4).

(ii) Suppose P_1 be non-trivial (c.f. Lemma 2.4 i)). If P_1 is holomorphic at some point, then P_1 and P_2 can be meromorphically continued to whole plane $\mathbb{C}^2$. The orders of poles of P_1 are at most two.

(iii) Suppose, moreover, that $v_2(z)$ has poles at three points $z = z_1, z_2, z_3$ such that $z_1 - z_2$ and $z_2 - z_3$ are linearly independent over $\mathbb{Q}$. Then the function v_2 can be expressed as

$$v_2(z) = \sum_{i=1}^{4} C_i \wp(z + \omega_i) + C_5,$$

with an elliptic function $\wp$ and constants $C_1, \ldots, C_5$.

2. Functional differential equation for type B_2

2.1. In this section we solve the functional differential equation (1.4)

$$(2.1) \quad \partial_2 \Big(v'(x_2)\big(u(x_1 + x_2) - u(x_1 - x_2)\big) + 2v(x_2)\big(u'(x_1 + x_2) + u'(x_1 - x_2)\big)\Big)$$

$$= \partial_1 \Big(v'(x_1)\big(u(x_1 + x_2) - u(x_1 - x_2)\big) + 2v(x_1)\big(u'(x_1 + x_2) - u'(x_1 - x_2)\big)\Big).$$

Remark 2.1. For even holomorphic functions u and v on $0 < |t| \ll 1$, there exist unique odd holomorphic functions U and V with $U' = u$ and $V' = v$ on $0 < |t| \ll 1$. Then the equation (2.1) is equivalent to

$$(2.2) \quad \partial_1 \partial_2 (\partial_1^2 - \partial_2^2)\Big(V(x_1)\big(U(x_1 + x_2) + U(x_1 - x_2)\big)$$

$$+ V(x_2)\big(U(x_1 + x_2) - U(x_1 - x_2)\big)\Big) = 0.$$

Lemma 2.2. *Odd holomorphic functions U and V on a small punctured disk satisfy the equation (2.2) if and only if there exist even holomorphic functions F and G on a small punctured disk such that*

$$(2.3) \quad \begin{aligned} V(x_1)\big(U(x_1 + x_2) + U(x_1 - x_2)\big) + V(x_2)\big(U(x_1 + x_2) - U(x_1 - x_2)\big) \\ = F(x_1 + x_2) + F(x_1 - x_2) + G(x_1) + G(x_2). \end{aligned}$$

Proof. The "if" part is clear. Now we assume (2.2) and set the left hand side of (2.3) to be $W(x_1, x_2) \in \mathcal{O}(\{(x_1, x_2) \in \mathbb{C}^2 \,|\, 0 < |x_1| < \varepsilon/2, \, 0 < |x_2| < \varepsilon/2, \, x_1 \neq \pm x_2\})$. Then the function $\partial_2(\partial_1^2 - \partial_2^2)W \in \mathcal{O}(\{(x_1, x_2) \in \mathbb{C}^2 \,|\, 0 < |x_1| < \varepsilon/2, \, 0 < |x_2| < \varepsilon/2, \, x_1 \neq \pm x_2\})$ is locally constant with respect to x_1 and consequently it is constant with respect to x_1. Then this is an element of $\mathcal{O}(\{x_2 \in \mathbb{C} \,|\, 0 < |x_2| < \varepsilon/2\})$. Moreover, the residue $\mathrm{Res}_{x_2=0} \, \partial_2(\partial_1^2 - \partial_2^2)W = \int_\gamma \partial_2(\partial_1^2 - \partial_2^2)W(x_1, x_2)dx_2 = 0$. Hence we have a holomorphic function $g_2(x_2) \in \mathcal{O}(\{x_2 \in \mathbb{C} \,|\, 0 < |x_2| < \varepsilon/2\})$ such that $\partial_2(\partial_1^2 - \partial_2^2)W(x_1, x_2) = \partial_2 g_2$. Then the difference $(\partial_1^2 - \partial_2^2)W - g_2$ is locally constant with respect to x_2. The same argument tells us that there exists a holomorphic function $g_1 \in \mathcal{O}(\{x_1 \in \mathbb{C} \,|\, 0 < |x_1| < \varepsilon/2\})$ such that $(\partial_1^2 - \partial_2^2)W = g_1 + g_2$.

Next we change the coordinates $\xi_1 = (x_1 + x_2)/2$, $\xi_2 = (x_1 - x_2)/2$ and write $\partial_1' = \frac{\partial}{\partial \xi_1}$, $\partial_2' = \frac{\partial}{\partial \xi_2}$ for short. Then $\partial_1' \partial_2' W = g_1(\xi_1 + \xi_2) + g_2(\xi_1 - \xi_2)$. The residue $\mathrm{Res}_{\xi_1 = -\xi_2} \, g_1(\xi_1 + \xi_2) = \int_\gamma \partial_1' \partial_2' W d\xi_1 - \int_\gamma g_2(\xi_1 - \xi_2)d\xi_1 = 0$. Then we have an integral $g_3(t) \in \mathcal{O}(\{t \in \mathbb{C} \,|\, 0 < |t| < \varepsilon/2\})$ such that $g_3' = g_1$. Similarly we have g_4 with $g_4' = g_2$, and $\partial_1'(\partial_2' W - g_3 - g_4) = 0$. Then $g_5 := \partial_2' W - g_3 - g_4$ is locally constant with respect to ξ_1, that is, g_5 is constant with respect to ξ_1. As before g_3, g_4 and g_5 have integrals G_3, G_4 and G_5, and the difference $G_6 := W - G_3 - G_4 - G_5$ depends only on ξ_1.

Taking the averages of G_3, G_4, G_5 and G_6 under the action of the Weyl group $W(B_2)$, we get functions F and G with required property. $\square$

This lemma can be generalized to the case when the Weyl group invariance is not imposed. In fact, the functional equation mentioned in Section 1.4(ii) can be expressed as

$$(2.4') \quad \partial_1 \partial_2 (\partial_1^2 - \partial_2^2)\Big(V_1(x_1)\big(U_1(x_1 + x_2) + U_2(x_1 - x_2)\big)$$
$$+ V_2(x_2)\big(U_1(x_1 + x_2) - U_2(x_1 - x_2)\big)\Big) = 0.$$

This can be integrated as

$$(2.5') \quad \begin{aligned} V_1(x_1)\big(U_1(x_1 + x_2) + U_2(x_1 - x_2)\big) + V_2(x_2)\big(U_1(x_1 + x_2) - U_2(x_1 - x_2)\big) \\ = F_1(x_1 + x_2) + F_2(x_1 - x_2) + G_1(x_1) + G_2(x_2). \end{aligned}$$

For detail, see Proposition 2.4 of [Oc].

Remark 2.3. The same argument holds for type A_2. The equation (1.6) with $u_1 = u_2 = u_3$ is equivalent to the equation

$$(2.6) \quad \partial_x \partial_y (\partial_x - \partial_y)\Big(\big(U(x) + U(y) + U(-x - y)\big)^2\Big) = 0,$$

where U is the odd primitive function of u. By the same argument as in the proof of the previous lemma this is also equivalent to

$$(2.7) \qquad \big((U(x) + U(y) + U(z)\big)^2 = F(x) + F(y) + F(z) \quad \text{for } x + y + z = 0$$

with some even function F. Remark that $u = \wp$ satisfies (1.6) and that $U = -\zeta$ and $F = \wp$ satisfy (2.7).

Lemma 2.4. i) *If u or v is constant, then (u,v) is a solution of* (2.1)*. A solution of this form is called a trivial solution.*

ii) *If there are functions F_1 and G_1 such that*

$$(2.8) \qquad \big(U(x_1 + x_2) + V(-x_1) + V(-x_2)\big)^2 = F_1(x_1 + x_2) + G_1(x_1) + G_1(x_2),$$

then (U,V) is a solution of (2.3)*.*

iii) *If $u = v = \wp$, then (u,v) is a solution of* (2.1)*.*

Proof. For ii), (U,V) satisfy (2.3) with $F(t) = \frac{1}{2}(U(t)^2 - F_1(t))$ and $G(t) = V(t)^2 - G_1(t)$. iii) follows from ii) and Remark 2.3. $\square$

We summarize several elementary properties of the equation (2.1).

Lemma 2.5. i) *The equation* (2.1) *is bilinear with respect to (u,v).*

ii) *For a solution $(u_0(t), v_0(t))$ of* (2.1) *and a non-zero constant C, $(u(t), v(t)) = (u_0(Ct), v_0(Ct))$ is also a solution.*

iii) *For a solution $(u_0(t), v_0(t))$ of* (2.1)*, $(u(t), v(t)) = (v_0(t), u_0(2t))$ is also a solution.*

iv) *For a solution $(u_0(t), v_0(t))$ of* (2.1) *with $u_0(t + 2\omega) = u_0(t)$ satisfying some constant ω, $(u(t), v(t)) = (u_0(t), v_0(t + \omega))$ is also a solution.*

Proof. All but iv) are shown in [OS, Proposition 6.3 iv)]. iv) follows from $u(x_1 - x_2) = u((x_1 + \omega) - (x_2 + \omega))$ and $u(x_1 + x_2) = u((x_1 + \omega) + (x_2 + \omega))$. $\square$

Remark 2.6. The equations (2.2) and (2.6) above are written in a uniform manner. Let the root system (E, Σ) be $(\mathbb{R}^2, \Sigma(A_2))$ or $(\mathbb{R}^2, \Sigma(B_2))$ with the Weyl group W. Consider an element $\mathbb{V}$ of the space of W-invariants $(\mathcal{O}(E) \otimes E^*)^W$ in $\mathcal{O}(E) \otimes E^*$. Extend the natural invariant inner bilinear form $\langle \, , \, \rangle$ on E^* to a $\mathcal{O}(E)$-linear form on this space of W-invariants. Consider the differential equations

$$(2.10) \qquad \begin{cases} \big(\displaystyle\prod_{\alpha \in \Sigma^+} \partial_\alpha \big) \mathbb{V} = 0, \\[2ex] \big(\displaystyle\prod_{\alpha \in \Sigma^+} \partial_\alpha \big) \langle \mathbb{V}, \mathbb{V} \rangle = 0. \end{cases}$$

Here differential operators act on the first factor of $\mathcal{O}(E) \otimes E^*$.

This is equivalent to the equations (2.2) or (2.6). In fact, if we set

$$(2.11) \qquad \mathbb{V} = \sum_{\alpha \in \Sigma^+} V_\alpha(\langle \alpha, \cdot \rangle) \otimes \alpha = \frac{1}{2} \sum_{\alpha \in \Sigma} V_\alpha(\langle \alpha, \cdot \rangle) \otimes \alpha$$

with V_α corresponding to the solutions (2.2) or (2.6), it satisfies the equation (2.10). On the other hand, any solution of the former equation of (2.10) is written in the form (2.11) with odd functions V_α, and the W-invariance and the latter equation of (2.10) are sufficient for the equations (2.2) or (2.6).

2.2. Elliptic functions. We summarize several well-known properties of the elliptic functions $\wp$ and ζ of Weierstrass type for latter convenience (cf. [WW]).

They are given by

$$(2.12) \qquad \wp(z) = \wp(z|2\omega_1, 2\omega_2) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right),$$

$$(2.13) \qquad \zeta(z) = \zeta(z|2\omega_1, 2\omega_2) = \frac{1}{z} + \sum_{\omega \neq 0} \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right),$$

where the sum ranges over all non-zero periods $2m_1\omega_1 + 2m_2\omega_2$ of $\wp$. They satisfy

$$\zeta'(z) = -\wp(z),$$
$$\wp(z + 2m_1\omega_1 + 2m_2\omega_2|2\omega_1, 2\omega_2) = \wp(z|2\omega_1, 2\omega_2),$$
$$\zeta(z + 2m_1\omega_1 + 2m_2\omega_2|2\omega_1, 2\omega_2) = \zeta(z|2\omega_1, 2\omega_2) + 2m_1\eta_1 + 2m_2\eta_2$$
$$\text{for } m_1, m_2 \in \mathbb{Z},$$
$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3).$$

Here the constants have the relations

$$g_2 = 60 \sum_{\omega \neq 0} \omega^{-4}, \quad g_3 = 140 \sum_{\omega \neq 0} \omega^{-6},$$
$$\omega_3 = -\omega_1 - \omega_2, \quad e_j = \wp(\omega_j), \quad \eta_j = \zeta(\omega_j),$$
$$e_1 + e_2 + e_3 = 0, \quad g_2 = -4(e_1 e_2 + e_2 e_3 + e_3 e_1), \quad g_3 = 4e_1 e_2 e_3,$$
$$\eta_1 + \eta_2 + \eta_3 = 0, \quad \eta_2\omega_1 - \eta_1\omega_2 = \pm\frac{\pi\sqrt{-1}}{2}.$$

The following are variants of addition formulas.

(2.14)

$$\left(\zeta(x) + \zeta(y) + \zeta(z) \right)^2 = \wp(x) + \wp(y) + \wp(z) \quad \text{when } x + y + z = 0,$$

(2.15)

$$\zeta(x+y) - \zeta(x) - \zeta(y) = \frac{1}{2} \frac{\wp'(x) - \wp'(y)}{\wp(x) - \wp(y)}.$$

The Laurent expansion at the origin is

$$(2.16) \qquad \wp(z|2\omega_1, 2\omega_2) = z^{-2} + \frac{g_2}{20}z^2 + \frac{g_3}{28}z^4 + \frac{g_2^2}{1200}z^6 + \cdots.$$

The complex numbers ω_1 and ω_2 are assumed to be linearly independent over $\mathbb{R}$ but we allow the period to be infinity. In other words, the numbers g_2 and g_3 are any complex numbers. For example we have

$$(2.17) \qquad \begin{array}{ll} \wp(z|\sqrt{-1}\pi, \infty) = \sinh^{-2} z + \frac{1}{3} & \text{when } g_2 = \frac{4}{3} \text{ and } g_3 = -\frac{8}{27}, \\ \wp(z|\infty, \infty) = z^{-2} & \text{when } g_2 = g_3 = 0. \end{array}$$

If ω_1 and ω_2 are finite, we have a formula

$$\wp(z + \omega_\nu|2\omega_1, 2\omega_2) = e_\nu + \frac{(e_\nu - e_\lambda)(e_\nu - e_\mu)}{\wp(z|2\omega_1, 2\omega_2) - e_\nu}$$

$(2.18) \qquad\qquad\qquad\qquad\qquad \text{with } \{\nu, \mu, \lambda\} = \{1, 2, 3\}$

and every function of the form $\wp'^{-2} \times ($a polynomial of $\wp$ of degree at most 4$)$ is written by a linear combination of 1, $\wp$, $(\wp - e_1)^{-1}$, $(\wp - e_2)^{-1}$ and $(\wp - e_3)^{-1}$, equivalently by a linear combination of 1, $\wp(z)$, $\wp(z + \omega_1)$, $\wp(z + \omega_2)$ and $\wp(z + \omega_3)$.

Lastly we quote the Landen transformation

$$(2.19) \quad \wp(z|\omega_1, 2\omega_2) = \wp(z|2\omega_1, 2\omega_2) + \wp(z + \omega_1|2\omega_1, 2\omega_2) - e_1 \quad \text{if } \omega_1 \text{ is finite.}$$

2.3. Solutions of the functional equation.

Theorem 2.7. *The functions*

$$(2.20) \quad \begin{aligned} u(t) &= c_6 \frac{(\wp(\frac{t}{2}) - e_3)^2}{\wp'(\frac{t}{2})^2} + c_7\wp(t) + c_8, \\ v(t) &= \frac{(\wp(t) - e_1)(\wp(t) - e_2)(c_1\wp(t)^2 + c_2\wp(t) + c_3) + c_4\wp(t) + c_5}{\wp'(t)^2} \end{aligned}$$

satisfy the equation (2.1) *if* $c_4 c_6 = c_5 c_6 = 0$.

Proof. Since the equation is bilinear, we may check for each *monomial* in u or v. Here we will give a proof for $\omega_1, \omega_2 < \infty$, which implies the theorem by the analytic continuation.

i) <u>Case $c_6 = c_7 = 0$</u>: It follows from Lemma 2.4 i).

ii) <u>Case $c_6 = c_8 = 0$, $c_7 = 1$</u>: We may assume that $v = \wp(t + a)$ with $a = 0$, ω_1, ω_2 or ω_3. Moreover we may assume $a = 0$ by Lemma 2.5 iv), that is, $u = v = \wp$. Then (2.3) follows from Lemma 2.4 ii) and Remark 2.3. This simplifies the proof of [OS, Proposition 7.3 ii)].

iii) <u>Case $c_7 = c_8 = 0$, $c_6 = 1$</u>: By §2.2 the function

$$\begin{aligned} v(t) &= \wp'(t)^{-2}(\wp(t) - e_1)(\wp(t) - e_2)(c_1\wp(t)^2 + c_2\wp(t) + c_3) \\ &= \frac{c_1\wp(t)^2 + c_2\wp(t) + c_3}{4(\wp(t) - e_3)} \end{aligned}$$

is a linear combination of 1, $\wp(t)$ and $\wp(t + \omega_3)$. Since

$$(2.21) \quad \begin{aligned} \frac{(\wp(\frac{t}{2}) - e_3)^2}{\wp'(\frac{t}{2})^2} &= \frac{1}{4}\left(\frac{e_1 - e_3}{e_1 - e_2}\frac{1}{\wp(\frac{t}{2}) - e_1} + \frac{e_2 - e_3}{e_2 - e_1}\frac{1}{\wp(\frac{t}{2}) - e_2}\right) \\ &= \frac{1}{4(e_1 - e_2)^2}\left(\wp(\frac{t}{2} + \omega_1) - e_1 + \wp(\frac{t}{2} + \omega_2) - e_2\right) \\ &= \frac{1}{4(e_1 - e_2)^2}\left(\wp(\frac{t}{2} + \omega_1|2\omega_1, \omega_3) + 2e_3\right) \\ &= \frac{1}{(e_1 - e_2)^2}\left(\wp(t + 2\omega_1|4\omega_1, 2\omega_3) + \frac{e_3}{2}\right) \end{aligned}$$

has a period $2\omega_3$, we may assume $v(t) = \wp(t)$ by Lemma 2.5 iv). By Lemma 2.5 iii) we can reduce to the case $u(t) = \wp(t)$ and $v(t) = \frac{1}{4(e_1 - e_2)^2}(\wp(t + \omega_1) + \wp(t + \omega_2) - e_1 - e_2)$, which has already treated in ii). $\square$

Remark 2.8. i) The solutions in §1.3 corresponds to (2.20) with the following conditions:

$$
\begin{array}{lll}
2) & c_6 = 0, & e_1 \neq e_2 \neq e_3 \neq e_1, \\
2') & c_6 = 0, & e_1 = -\tfrac{2}{3}\lambda^2 \neq 0, \quad e_2 = e_3 = \tfrac{1}{3}\lambda^2, \\
2'') & c_6 = 0, & e_1 = e_2 = e_3 = 0, \\
3) & c_4 = c_5 = 0, & e_1 \neq e_2 \neq e_3 \neq e_1, \\
3') & c_4 = c_5 = 0, & e_1 = -\tfrac{2}{3}\lambda^2 \neq 0, \quad e_2 = e_3 = \tfrac{1}{3}\lambda^2, \\
3')^d & c_4 = c_5 = 0, & e_1 = e_2 = \tfrac{1}{3}\lambda^2 \neq 0, \quad e_3 = -\tfrac{2}{3}\lambda^2, \\
3'') & c_4 = c_5 = 0, & e_1 = e_2 = e_3 = 0.
\end{array}
$$

ii) The family of solutions with $c_4 = c_5 = 0$ are written in a more symmetric form under the symmetry in Lemma 2.5 iii). By the proof of Theorem 2.7 iii) we can write

$$
(2.22) \qquad
\begin{aligned}
u(t) &= a_1 \wp(t|4\omega_1, 2\omega_3) + a_2 \wp(t|2\omega_1, 2\omega_3) + a_3, \\
v(t) &= b_1 \wp(t|2\omega_1, 2\omega_3) + b_2 \wp(t|2\omega_1, \omega_3) + b_3.
\end{aligned}
$$

Then the solution $(\bar{u}(t), \bar{v}(t)) = (v(t), u(2t))$ can be expressed in the same form as (2.22) by replacing $2\bar\omega_1 = \omega_3$, $2\bar\omega_3 = 2\omega_1$, $\bar{a}_1 = b_1$, $\bar{a}_2 = b_2$, $\bar{a}_3 = b_3$, $\bar{b}_1 = a_1/4$, $\bar{b}_2 = a_2/4$ and $\bar{b}_3 = a_3$.

2.4. The main theorem.

In this subsection we shall solve the functional differential equation (2.1) by the aid of a computer with the algebraic programming system REDUCE Ver.3.4. The following is the main result in §2, which is proved at the end of § 2.5.4:

Theorem 2.9. *Any solution $(u(t), v(t))$ of the equation (2.1) such that $u(t)$ and $v(t)$ are real analytic on $\{t \in \mathbb{R} \mid 0 < |t| \ll 1\}$ is one of the following form.*

   i) *Functions $(u(t), v(t))$ is of the form in* Theorem 2.7 *with $c_4 c_6 = c_5 c_6 = 0$.*

   ii) *Functions $(v(t), u(2t))$ is of the form in* Theorem 2.7 *with $c_4 c_6 = c_5 c_6 = 0$.*

   iii) *Either u or v is constant.*

   iv) *$u' = 0$ and v'' is constant.*

   v) *$v' = 0$ and u'' is constant.*

Here we note that if $u(t)$ and $v(t)$ are even or they are holomorphic on $\{t \in \mathbb{C} \mid 0 < |t| \ll 1\}$, then iv) and v) are reduced to iii).

2.4.1. The following lemma is a generalization of [OS, Lemma 7.1 i)].

Lemma 2.10. *Let $u(t)$ and $v(t)$ be real analytic functions on $\{t \in \mathbb{R} \mid 0 < |t| \ll 1\}$ which satisfy (2.1). Suppose $u' \neq 0$ and $v' \neq 0$. Then $u(t)$ and $v(t)$ can be extended to even meromorphic functions on $\{t \in \mathbb{C} \mid 0 < |t| \ll 1\}$ with poles of order at most 2 at the origin.*

Proof. We may assume $v'|_{t>0} \neq 0$ by replacing the following x by $-x$ if necessary. Fix x with $0 < x \ll 1$ and consider the Laurent expansion for $0 < |y| \ll x$

$$
(2.23) \qquad u(x+y) - u(x-y) = 2\left(\frac{u^{(1)}(x)}{1!} y + \frac{u^{(3)}(x)}{3!} y^3 + \cdots \right).
$$

Then we have

$$
(2.24)
\begin{aligned}
&\frac{\partial}{\partial x}\left(v'(x) \sum_{k=0}^{\infty} \frac{u^{(2k+1)}(x)}{(2k+1)!} y^{2k+1} + 2v(x) \sum_{k=0}^{\infty} \frac{u^{(2k+2)}(x)}{(2k+1)!} y^{2k+1} \right) \\
&\qquad - \frac{\partial}{\partial y}\left(v'(y) \sum_{k=0}^{\infty} \frac{u^{(2k+1)}(x)}{(2k+1)!} y^{2k+1} + 2v(y) \sum_{k=0}^{\infty} \frac{u^{(2k+1)}(x)}{(2k)!} y^{2k} \right) = 0
\end{aligned}
$$

and for $0 < |y| \ll x$

(2.25) $\quad f(x,y) = y\big(u'(x) + yc_2(x,y)\big)v''(y) + 3\big(u'(x) + yc_1(x,y)\big)v'(y) + c_0(x,y)v(y)$

with a suitable holomorphic functions $f(x,y)$, $c_0(x,y)$, $c_1(x,y)$ and $c_2(x,y)$ of y defined on a neighborhood of the origin. Since this equation for $v(y)$ has regular singularities at the origin with the characteristic exponents 0 and -2,

(2.26) $\qquad\qquad v(t) = a_{-1}t^{-2} + v_0(t) + v_1(t)\log t \quad$ for $0 < t \ll 1$.

Here $v_0(t)$ and $v_1(t)$ are holomorphic function defined in a neighborhood of the origin and moreover $v_1(0) = 0$ means $v_1 = 0$.

By the analytic continuation of (2.24) for the variable y around the origin we have

(2.27) $\qquad \dfrac{\partial}{\partial y}\left(v_1'(y)\displaystyle\sum_{k=0}^{\infty}\dfrac{u^{(2k+1)}(x)}{(2k+1)!}y^{2k+1} + 2v_1(y)\displaystyle\sum_{k=0}^{\infty}\dfrac{u^{(2k+1)}(x)}{(2k)!}y^{2k}\right) = 0.$

The coefficients of y^1 in this equation mean

(2.28) $\qquad\qquad 2v_1(0)u^{(3)}(x) + 4v_1''(0)u'(x) = 0.$

Suppose $v_1 \neq 0$. Let λ be a complex number with $\lambda^2 = -2v_1''(0)/v_1(0)$.

(2.29) $\qquad\qquad u^{(3)}(x) = \lambda^2 u'(x).$

Then (2.27) is

(2.30) $\qquad \dfrac{\partial}{\partial y}\left(v_1'(y)u'(x)\dfrac{\sinh \lambda y}{\lambda} + 2v_1(y)u'(x)\cosh \lambda y\right) = 0.$

For $u'(x_0) \neq 0$

$$\frac{\partial}{\partial y}\left(v_1'(y)\left(\frac{\sinh \lambda y}{\lambda}\right)^2\right) = 0,$$

$$v_1'(y)\left(\frac{\sinh \lambda y}{\lambda}\right)^2 = v_1'(0)0 = 0,$$

then $v_1 = 0$, which contradicts to the assumption $v_1 \neq 0$.

Thus we have proved that $v_1 = 0$. By (2.26) we can put

$$v(t) = a_{-1}t^{-2} + \sum_{j=0}^{\infty}\big(a_j t^{2j} + c_j t^{2j+1}\big)$$

with suitable a_j, $c_j \in \mathbb{C}$ on $0 < t \ll 1$. Suppose there exist c_k satisfying $c_k \neq 0$ and $c_j = 0$ for $j = 0,\dots,k-1$. Then the coefficients of y^{2k} in (2.24) shows

$$-\big((2k+1)^2 + 2(2k+1)\big)c_k u^{(1)}(x) = 0,$$

which contradicts to the assumption $c_k \neq 0$ and hence $v(t) = a_{-1}t^{-2} + \sum_{j=0}^{\infty}a_j t^{2j}$ on $0 < t \ll 1$. Here we note that $v'' \neq 0$ and that $u'' \neq 0$ by the symmetry of u and v.

Substituting (x_1, x_2) in (2.1) by (x, y) and $(x, -y)$, respectively, and summing up the resulting equations, we have

$$\frac{\partial}{\partial y}\Big(\big(v'(y)+v'(-y)\big)\big(u(x+y)-u(x-y)\big)+2\big(v(y)-v(-y)\big)\big(u'(x+y)+u'(x-y)\big)\Big) = 0$$

and hence

$$\frac{\partial^2}{\partial y^2}\Big(\big(v(y) - v(-y)\big)\big(u(x + y) - u(x - y)\big)^2\Big) = 0.$$

Thus we have $v(-y) = v(y)$ because $u'' \neq 0$.

By the symmetry of $u(t)$ and $v(t)$ we have the lemma. $\square$

First suppose that $u(t)$ and $v(t)$ are real analytic functions on $\{t \in \mathbb{R} \,|\, 0 < |t| \ll 1\}$. It is clear that (u, v) given by iii) or iv) or v) in Theorem 2.9 satisfies (2.1). Assume $u' = 0$. Then there exist C_1, $C_2 \in \mathbb{C}$ such that $u(t) = C_1$ and $u(-t) = C_2$ for $0 < t \ll 1$. Suppose (u, v) satisfies (2.1) and suppose $C_1 \neq C_2$ and let $0 < x < y \ll 1$. Substituting (x_1, x_2) in (2.1) by (x, y), $(-x, -y)$ and $(-x, y)$, we have $v''(y) = v''(x)$, $v''(-y) = v''(-x)$ and $v''(y) = v''(-x)$, respectively, and therefore v'' is constant. In the same way, if $v' = 0$ and (u, v) satisfies (2.1), then v is constant or u'' is constant.

Then owing to Lemma 2.10 we assume $u(t)$ and $v(t)$ are holomorphic on e $\{t \in \mathbb{C} \,|\, 0 < |t| \ll 1\}$ and satisfy (2.1) to the end of this section. By Lemma 2.10, the Laurent expansion at the origin can be assumed as follows.

$$(2.31) \qquad u(t) = a_{-1}t^{-2} + \sum_{j=1}^{\infty} a_j t^{2j}, \quad v(t) = b_{-1}t^{-2} + \sum_{j=1}^{\infty} b_j t^{2j}.$$

Suppose $0 < |y| \ll |x| \ll 1$. It follows from (2.23) that

$$(2.32) \qquad \frac{\partial^2}{\partial x \partial y}\Bigg(v'(x)\sum_{k=0}^{\infty}\frac{u^{(2k+1)}(x)}{(2k+2)!}y^{2k+2} + 2v(x)\sum_{k=0}^{\infty}\frac{u^{(2k+2)}(x)}{(2k+2)!}y^{2k+2}$$

$$- \Big(\sum_{j=-1}^{\infty}2jb_j y^{2j-1}\Big)\sum_{k=0}^{\infty}\frac{u^{(2k)}(x)}{(2k+1)!}y^{2k+1}$$

$$- \Big(\sum_{j=-1}^{\infty}2b_j y^{2j}\Big)\sum_{k=0}^{\infty}\frac{u^{(2k)}(x)}{(2k)!}y^{2k}\Bigg) = 0.$$

Since the coefficient of the term $b_j u^{(2m-2j)}(x)y^{2m}$ inside the above $\Big(\quad\Big)$ equals

$$-\frac{2j}{(2m - 2j + 1)!} - \frac{2}{(2m - 2j)!} = -2\frac{2m - j + 1}{(2m - 2j + 1)!},$$

for any positive integer m, we obtain
(2.33)

$$u^{(2m-1)}(x)v'(x) + 2u^{(2m)}(x)v(x) - \sum_{j=-1}^{m}\frac{2(2m)!(2m - j + 1)}{(2m - 2j + 1)!}b_j u^{(2m-2j)}(x) = C_m$$

with suitable constant numbers C_m.

Let $X(m,k)$ denote the the coefficients of x^{2k} in the left hand side of (2.33). Then the condition $X(m,k) = 0$ for all $m \geq 1$ and $k \geq 1$ is equivalent to (2.33), and so is to (2.1).

For example, we have the following, all of which will be used in the proof of Theorem 2.9.

$$X(1,1) = 0,$$

$$X(1,2) = 4(3a_1b_2 + 6a_2b_1 - 32a_4b_{-1} - a_{-1}b_4),$$

$$X(1,3) = 8(2a_1b_3 + 5a_2b_2 + 8a_3b_1 - 64a_5b_{-1} - a_{-1}b_5),$$

$$X(1,4) = 4(5a_1b_4 + 12a_2b_3 + 21a_3b_2 + 30a_4b_1 - 336a_6b_{-1} - 3a_{-1}b_6),$$

$$X(1,5) = \tfrac{8}{5}(15a_1b_5 + 35a_2b_4 + 60a_3b_3 + 90a_4b_2 + 120a_5b_1$$
$$- 1792a_7b_{-1} - 10a_{-1}b_7),$$

$$X(1,6) = 4(7a_1b_6 + 16a_2b_5 + 27a_3b_4 + 40a_4b_3 + 55a_5b_2 + 70a_6b_1$$
$$- 1344a_8b_{-1} - 5a_{-1}b_8),$$

$$X(1,7) = 8(4a_1b_7 + 9a_2b_6 + 15a_3b_5 + 22a_4b_4 + 30a_5b_3 + 39a_6b_2 + 48a_7b_1$$
$$- 1152a_9b_{-1} - 3a_{-1}b_9),$$

$$X(1,8) = 4(9a_1b_8 + 20a_2b_7 + 33a_3b_6 + 48a_4b_5 + 65a_5b_4 + 84a_6b_3 + 105a_7b_2$$
$$+ 126a_8b_1 - 3696a_{10}b_{-1} - 7a_{-1}b_{10}),$$

$$X(1,9) = 8(5a_1b_9 + 11a_2b_8 + 18a_3b_7 + 26a_4b_6 + 35a_5b_5 + 45a_6b_4 + 56a_7b_3,$$
$$+ 68a_8b_2 + 80a_9b_1 - 2816a_{11}b_{-1} - a_{-1}4b_{11}),$$

$$X(2,1) = 48(-3a_1b_2 - 6a_2b_1 + 32a_4b_{-1} + a_{-1}b_4),$$

$$X(2,2) = 0,$$

$$X(2,3) = 16(12a_2b_3 + 66a_3b_2 + 140a_4b_1 - 1056a_6b_{-1} - 3a_{-1}b_6),$$

$$X(2,4) = 48(5a_2b_4 + 30a_3b_3 + 95a_4b_2 + 180a_5b_1 - 1664a_7b_{-1} - 2a_{-1}b_7),$$

$$X(2,5) = 48(6a_2b_5 + 35a_3b_4 + 112a_4b_3 + 267a_5b_2 + 462a_6b_1$$
$$- 5184a_8b_{-1} - 3a_{-1}b_8),$$

$$X(2,6) = 16(21a_2b_6 + 120a_3b_5 + 378a_4b_4 + 900a_5b_3 + 1806a_6b_2 + 2912a_7b_1$$
$$- 39168a_9b_{-1} - 12a_{-1}b_9),$$

$$X(2,7) = \tfrac{48}{7}(56a_2b_7 + 315a_3b_6 + 980a_4b_5 + 2310a_5b_4 + 4620a_6b_3$$
$$+ 8260a_7b_2 + 12600a_8b_1 - 200640a_{10}b_{-1} - 35a_{-1}b_{10}),$$

$$X(2,8) = 48(+9a_2b_8 + 50a_3b_7 + 154a_4b_6 + 360a_5b_5 + 715a_6b_4 + 1274a_7b_3$$
$$+ 2097a_8b_2 + 3060a_9b_1 - 57024a_{11}b_{-1} - 6a_{-1}b_{11}),$$

$$X(3,1) = 2880(-2a_1b_3 - 5a_2b_2 - 8a_3b_1 + 64a_5b_{-1} + a_{-1}b_5),$$

$$X(3,2) = 480(-12a_2b_3 - 66a_3b_2 - 140a_4b_1 + 1056a_6b_{-1} + 3a_{-1}b_6),$$

$$X(3,3) = 0,$$

$$X(3,4) = 1440(5a_3b_4 + 52a_4b_3 + 219a_5b_2 + 462a_6b_1 - 4160a_8b_{-1} - a_{-1}b_8),$$

$$X(3,5) = 192(45a_3b_5 + 490a_4b_4 + 2490a_5b_3 + 8085a_6b_2 + 16016a_7b_1$$
$$- 163200a_9b_{-1} - 15a_{-1}b_9),$$

$$X(3,6) = 480(21a_3b_6 + 224a_4b_5 + 1134a_5b_4 + 3948a_6b_3 + 10556a_7b_2$$
$$+ 19656a_8b_1 - 227392a_{10}b_{-1} - 9a_{-1}b_{10}),$$

$$X(3,7) = 5760(2a_3b_7 + 21a_4b_6 + 105a_5b_5 + 363a_6b_4 + 1000a_7b_3$$
$$+ 2316a_8b_2 + 4080a_9b_1 - 53504a_{11}b_{-1} - a_{-1}b_{11}),$$

$$X(4,1) = 80640(-5a_1b_4 - 12a_2b_3 - 21a_3b_2 - 30a_4b_1 + 336a_6b_{-1} + 3a_{-1}b_6),$$

$$X(4,2) = 80640(-5a_2b_4 - 30a_3b_3 - 95a_4b_2 - 180a_5b_1 + 1664a_7b_{-1} + a_{-1}2b_7),$$

$$X(4,3) = 80640(-5a_3b_4 - 52a_4b_3 - 219a_5b_2 - 462a_6b_1$$
$$+ 4160a_8b_{-1} + a_{-1}b_8),$$

$$X(4,4) = 0,$$

$$X(4,5) = 16128(30a_4b_5 + 500a_5b_4 + 3300a_6b_3 + 12298a_7b_2 + 25740a_8b_1$$
$$- 258400a_{10}b_{-1} - 5a_{-1}b_{10}),$$

$$X(4,6) = 80640(7a_4b_6 + 120a_5b_5 + 886a_6b_4 + 4108a_7b_3 + 13182a_8b_2$$
$$+ 26520a_9b_1 - 289408a_{11}b_{-1} - 2a_{-1}b_{11}),$$

We borrow the following notation from REDUCE. For a polynomial function p, we denote by $\mathrm{coeffn}(p, x, k)$ the coefficient of the term x^k of p with respect to one specific variable x. For example, $\mathrm{coeffn}(x^2 + 2xy + 3x + y^2, x, 1) = 2y + 3$.

2.4.2. Now we shall prove Theorem 2.9 dividing into the cases classified by the order of zeros of $(u(t), v(t))$. Owing to the symmetry between u and v, [OS, Lemma 7.1 ii)] shows that we may assume the pair of orders of the zeros equal $(-2, 6)$, $(-2, 4)$, $(2, 2)$, $(-2, 2)$ or $(-2, -2)$.

Type (-2,6).

We may assume $a_{-1} = b_3 = 1$ and $b_{-1} = b_1 = b_2 = 0$. For $k \geq 5$ we have

$$\begin{pmatrix} \mathrm{coeffn}(X(1, k-2), a_{k-4}, 1) & \mathrm{coeffn}(X(1, k-2), b_k, 1) \\ \mathrm{coeffn}(X(2, k-3), a_{k-4}, 1) & \mathrm{coeffn}(X(2, k-3), b_k, 1) \end{pmatrix}$$
$$= \begin{pmatrix} (2k-8)(2k-6) & (-2)(2k-6) \\ (2k-8)(2k-9)(2k-8)(2k-10) & (-2)(-3)(-4)(2k-10) \end{pmatrix}.$$

The determinant of this matrix equals

$$4(2k-5)(2k-6)(2k-8)(2k-10)(2k-12).$$

Hence if $k \geq 7$, the equations $X(1, k-2) = X(2, k-3) = 0$ assure that a_{k-4} and b_k are expressed suitable linear combinations of $a_{k-j-1}b_j$ with $j = 4, \ldots, k-2$, which proves that a_{k-4} and b_k with $k \geq 7$ are expressed by polynomial functions of $(a_1, a_2, b_1, b_4, b_5, b_6)$ by the induction on k.

Now we note that $X(1,2) = 0$ implies $b_4 = 0$. Moreover it follows from $X(1,3) = X(1,4) = 0$ that b_5 and b_6 are expressed by polynomial functions of (a_1, a_2, b_4). Hence we have proved that all the coefficients a_j and b_j are uniquely expressed by polynomial functions of (a_1, a_2). In particular for any given $(a_1, a_2) \in \mathbb{C}^2$ the solution is unique if it exists.

On the other hand we have the solution

$$u(t) = \wp(t) = t^{-2} + \frac{g_2}{20}t^2 + \frac{g_3}{28}t^4 + \cdots,$$

$$v(t) = \frac{4}{\wp'(t)^2} = t^6 + \frac{g_2}{10}t^{10} + \cdots.$$

Hence the coefficients a_j and b_j which are uniquely determined by (a_1, a_2) equal the coefficients of the above $u(t)$ and $v(t)$ with $g_2 = 20a_1$ and $g_3 = 28a_2$.

2.4.3. Type (−2,4).

We may assume $a_{-1} = b_2 = 1$ and $b_{-1} = b_1 = 0$. Then for $k \geq 4$

$$
\begin{pmatrix}
\mathrm{coeffn}(X(1, k-2), a_{k-3}, 1) & \mathrm{coeffn}(X(1, k-2), b_k, 1) \\
\mathrm{coeffn}(X(2, k-3), a_{k-3}, 1) & \mathrm{coeffn}(X(2, k-3), b_k, 1)
\end{pmatrix}
$$

$$
= \begin{pmatrix}
2(2k-6)(2k-5) & (-2)(2k-6) \\
2(2k-4)(2k-10)(4k^2 - 28k + 57) & (-2)(-3)(-4)(2k-10)
\end{pmatrix}.
$$

Since the determinant of this matrix is

$$
4(2k-3)(2k-6)(2k-7)(2k-8)(2k-10),
$$

a_{k-3} and b_k for $k \geq 6$ are uniquely determined by $(a_1, a_2, b_3, b_4, b_5)$. Moreover $X(1,2) = X(1,3) = 0$ imply that b_4 and b_5 are uniquely determined by (a_1, a_2, b_3). On the other hand, we have the solution

$$
u(t) = \wp(t) = t^{-2} + \frac{g_2}{20} t^2 + \frac{g_3}{28} t^4 + \cdots,
$$
$$
v(t) = \frac{4(\wp(t) + C_5)}{\wp'(t)^2} = t^4 + C_5 t^6 + \cdots
$$

with parameters g_2, g_3 and C_5. Thus the coefficients a_k and b_k uniquely determined by (a_1, a_2, b_3) corresponds to this solution with $g_2 = 20a_1$, $g_3 = 28a_2$ and $C_5 = b_3$.

2.4.4. Type (2,2).

We may assume $a_1 = b_1 = 1$ and $a_{-1} = b_{-1} = 0$. For $k \geq 4$

$$
\begin{pmatrix}
\mathrm{coeffn}(X(1, k-2), a_{k-2}, 1) & \mathrm{coeffn}(X(1, k-2), b_{k-2}, 1) \\
\mathrm{coeffn}(X(2, k-3), a_{k-2}, 1) & \mathrm{coeffn}(X(2, k-3), b_{k-2}, 1)
\end{pmatrix}
$$

$$
= \begin{pmatrix}
2(2k-2)(2k-6) & 2(2k-2) \\
2(2k-2)(2k-4)(2k-5)(2k-10) & 0
\end{pmatrix}
$$

and the determinant of this matrix equals

$$
-4(2k-2)^2(2k-4)(2k-5)(2k-10).
$$

Hence a_{k-2} and b_{k-2} for $k \geq 6$ are uniquely determined by (a_2, a_3, b_2, b_3).

Moreover since $X(1,2) = X(1,3) = 0$, for any given (a_2, a_3) the solution is unique if it exists and therefore it corresponds to the solution

$$
u(t) = \frac{16\big(\wp(\frac{t}{2}) - e_3\big)^2}{\wp'(\frac{t}{2})^2} = t^2 - \frac{e_3}{2} t^4 + \frac{1}{16}\Big(\frac{g_2}{5} + e_3^2\Big) t^6 + \cdots,
$$
$$
v(t) = \frac{1}{\wp(t) - e_3} = t^2 + \cdots.
$$

with $e_3 = -2a_2$ and $g_2 = 5(16a_3 - e_3^2)$.

2.4.5. Type (–2,2).

We may assume $a_{-1} = b_1 = 1$ and $b_{-1} = 0$. For $k \geq 4$,

$$
\begin{pmatrix}
\mathrm{coeffn}(X(1, k - 2), a_{k-2}, 1) & \mathrm{coeffn}(X(1, k - 2), b_k, 1) \\
\mathrm{coeffn}(X(2, k - 3), a_{k-2}, 1) & \mathrm{coeffn}(X(2, k - 3), b_k, 1)
\end{pmatrix}
$$

$$
= \begin{pmatrix}
2(2k - 2)(2k - 6) & (-2)(2k - 6) \\
2(2k - 2)(2k - 4)(2k - 5)(2k - 10) & (-2)(-3)(-4)(2k - 10)
\end{pmatrix}
$$

and the determinant of this matrix equals

$$
4(2k - 1)(2k - 2)(2k - 6)(2k - 8)(2k - 10).
$$

Hence a_{k-2} and b_k for $k \geq 6$ are uniquely determined by $(a_1, a_2, a_3, a_4, a_5, b_2, b_3)$. Owing to this with $X(1, 2) = X(1, 3) = 0$, for any given $(a_1, a_2, a_3, b_2, b_3)$ the solution is unique if it exists.

Now putting

$$
a_3 = c_3 + \tfrac{1}{3} a_1^2,
$$

we write $X(3, 4)$ and $X(3, 5)$ by the variables $(a_1, a_2, c_3, b_2, b_3)$:

$$
X(3, 4) = 12096 c_3(-7 a_1 b_2 - 14 a_2 - b_2^3 + 3 b_2 b_3),
$$

$$
X(3, 5) = 12096 c_3(-16 a_1^2 + 29 a_1 b_2^2 - 12 a_1 b_3 + 28 a_2 b_2 + 3 b_2^4 - 11 b_2^2 b_3
$$

$$
+ 4 b_3^2 - 48 c_3).
$$

First suppose $c_3 = 0$. Then the solution is uniquely determined by (a_1, a_2, b_2, b_3), which corresponds to the solution

$$
u(t) = \wp(t) = t^{-2} + \frac{g_2}{20} t^2 + \frac{g_3}{28} t^4 + \cdots,
$$

$$
v(t) = \frac{1}{\wp(t) - e_3} + \frac{4(C_4 \wp(t) + C_5)}{\wp'(t)^2}
$$

$$
= t^2 + (e_3 + C_4) t^4 + (C_5 + e_3^2 - \frac{g_2}{20}) t^6 + \cdots
$$

with $g_2 = 20 a_1$, $g_3 = 28 a_2$, $C_4 = b_2 - e_3$ and $C_5 = b_3 - e_3^2 + \frac{g_2}{20}$.

Next suppose $c_3 \neq 0$. Then it follows from $X(3, 4) = X(3, 5) = 0$ that (a_2, c_3) is uniquely determined by (a_1, b_2, b_3). Hence the solution is uniquely determined by (a_1, b_2, b_3), which corresponds to the solution

$$
u(t) = \wp(t) + 16 C_6 \frac{\left(\wp(\frac{t}{2}) - e_3\right)^2}{\wp'(\frac{t}{2})^2} = t^{-2} + (C_6 + \frac{g_2}{20}) t^2 + \cdots,
$$

$$
v(t) = \frac{1}{\wp(t) - e_3} = t^2 + e_3 t^4 + (e_3^2 - \frac{g_2}{20}) t^6 + \cdots
$$

with $e_3 = b_2$, $g_2 = 20(e_3^2 - b_3)$ and $C_6 = a_1 - \frac{g_2}{20}$.

2.5. Type (–2,–2).

We shall do a similar but more complicated calculation for the type $(-2, -2)$. In § 2.5.3 and 2.5.6 we also use REDUCE.

2.5.1. We may assume $a_{-1} = b_{-1} = 1$. For $k \geq 4$

$$\begin{pmatrix} \mathrm{coeffn}(X(1,k-2),a_k,1) & \mathrm{coeffn}(X(1,k-2),b_k,1) \\ \mathrm{coeffn}(X(2,k-3),a_k,1) & \mathrm{coeffn}(X(2,k-3),b_k,1) \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{4}{15}k(2k-2)(2k-6)(2k+2) & (-2)(2k-6) \\ \frac{4}{35}k(2k-1)(2k-2)(2k-4)(2k+2)(2k-10) & (-2)(-3)(-4)(2k-10) \end{pmatrix}.$$

The determinant of this matrix equals

$$-\frac{4}{35}2k(2k+2)(2k+3)(2k-2)(2k-6)(2k-8)(2k-10).$$

Hence a_k and b_k with $k \geq 6$ are uniquely determined by $(a_1, \ldots, a_5, b_1, \ldots, b_5)$. Moreover b_4 and b_5 are expressed by polynomial functions of $(a_1, \ldots, a_5, b_1, b_2, b_3)$ by using the equations $X(1,2) = X(1,3) = 0$.

Thus a_i and b_j with $i \geq 6$ and $j \geq 4$ are determined by polynomial functions of

$$(a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3).$$

Here we have used all of $X(1,1), X(1,2), \ldots$ and $X(2,1), X(2,2), \ldots$.
   We put

$$(2.34) \qquad \begin{aligned} a_3 &= c_3 + \tfrac{1}{3}a_1^2, \\ b_3 &= d_3 + \tfrac{1}{3}b_1^2, \\ a_4 &= c_4 + \tfrac{3}{11}a_1a_2, \\ a_5 &= c_5 + \tfrac{2}{39}a_1^3 + \tfrac{1}{13}a_2^2. \end{aligned}$$

Then all coefficients are suitable polynomial functions of

$$(a_1, a_2, c_3, c_4, c_5, b_1, b_2, d_3).$$

   Similarly by denoting

$$\begin{aligned} b_4 &= d_4 + \tfrac{3}{11}b_1b_2, \\ b_5 &= d_5 + \tfrac{2}{39}b_1^3 + \tfrac{1}{13}b_2^2, \end{aligned}$$

we have

$$\begin{aligned} d_4 &= \tfrac{3}{11}\left(-32a_1a_2 + 11a_1b_2 + 22a_2b_1 - b_1b_2\right) - 32c_4, \\ d_5 &= \tfrac{1}{39}\left(-128a_1^3 + 104a_1^2b_1 + 26a_1b_1^2 + 78a_1d_3 - 192a_2^2 + 195a_2b_2 \right. \\ &\quad \left. - 2b_1^3 + 312b_1c_3 - 3b_2^2\right) - 64c_5. \end{aligned}$$

Here we remark that $c_3 = c_4 = c_5 = 0$ (resp. $d_3 = d_4 = d_5 = 0$) if u (resp. v) is the Weierstrass function $\wp$.

2.5.2. Before going into the detail we prepare several notations.

$$V := \{(a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3) \in \mathbb{C}^8 \mid \text{a solution } (u,v)$$
$$\text{of the form } (2.31) \text{ with } a_{-1} = b_{-1} = 1 \text{ satisfies } (2.1)\}.$$

Since the map defined by (2.31) and (2.34)

$$V \ni (u,v) \mapsto (a_1, a_2, c_3, c_4, c_5, b_1, b_2, d_3) \in \mathbb{C}^8$$

is injective, we will consider V as a subset of $\mathbb{C}^8$, which is a closed subvariety.

Lemma 2.11. i) *The solutions*

$$
\begin{aligned}
u(t) &= \wp(t) + 16C\frac{(\wp(\frac{t}{2})-e_3)^2}{\wp'(\frac{t}{2})^2} \\
&= t^{-2} + (\tfrac{g_2}{20} + C)t^2 + (\tfrac{g_3}{28} - \tfrac{e_3 C}{2})t^4 + (\tfrac{g_2^2}{1200} + \tfrac{1}{16}(e_3^2 + \tfrac{g_2}{5})C)t^6 \\
&\quad + (\tfrac{3g_2 g_3}{6160} + \tfrac{3}{64}(\tfrac{g_3}{14} - \tfrac{g_2 e_3}{10})C)t^8 + \cdots, \\
v(t) &= \wp(t) + \frac{C'}{\wp(t)-e_3} \\
&= t^{-2} + (\tfrac{g_2}{20} + C')t^2 + (\tfrac{g_3}{28} + e_3 C')t^4 + (\tfrac{g_2^2}{1200} + (e_3^2 - \tfrac{g_2}{20})C')t^6 \\
&\quad + (\tfrac{3g_2 g_3}{6160} + \tfrac{3}{64}(e_3^3 - \tfrac{g_3}{28} - \tfrac{g_2 e_3}{10})C')t^8 + \cdots
\end{aligned}
$$

(2.35)

belong to V. Therefore we can define a map

$$\Psi_3 : (e_1 + e_2, e_1 e_2, C, C') \in \mathbb{C}^4 \to V.$$

ii) *We can define a $\mathbb{C}^\times$-action by*

(2.36)
$$
\begin{aligned}
&\lambda.(A, B, C, C') = (\lambda A, \lambda^2 B, \lambda^2 C, \lambda^2 C'), \\
&\lambda.(a_1, a_2, c_3, c_4, c_5, b_1, b_2, d_3) = (\lambda^2 a_1, \lambda^3 a_2, \lambda^4 c_3, \lambda^5 c_4, \lambda^6 c_5, \lambda^2 b_1, \lambda^3 b_2, \lambda^4 d_3)
\end{aligned}
$$

so that Ψ_3 is $\mathbb{C}^\times$-equivariant. Moreover $\Psi_3^{-1}(0) = 0$.

iii) *The solutions*

(2.37)
$$
\begin{aligned}
u(t) &= \wp(t), \\
v(t) &= \frac{4\wp(t)^4 + C\wp(t)^2 + C'\wp(t) + C''}{\wp'(t)^2}
\end{aligned}
$$

belong to V. Then we have a map

$$\Psi_1 : (g_2, g_3, C, C', C'') \in \mathbb{C}^5 \to V.$$

Similarly the solutions

(2.38)
$$
\begin{aligned}
u(t) &= \frac{\wp(\frac{t}{2})^4 + C\wp(\frac{t}{2})^2 + C'\wp(\frac{t}{2}) + C''}{\wp'(\frac{t}{2})^2}, \\
v(t) &= \wp(t)
\end{aligned}
$$

belong to V, which define a map

$$\Psi_2 : (g_2, g_3, C, C', C'') \in \mathbb{C}^5 \to V.$$

We have a $\mathbb{C}^\times$-action

$$\lambda.(g_2, g_3, C, C', C'') = (\lambda^2 g_2, \lambda^3 g_3, \lambda^2 C, \lambda^3 C', \lambda^4 C'')$$

so that Ψ_1 and Ψ_2 are $\mathbb{C}^\times$-equivariant.

iv) *Cf.* [OS, Proposition 7.3 ii)]

$$
\begin{aligned}
\operatorname{Im}\Psi_1 &= V \cap \{c_3 = c_4 = c_5 = 0\}, \\
\operatorname{Im}\Psi_2 &= V \cap \{d_3 = d_4 = d_5 = 0\}.
\end{aligned}
$$

v) *The maps Ψ_1 and Ψ_2 are injective.*

Proof. i) follows from Theorem 2.7.

ii) The $\mathbb{C}^{\times}$-equivariance is easy. To prove $\Psi_3^{-1}(0) = 0$, suppose $u(t) = v(t) = t^{-2}$ of the form (2.35). If one of e_i is not zero, $u(t)$ and $v(t)$ have a finite period. Since t^{-2} has no period, $e_1 = e_2 = e_3 = 0$. This means $\wp(t) = t^{-2}$. Then one should have $C = C' = 0$.

iii) is similarly proved as in the case of i) and ii).

iv) The Laurent expansion (2.16) of $\wp(t)$ implies that the left hand side is contained in the right hand side. Conversely, for any (a_1, a_2) we can take $g_2 = 20a_1$ and $g_3 = 28a_2$ and moreover for any (b_1, b_2, b_3), we can take (C, C', C'') so that the expansion of $v(t)$ has desired coefficients.

v) For Ψ_1, the Taylor expansion (2.16) of $u(t)$ determines g_2 and g_3. The other coefficients C, C', C'' are determined by the Taylor expansion of $v(t)$. $\square$

2.5.3. By direct calculations we obtain that the vanishing of $X(3,4)$, $X(3,5)$, $X(3,6)$, $X(4,5)$, $X(3,7)$ and $X(4,6)$ are equivalent to

(2.39)
$$f_1 := 96a_1a_2c_3 - 33a_1b_2c_3 - 66a_2b_1c_3 + 3b_1b_2c_3 + 352c_3c_4 - 22c_4d_3 = 0,$$

(2.40)
$$\begin{aligned} f_2 := &\, 128a_1^3c_3 - 104a_1^2b_1c_3 + 1056a_1a_2c_4 - 26a_1b_1^2c_3 - 363a_1b_2c_4 - 54a_1c_3d_3 \\ &+ 192a_2^2c_3 - 726a_2b_1c_4 - 195a_2b_2c_3 + 2b_1^3c_3 + 33b_1b_2c_4 - 312b_1c_3^2 \\ &+ 6b_1c_3d_3 + 3b_2^2c_3 + 2496c_3c_5 + 3872c_4^2 - 156c_5d_3 = 0, \end{aligned}$$

(2.41)
$$\begin{aligned} f_3 := &\, 394240a_1^3c_4 + 446976a_1^2a_2c_3 - 320320a_1^2b_1c_4 - 153648a_1^2b_2c_3 \\ &- 272736a_1a_2b_1c_3 + 1946880a_1a_2c_5 - 80080a_1b_1^2c_4 + 2088a_1b_1b_2c_3 \\ &- 669240a_1b_2c_5 + 1638912a_1c_3c_4 - 268752a_1c_4d_3 + 591360a_2^2c_4 \\ &- 23760a_2b_1^2c_3 - 1338480a_2b_1c_5 - 600600a_2b_2c_4 - 116640a_2c_3d_3 \\ &+ 6160b_1^3c_4 + 1080b_1^2b_2c_3 + 60840b_1b_2c_5 - 834240b_1c_3c_4 \\ &- 16170b_1c_4d_3 + 9240b_2^2c_4 - 10935b_2c_3d_3 + 14826240c_4c_5 = 0, \end{aligned}$$

(2.42)
$$\begin{aligned} f_4 := &\, 305536a_1^3c_4 + 257760a_1^2a_2c_3 - 248248a_1^2b_1c_4 - 88605a_1^2b_2c_3 \\ &- 165978a_1a_2b_1c_3 + 1812096a_1a_2c_5 - 62062a_1b_1^2c_4 + 4194a_1b_1b_2c_3 \\ &- 622908a_1b_2c_5 + 945120a_1c_3c_4 - 225390a_1c_4d_3 + 458304a_2^2c_4 \\ &- 7722a_2b_1^2c_3 - 1245816a_2b_1c_5 - 465465a_2b_2c_4 - 68526a_2c_3d_3 \\ &+ 4774b_1^3c_4 + 351b_1^2b_2c_3 + 56628b_1b_2c_5 - 703560b_1c_3c_4 \\ &- 20328b_1c_4d_3 + 7161b_2^2c_4 - 13122b_2c_3d_3 + 12602304c_4c_5 = 0, \end{aligned}$$

(2.43)
$$\begin{aligned} f_5 := &\, 122880a_1^4c_3 - 89600a_1^3b_1c_3 + 1198080a_1^3c_5 + 4308480a_1^2a_2c_4 \\ &- 33280a_1^2b_1^2c_3 - 973440a_1^2b_1c_5 - 1481040a_1^2b_2c_4 - 120960a_1^2c_3d_3 \\ &+ 1331712a_1a_2^2c_3 - 2085600a_1a_2b_1c_4 - 303696a_1a_2b_2c_3 - 160a_1b_1^3c_3 \\ &- 243360a_1b_1^2c_5 - 166650a_1b_1b_2c_4 - 299520a_1b_1c_3^2 + 7920a_1b_1c_3d_3 \\ &- 92655a_1b_2^2c_3 + 2396160a_1c_3c_5 + 15797760a_1c_4^2 - 767520a_1c_5d_3 \\ &- 773472a_2^2b_1c_3 + 1797120a_2^2c_5 - 602580a_2b_1^2c_4 - 170814a_2b_1b_2c_3 \end{aligned}$$

$$- 1825200a_2b_2c_5 + 4207104a_2c_3c_4 - 690624a_2c_4d_3 + 160b_1^4c_3$$

$$+ 18720b_1^3c_5 + 27390b_1^2b_2c_4 - 24960b_1^2c_3^2 - 1140b_1^2c_3d_3$$

$$+ 8925b_1b_2^2c_3 - 2720640b_1c_3c_5 + 3213760b_1c_4^2 + 1560b_1c_5d_3$$

$$+ 28080b_2^2c_5 + 1019040b_2c_3c_4 - 10230b_2c_4d_3 - 155520c_3^2d_3$$

$$- 4860c_3d_3^2 + 23362560c_5^2 = 0,$$

(2.44)

$$f_6 := 2826240a_1^4c_3 - 2245120a_1^3b_1c_3 + 49121280a_1^3c_5 + 164482560a_1^2a_2c_4$$

$$- 615680a_1^2b_1^2c_3 - 39911040a_1^2b_1c_5 - 56540880a_1^2b_2c_4 - 2782080a_1^2c_3d_3$$

$$+ 33864192a_1a_2^2c_3 - 74865120a_1a_2b_1c_4 - 917136a_1a_2b_2c_3 + 33760a_1b_1^3c_3$$

$$- 9977760a_1b_1^2c_5 - 7996890a_1b_1b_2c_4 - 6888960a_1b_1c_3^2 + 532080a_1b_1c_3d_3$$

$$- 4599135a_1b_2^2c_3 + 55111680a_1c_3c_5 + 603102720a_1c_4^2 - 32816160a_1c_5d_3$$

$$- 20290272a_2^2b_1c_3 + 73681920a_2^2c_5 - 26273940a_2b_1^2c_4 - 8482974a_2b_1b_2c_3$$

$$- 74833200a_2b_2c_5 + 108624384a_2c_3c_4 - 24323904a_2c_4d_3 + 800b_1^4c_3$$

$$+ 767520b_1^3c_5 + 1194270b_1^2b_2c_4 - 124800b_1^2c_3^2 - 102900b_1^2c_3d_3$$

$$+ 425325b_1b_2^2c_3 - 118734720b_1c_3c_5 + 1401276680b_1c_4^2 - 497640b_1c_5d_3$$

$$+ 1151280b_2^2c_5 + 49764000b_2c_3c_4 - 918390b_2c_4d_3 - 4510080c_3^2d_3$$

$$- 315900c_3d_3^2 + 957864960c_5^2 = 0,$$

respectively.

Note that $f_1 = 0$ is equivalent to

(2.45)
$$c_3d_4 + 2d_3c_4 = 0.$$

Lemma 2.12. i) $V \cap \{c_3 = d_3 = 0\} \subset \operatorname{Im} \Psi_1 \cup \operatorname{Im} \Psi_2.$

ii) $V \cap \{c_3 = 0, d_3 \neq 0\} \subset \operatorname{Im} \Psi_1.$

iii) $V \cap \{c_3 \neq 0, d_3 = 0\} \subset \operatorname{Im} \Psi_2.$

Proof. We examine the left hand sides.

i) First note that $f_2 = -121c_4d_4$ when $c_3 = d_3 = 0$. Hence we may assume $c_3 = c_4 = d_3 = 0$ by the symmetry of u and v. In this case $f_3 = -223080c_5d_4$ and $f_5 = -365040c_5d_5$. Hence we have $c_5 = 0$ or $d_4 = d_5 = 0$. By Lemma 2.11 iv) the result holds.

ii) Since $c_4 = 0$ by (2.45), we have $f_2 = -156c_5d_3$. Then we have $c_3 = c_4 = c_5 = 0$.

iii) By the symmetry between u and v, it is reduced to ii). $\square$

2.5.4. The remaining case is $c_3d_3 \neq 0$. Since $V \cap \{c_3d_3 \neq 0\} \cap (\operatorname{Im} \Psi_1 \cup \operatorname{Im} \Psi_2) = \emptyset$, we have to prove

(2.46)
$$V \cap \{c_3d_3 \neq 0\} \subset \operatorname{Im} \Psi_3,$$

which will be proved at the end of this subsection.

Proposition 2.13. *Recall the map* $\Psi_3 : \mathbb{C}^4 \to \mathbb{C}^8$ *in Lemma 2.11. Let Y be a d-dimensional subspace of $\mathbb{C}^4$, L a subspace of $\mathbb{C}^8$ and Ω a Zariski open subset of L such that*

a) $\Psi_3(Y) \subset L$.

b) $\Omega \cap \Psi_3(Y) \neq \emptyset$ *and* $\Psi_3|_{\Psi_3^{-1}(\Omega)\cap Y}$ *is locally injective at a certain point.*

c) $\Omega \cap V$ *is contained in an irreducible d-dimensional subvariety of $V \cap L$.*

Then $\Omega \cap V \subset \operatorname{Im}\Psi_3$.

Proof. By Lemma 2.11 ii),

$$\bar{\Psi}_3 : (Y - \{0\})/\mathbb{C}^\times \to (L - \{0\})/\mathbb{C}^\times$$

is well defined. Then the image of $\bar{\Psi}_3$ is compact, $\Psi_3(Y - \{0\})$ is closed in $L - \{0\}$, $\Psi_3(Y)$ is closed in L and then $\Omega \cap \Psi_3(Y)$ is closed in $\Omega \cap Y$. On the other hand, by the assumption c),

$$\Psi_3(\Psi_3^{-1}(\Omega) \cap Y) \subset \Omega \cap \Psi_3(Y) \subset \Omega \cap V \subset (\text{a } d\text{-dimensional irreducible variety}).$$

By the assumption b), the first term is dense in the last term and then $\Omega \cap \Psi_3(Y)$ is dense in $\Omega \cap V$. Hence $\Omega \cap \Psi_3(Y) = \Omega \cap V$. $\square$

Proposition 2.14. *The following Y, L and Ω satisfy the assumptions* a), b) *and* c) *in Proposition 2.13. Here (A, B, C, C') and $(a_1, a_2, c_3, c_4, c_5, b_1, b_2, d_3)$ are the coordinates of $\mathbb{C}^4$ and $\mathbb{C}^8$, respectively.*

i) $Y = \mathbb{C}^4$, $L = \mathbb{C}^8$ *and* $\Omega = \{c_3 d_3 c_4 d_4 \neq 0\}$.

ii) $Y = \{A = 0\}$, $L = \{a_2 = b_2 = c_4 = 0\}$ *and* $\Omega = \{c_3 d_3(16c_3 - d_3) \neq 0\} \cap L$.

iii) $Y = \{A = 4C - C' = 0\}$, $L = \{a_2 = b_2 = c_4 = 16c_3 - d_3 = 0\}$ *and* $\Omega = \{c_3 d_3 \neq 0, 4a_1 + b_1 \neq 0\} \cap L$.

iv) $Y = \{A = B - 4C - C' = 0\}$, $L = \{a_2 = b_2 = c_4 = 16c_3 - d_3 = 4a_1 + b_1 = 0\}$ *and* $\Omega = \{c_3 d_3 \neq 0\} \cap L$.

Proof. The explicit expression of Ψ_3 shows

$$(2.47) \quad
\begin{aligned}
a_1 &= \tfrac{g_2}{20} + C & &= \tfrac{1}{5}(A^2 - B + 5C), \\
a_2 &= \tfrac{g_3}{28} - \tfrac{e_3 C}{2} & &= \tfrac{1}{14}A(-2B + 7C), \\
c_3 &= -\tfrac{1}{3}C\left(C + \tfrac{1}{16}(g_2 - 3e_3^2)\right) & &= \tfrac{1}{48}C(-A^2 + 4B - 16C), \\
c_4 &= \tfrac{3}{22}e_3 C\left(C + \tfrac{1}{16}(e_1 - e_2)^2\right) & &= \tfrac{3}{352}AC(-A^2 + 4B - 16C), \\
b_1 &= \tfrac{g_2}{20} + C' & &= \tfrac{1}{5}(A^2 - B + 5C'), \\
b_2 &= \tfrac{g_3}{28} + e_3 C' & &= -\tfrac{1}{7}A(B + 7C'), \\
d_3 &= -\tfrac{1}{3}C'(C' + \tfrac{1}{4}(g_2 - 12e_3^2)) & &= \tfrac{1}{3}C'(2A^2 + B - C').
\end{aligned}$$

Hence if $A = 0$, we have

$$(2.48) \quad
\begin{aligned}
a_1 &= C - \tfrac{1}{5}B, \\
c_3 &= -\tfrac{1}{3}C^2 + \tfrac{1}{12}BC, \\
b_1 &= C' - \tfrac{1}{5}B, \\
d_3 &= -\tfrac{1}{3}C'^2 + \tfrac{1}{3}BC', \\
a_2 &= c_4 = b_2 = 0, \\
4a_1 + b_1 &= C' + 4C - B, \\
d_3 - 16c_3 &= \tfrac{1}{3}(4C - C')(4a_1 + b_1)
\end{aligned}$$

which proves a) for ii), iii) and iv). The assumption b) is also clear from (2.47) and (2.48). The assumption c) will be proved in Lemma 2.16, 2.18, 2.19 and 2.20, respectively. □

Proof of Theorem 2.9. As we have already remarked, we have to prove (2.46). By Proposition 2.13 with the help of Proposition 2.14, it is enough to show

$$(2.49) \qquad V \cap \{c_3 d_3 \neq 0, c_4 d_4 = 0\} \subset V \cap \{c_3 d_3 \neq 0, a_2 = b_2 = c_4 = 0\}.$$

This is proved as follows: Take an element in V such that $c_3 d_3 \neq 0$, and $c_4 d_4 = 0$ First note that we have $c_4 = d_4 = 0$ by (2.45). Putting $c_4 = 0$, we have

$$(2.50) \qquad \begin{aligned} &\operatorname{coeffn}(f_1, b_2, 1) f_3 - \operatorname{coeffn}(f_3, b_2, 1) f_1 \\ &= c_3(-11a_1 + b_1) f_3 - 3(-17072a_1^2 c_3 + 232a_1 b_1 c_3 - 74360a_1 c_5 \\ &\quad + 120b_1^2 c_3 + 6760b_1 c_5 - 1215c_3 d_3) f_1 \\ &= 51030a_2 c_3^2 d_3 (32a_1 - 7b_1). \end{aligned}$$

Suppose $a_2 \neq 0$. Then we have $b_1 = \frac{32}{7}a_1$ and therefore $f_1 = -\frac{45}{7}a_1 c_3(32a_2 + 3b_2)$. Since the assumption $b_2 = -\frac{32}{3}a_2$ implies $f_4 = 71442a_2 c_3 d_3 \neq 0$, we have $a_1 = b_1 = 0$ and hence $f_4 = 1458c_3 d_3(-47a_2 - 9b_2)$. Applying $a_1 = b_1 = c_4 = 0$ and $b_2 = -\frac{47}{9}a_2$ to f_3, we obtain $f_3 = -59535a_2 c_3 d_3 \neq 0$, which means a contradiction.

Thus we can conclude $a_2 = 0$ and by the symmetry between u and v, we have $a_2 = b_2 = c_4 = 0$. □

Corollary 2.15. *The subset* $\operatorname{Im} \Psi_1$, $\operatorname{Im} \Psi_2$ *and* $\operatorname{Im} \Psi_3$ *are closed subvarieties.*

Proof. Lemma 2.11 iv) shows that $\operatorname{Im} \Psi_1$ and $\operatorname{Im} \Psi_2$ are closed. Proposition 2.14 and the proof of Proposition 2.13 imply that $\operatorname{Im} \Psi_3$ is closed. □

2.5.5. We shall examine the assumption c).

Lemma 2.16. *The restriction of the projection*

$$(2.51) \qquad V \cap \{c_3 d_3 c_4 d_4 \neq 0\} \ni (a_1, \dots, d_3) \mapsto (a_1, a_2, c_3, c_4, b_1) \in \mathbb{C}^5$$

is injective. Its image is contained in $\{h_1 = 0\}$ *with an irreducible polynomial* $h_1(a_1, a_2, c_3, c_4)$ *in* (2.59).

Proof. If $16c_3 \neq d_3$, then we have

$$(2.52) \qquad \begin{aligned} c_5 = \frac{1}{156(16c_3 - d_3)} \Big(&- 128a_1^3 c_3 + 104a_1^2 b_1 c_3 - 1056a_1 a_2 c_4 + 26a_1 b_1^2 c_3 \\ &+ 363a_1 b_2 c_4 + 54a_1 c_3 d_3 - 192a_2^2 c_3 + 726a_2 b_1 c_4 + 195a_2 b_2 c_3 - 2b_1^3 c_3 \\ &- 33b_1 b_2 c_4 + 312b_1 c_3^2 - 6b_1 c_3 d_3 - 3b_2^2 c_3 - 3872c_4^2 \Big) \end{aligned}$$

from $f_2 = 0$. If $d_3 = 16c_3$, then we have

$$(2.53) \qquad \begin{aligned} c_5 = \tfrac{1}{1853280} c_4^{-1} \Big(&- 49280a_1^3 c_4 - 55872a_1^2 a_2 c_3 + 40040a_1^2 b_1 c_4 + 19206a_1^2 b_2 c_3 \\ &+ 34092a_1 a_2 b_1 c_3 + 10010a_1 b_1^2 c_4 - 261a_1 b_1 b_2 c_3 + 332640a_1 c_3 c_4 - 73920a_2^2 c_4 \\ &+ 2970a_2 b_1^2 c_3 + 75075a_2 b_2 c_4 + 233280a_2 c_3^2 - 770b_1^3 c_4 - 135b_1^2 b_2 c_3 \\ &+ 136620b_1 c_3 c_4 - 1155b_2^2 c_4 + 21870b_2 c_3^2 \Big) \end{aligned}$$

from the relation $c_3 f_3 - 20280 c_5 f_1 = 0$. In either case, the relation (2.52) or (2.53) shows that c_5 is uniquely determined by $(a_1, a_2, c_3, c_4, b_1, b_2, d_3)$.

Next we will do eliminations of variables in $f_1 = \cdots = f_4 = 0$. Put

$$r_1 = \mathrm{coeffn}(f_1, d_3, 1) f_2 - \mathrm{coeffn}(f_2, d_3, 1) f_1,$$
$$r_2 = 17 f_3 - 20 f_4.$$

Then

$$r_1 = (-22 c_4) f_2 - 6(-9 a_1 c_3 + b_1 c_3 - 26 c_5) f_1.$$

Moreover motivated by $\mathrm{coeffn}(r_1, c_5, 1) / \mathrm{coeffn}(r_2, c_5, 1) = -c_3 / 210$, we put

$$r_3 = -c_3 r_2 - 210 r_1,$$
$$r_4 = \mathrm{coeffn}(f_1, d_3, 1) r_3 - \mathrm{coeffn}(r_3, d_3, 1) f_1,$$
$$= -22 c_4 r_3 - 63 c_3 (968 a_1 c_4 + 9720 a_2 c_3 - 2090 b_1 c_4 - 1215 b_2 c_3) f_1.$$

Then r_4 is a polynomial function of $(a_1, a_2, c_3, c_4, b_1, b_2)$ and it is factored into

$$(2.54) \quad r_4 = -105 d_4 (7128 a_1 c_3^2 c_4 - 5832 a_2 c_3^3 + 1782 b_1 c_3^2 c_4 + 729 b_2 c_3^3 - 10648 c_4^3).$$

Then we have

$$(2.55) \qquad 7128 a_1 c_3^2 c_4 - 5832 a_2 c_3^3 + 1782 b_1 c_3^2 c_4 + 729 b_2 c_3^3 - 10648 c_4^3 = 0$$

and hence

$$(2.56) \qquad b_2 = \tfrac{2}{729} c_3^{-3}(-3564 a_1 c_3^2 c_4 + 2916 a_2 c_3^3 - 891 b_1 c_3^2 c_4 + 5324 c_4^3).$$

Finally from $f_1 = 0$ we have

$$(2.57) \qquad d_3 = \tfrac{1}{22} c_3 c_4^{-1}(96 a_1 a_2 - 33 a_1 b_2 - 66 a_2 b_1 + 3 b_1 b_2 + 352 c_4).$$

Since c_5, b_2 and d_3 is given by (2.52) or (2.53), (2.56) and (2.57), all coefficients are uniquely determined by $(a_1, a_2, c_3, c_4, b_1)$. This proves the injectivity.

By substituting (2.56) and (2.57) we have

(2.58)
$$\mathrm{coeffn}(f_2, c_5, 1) f_3 - \mathrm{coeffn}(f_3, c_5, 1) f_2$$
$$= 156(16 c_3 - d_3) f_3 - 4680(416 a_1 a_2 - 143 a_1 b_2 - 286 a_2 b_1 + 13 b_1 b_2 + 3168 c_4) f_2$$
$$= \tfrac{1040}{24057} c_3^{-5} c_4^{-2} d_4 h_1 (108 a_1 c_3^2 + 27 b_1 c_3^2 - 484 c_4^2)$$

with

(2.59)
$$h_1 = -39694050 a_1^2 c_3^4 c_4^2 - 22733865 a_1 a_2 c_3^5 c_4 + 59296050 a_1 c_3^2 c_4^4$$
$$+ 26040609 a_2^2 c_3^6 + 47544651 a_2 c_3^3 c_4^3 - 85739148 c_3^5 c_4^2 - 14172488 c_4^6.$$

Suppose $h_1 \neq 0$. Then from (2.58) we have

$$(2.60) \qquad\qquad b_1 = \tfrac{4}{27} c_3^{-2}(-27 a_1 c_3^2 + 121 c_4^2).$$

When $16c_3 \neq d_3$, it follows from (2.52), (2.56), (2.57) and (2.60) that

$$f_5 = \tfrac{80}{29403} c_3^{-3} c_4^{-2} d_3 h_1 \neq 0,$$

which contradicts to the fact that $f_5 = 0$.

When $16c_3 = d_3$, substituting (2.53), (2.56), (2.60) and $d_3 = 16c_3$ to f_1 we have

$$f_1 = \tfrac{968}{6561} c_3^{-4} c_4^2 (8910 a_1 c_3^2 c_4 - 5103 a_2 c_3^3 - 10648 c_4^3)$$

and therefore

$$a_2 = \tfrac{22}{5103} c_3^{-3} c_4 (405 a_1 c_3^2 - 484 c_4^2).$$

Combining this with (2.53), (2.56), (2.60) and $d_3 = 16c_3$, we have

$$f_5 = -3732480 c_3^3 \neq 0,$$

which also leads a contradiction.

Hence we obtain

(2.61) $$h_1(a_1, a_2, c_3, c_4) = 0. \quad \square$$

Lemma 2.17. *On $V \cap \{c_3 d_3 \neq 0, a_2 = b_2 = c_4 = d_4 = 0\}$, (a_1, c_3, b_1, d_3) satisfy an equation $h_2(a_1, c_3, b_1, d_3) = 0$, which is given in (2.63).*

Proof. By $f_2 = 0$, we have

(2.62) $$156 c_5 (16 c_3 - d_3) =$$
$$- c_3 (128 a_1^3 - 104 a_1^2 b_1 - 26 a_1 b_1^2 - 54 a_1 d_3 + 2 b_1^3 - 312 b_1 c_3 + 6 b_1 d_3).$$

Now applying $a_2 = b_2 = c_4 = 0$ and (2.62) to $(16c_3 - d_3)^2 f_5$ and $(16c_3 - d_3)^2 f_6$, we obtain

$$(16c_3 - d_3)^2 f_5 = 60 c_3 d_3 h_2 r_5,$$
$$(16c_3 - d_3)^2 f_6 = 60 c_3 d_3 h_2 r_6$$

with

$$h_2 = 256 a_1^4 - 144 a_1^3 b_1 - 104 a_1^2 b_1^2 + 1536 a_1^2 c_3 - 204 a_1^2 d_3 - 9 a_1 b_1^3 - 432 a_1 b_1 c_3$$
$$- 27 a_1 b_1 d_3 + b_1^4 - 204 b_1^2 c_3 + 6 b_1^2 d_3 + 2304 c_3^2 - 288 c_3 d_3 + 9 d_3^2,$$
$$r_5 = 64 a_1^2 - 68 a_1 b_1 + 4 b_1^2 - 288 c_3 - 9 d_3,$$
$$r_6 = 2624 a_1^2 - 2788 a_1 b_1 + 164 b_1^2 - 8352 c_3 - 585 d_3,$$

respectively. We have $h_2 r_5 = h_2 r_6 = 0$. Moreover because of the identity

$$5184 h_2 = 1681 r_5^2 - 82 r_5 r_6 + r_6^2$$
$$- (144576 a_1^2 - 1512 a_1 b_1 - 9414 b_1^2) r_5 + (4032 a_1^2 + 216 a_1 b_1 - 198 b_1^2) r_6$$

we can conclude

(2.63) $$h_2(a_1, c_3, b_1, d_3) = 0. \quad \square$$

Lemma 2.18. *The map* $V \cap \{c_3 d_3 (16 c_3 - d_3) \neq 0, a_2 = b_2 = c_4 = d_4 = 0\} \ni$
$(a_1, \ldots, d_3) \mapsto (a_1, c_3, b_1, d_3) \in \mathbb{C}^4$ *is injective. Its image is contained in* $\{h_2 = 0\}$.

Proof. Since $d_3 \neq 16 d_3$, c_5 is uniquely determined by (2.62) and the lemma is clear
from Lemma 2.17. $\quad\square$

Lemma 2.19. *The map*

$$V \cap \{c_3 (4 a_1 + b_1) \neq 0, a_2 = b_2 = c_4 = d_4 = 16 c_3 - d_3 = 0\} \ni (a_1, \ldots, d_3) \mapsto (a_1, b_1)$$

is injective.

Proof. For an element of V such that $a_2 = b_2 = c_4 = d_4 = 16 c_3 - d_3 = 0$, we have

$$h_2 = (4 a_1 + b_1)^2 (16 a_1^2 - 17 a_1 b_1 + b_1^2 - 108 c_3).$$

Moreover assume $c_3 (4 a_1 + b_1) \neq 0$, then we have

$$c_3 = \tfrac{1}{180} (16 a_1^2 - 17 a_1 b_1 + b_1^2).$$

Then

$$f_5 = \tfrac{20}{27} h_4 r_7,$$
$$f_6 = \tfrac{20}{27} h_4 r_8$$

with

$$h_4 = 128 a_1^3 - 152 a_1^2 b_1 + 25 a_1 b_1^2 - b_1^3 - 2808 c_5,$$
$$r_7 = -384 a_1^3 + 520 a_1^2 b_1 - 143 a_1 b_1^2 + 7 b_1^3 - 11232 c_5,$$
$$r_8 = -11136 a_1^3 + 17576 a_1^2 b_1 - 6799 a_1 b_1^2 + 359 b_1^3 - 460512 c_5.$$

Now by the equality

$$r_8 - 41 r_7 = 7776 c_3 (4 a_1 + b_1) \neq 0,$$

we can conclude $h_4 = 0$. Then c_5 is determined by (a_1, b_1). $\quad\square$

Lemma 2.20. *The map* $V \cap \{c_3 \neq 0, a_2 = b_2 = c_4 = d_4 = 16 c_3 - d_3 = 4 a_1 + b_1 =$
$0\} \ni (a_1, \ldots, d_3) \mapsto (a_1, c_3, c_5) \in \mathbb{C}^3$ *is injective. Its image is contained in* $\{f_5 = 0\}$.

Proof. In this case,

$$f_5 = 69120 (338 c_5^2 + 13 a_1 c_3 c_5 - 28 a_1^2 c_3^2 - 54 c_3^3)$$

is an irreducible polynomial. $\quad\square$

3. Reducible systems of type B_2

3.1. For our commuting differential operators P_1 and P_2 we can consider the simultaneous eigenvalue problem

$$(3.1) \qquad P_j u(x) = \lambda_j u(x) \quad \text{for} \quad j = 1 \text{ and } 2$$

with $\lambda_j \in \mathbb{C}$. If the potential function of P_1 is generic, the study of this problem seems to be difficult. For the first step to analyze (3.1) we examine the case when the system (3.1) is reducible. To be precise we study the operators P and Q in the following lemma such that $P = P_1$ and $P_2 = {}^t QQ$.

Lemma 3.1. *Let P be a self-adjoint differential operator and let Q be a differential operator satisfying*

$$(3.2) \qquad [P, Q] = BQ$$

with a self-adjoint operator B. Then

$$[P, {}^t QQ] = 0.$$

Proof. The assumption implies $[P, {}^t QQ] = [P, {}^t Q]Q + {}^t Q[P, Q] = -{}^t[{}^t P, Q]Q + {}^t Q[P, Q] = -{}^t QBQ + {}^t QBQ = 0$. $\square$

Theorem 3.2. *Let ε be the one dimensional representation $\varepsilon : W(B_2) \to \{\pm 1\}$ such that $g(x_1 x_2) = \varepsilon(g)x_1 x_2$ for $g \in W(B_2)$. Let P and Q be holomorphic differential operators of the form*

$$(3.3) \qquad \begin{cases} P = \partial_1^2 + \partial_2^2 + R(x_1, x_2), \\ Q = \partial_1 \partial_2 + a_1(x_1, x_2)\partial_1 + a_2(x_1, x_2)\partial_2 + a_0(x_1, x_2) \end{cases}$$

defined on a Zariski open subset of a connected open neighborhood of the origin of $\mathbb{C}^2$. Suppose

$$(3.4) \qquad g(P) = P, \ g(Q) = \varepsilon(g)Q \quad \text{for} \quad g \in W(B_2)$$

and

$$(3.5) \qquad [P, Q] = b(x_1, x_2)Q$$

with a function $b(x_1, x_2)$. Then

$$(3.6) \qquad \begin{cases} R(x_1, x_2) = u(x_1 + x_2) + u(x_1 - x_2) + w(x_1) + w(x_2), \\ w(t) = V'(t) - V^2(t), \\ a_0(x_1, x_2) = V(x_1)V(x_2) + \frac{1}{2}(u(x_1 + x_2) - u(x_1 - x_2)), \\ a_1(x_1, x_2) = V(x_2), \\ a_2(x_1, x_2) = V(x_1), \\ b(x_1, x_2) = 2V'(x_1) + 2V'(x_2), \end{cases}$$

where

$$(3.7) \quad \begin{cases} u(t) = c_4 \dfrac{(\wp(\frac{t}{2}) - e_3)^2}{\wp'(\frac{t}{2})^2} + c_5 \wp(t) + c_6, \\ V(t) = \dfrac{c_1(\wp(t) - e_1)(\wp(t) - e_2) + c_2 \wp(t) + c_3}{\wp'(t)} \end{cases}$$

with suitable complex numbers $c_1, \dots, c_6$ *satisfying*

$$(3.8) \quad c_2 c_4 = c_3 c_4 = 0$$

or

$$(3.9) \quad u(t) = c, \ V(t) \text{ is any odd function with } c \in \mathbb{C}$$

or

$$(3.10) \quad u(t) \text{ is any even function, } V(t) = 0.$$

On the other hand the operators P and Q given by (3.6) satisfy the relation (3.5) by putting (3.7) for any complex numbers $c_1, \dots, c_6$ with (3.8) and any periods of $\wp(t)$ or by putting (3.9) or by putting (3.10).

The following Remark 3.3 and Remark 3.4 are easily obtained by direct calculations.

Remark 3.3. Under the notation of Theorem 3.2

$$^tQQ = \left(\partial_1 \partial_2 + \frac{(u(x_1 + x_2) - u(x_1 - x_2))}{2}\right)^2 + w(x_2)\partial_1^2 + w(x_1)\partial_2^2$$
$$+ w(x_1)w(x_2) + V(x_1)V(x_2)(u(x_1 + x_2) - u(x_1 - x_2))$$
$$- \frac{1}{2}\Big(V(x_1)(u'(x_1 + x_2) + u'(x_1 - x_2)) + V(x_2)(u'(x_1 + x_2) - u'(x_1 - x_2))\Big).$$

Remark 3.4. In Theorem 3.2 we have the following from (3.7) with complex numbers $C_1, \dots$.

i) If the fundamental half periods ω_1 and ω_2 of $\wp$ are finite and $c_4 = 0$, then

$$\begin{cases} u(t) = c_5 \wp(t) + c_6, \\ V(t) = \sum_{j=1}^3 \frac{1}{2}C_j \frac{\wp'(t)}{\wp(t) - e_j}, \\ w(t) = -\sum_{j=1}^4 (C_j + C_j^2)\wp(t + \omega_j) \\ \qquad -(C_1^2 - 2C_2 C_3)e_1 - (C_2^2 - 2C_3 C_1)e_2 - (C_3^2 - 2C_1 C_2)e_3, \\ C_4 = -(C_1 + C_2 + C_3). \end{cases}$$

ii) If ω_1 and ω_2 are finite and $c_2 = c_3 = 0$, then

$$\begin{cases} u(t) = C_2 \wp(t) + C_3\left(\wp(\frac{t}{2} + \omega_1) + \wp(\frac{t}{2} + \omega_2)\right) + C_4, \\ V(t) = \frac{1}{2}C_1 \frac{\wp'(t)}{\wp(t) - e_3}, \\ w(t) = -(C_1 + C_1^2)\wp(t + \omega_3) + (C_1 - C_1^2)\wp(t) - C_1^2 e_3 \end{cases}$$

iii) If $e_1 = e_2 = \frac{1}{3}\lambda^2 \neq 0$ and $c_4 = 0$, then

$$\begin{cases} u(t) &= C_4 \sinh^{-2} \lambda t + C_5, \\ V(t) &= C_1 \coth \lambda t + C_2 \tanh \lambda t + C_3 \sinh 2\lambda t, \\ w(t) &= -(C_1\lambda + C_1^2)\sinh^{-2}\lambda t + (C_2\lambda + C_2^2)\cosh^{-2}\lambda t \\ &\quad + 2(C_3\lambda - C_1C_3 - C_2C_3)\cosh 2\lambda t - C_3^2 \cosh^2 2\lambda t \\ &\quad - (C_1^2 + C_2^2 - C_3^2 + 2C_1C_2 + 2C_1C_3 - 2C_2C_3). \end{cases}$$

iv) If $e_2 = e_3 = \frac{1}{3}\lambda^2 \neq 0$ and $c_2 = c_3 = 0$, then

$$\begin{cases} u(t) &= C_2 \sinh^{-2} \lambda t + C_3 \sinh^{-2} \frac{\lambda}{2} t + C_4, \\ V(t) &= C_1 \coth \lambda t, \\ w(t) &= -(C_1\lambda + C_1^2)\sinh^{-2}\lambda t - C_1^2. \end{cases}$$

v) If $e_1 = e_2 = \frac{1}{3}\lambda^2 \neq 0$ and $c_2 = c_3 = 0$, then

$$\begin{cases} u(t) &= C_2 \sinh^{-2} \lambda t + C_3 \cosh 2\lambda t + C_4, \\ V(t) &= C_1 \sinh^{-1} 2\lambda t, \\ w(t) &= -C_1\lambda \sinh^{-2}\lambda t + (2C_1\lambda - C_1^2)\sinh^{-1} 2\lambda t. \end{cases}$$

vi) If $e_1 = e_2 = c_4 = 0$, then

$$\begin{cases} u(t) &= C_4 t^{-2} + C_5, \\ V(t) &= C_1 t^{-1} + C_2 t + C_3 t^3, \\ w(t) &= -(C_1 + C_1^2)t^{-2} - (2C_1C_2 - C_2) - (2C_1C_3 + C_2^2 - 3C_3)t^2 \\ &\quad - 2C_2C_3 t^4 - C_3^2 t^6. \end{cases}$$

vii) If $e_1 = e_2 = c_2 = c_3 = 0$, then

$$\begin{cases} u(t) &= C_2 t^{-2} + C_3 t^2 + C_4, \\ V(t) &= C_1 t^{-1}, \\ w(t) &= -(C_1 + C_1^2)t^{-2}. \end{cases}$$

3.2. To prove Theorem 3.2 we will translate the reducibility into a functional equation. The coefficients of ∂_1^2 and ∂_2^2 in (3.5) mean $2\partial_1 a_2 = 2\partial_2 a_1 = 0$ and therefore

$$a_1 = V(x_2) \quad \text{and} \quad a_2 = V(x_1)$$

with a suitable odd function $V(t)$. The coefficient of $\partial_1\partial_2$ in (3.5) proves

(3.11) $$b = 2(\partial_2 a_1 + \partial_1 a_2) = 2\big(V'(x_2) + V'(x_1)\big).$$

The coefficients of ∂_1 and ∂_2 in (3.5) are

$$\begin{cases} V''(x_2) + 2\partial_1 a_0 - \partial_2 R = 2V(x_2)\big(V'(x_1) + V'(x_2)\big), \\ V''(x_1) + 2\partial_2 a_0 - \partial_2 R = 2V(x_1)\big(V'(x_1) + V'(x_2)\big) \end{cases}$$

and equivalently

$$\begin{cases} (\partial_1 + \partial_2)\big(R - 2a_0 - V'(x_1) - V'(x_2) + (V(x_1) + V_2(x_2))^2\big) = 0, \\ (\partial_1 - \partial_2)\big(R + 2a_0 - V'(x_1) - V'(x_2) + (V(x_1) - V_2(x_2))^2\big) = 0. \end{cases}$$

Hence there exist functions $u_+(t)$ and $u_-(t)$ such that

$$(3.12) \qquad \begin{aligned} R &= V'(x_1) + V'(x_2) - V(x_1)^2 - V(x_2)^2 + u_+(x_1 + x_2) + u_-(x_1 - x_2), \\ a_0 &= V(x_1)V(x_2) + \tfrac{1}{2}\big(u_+(x_1 + x_2) - u_-(x_1 - x_2)\big). \end{aligned}$$

Since $g(a_0) = \varepsilon(g)a_0$, we have

$$(3.13) \qquad\qquad u_+(t) = u_-(t) = u(t)$$

with a suitable even function $u(t)$. This proves (3.6).

Let $U(t)$ be the odd function with $U'(t) = u(t)$. Then

$$[P, Q] - bP$$

$$= -V(x_1)\big(u(x_1 + x_2) + u(x_1 - x_2)\big) - V(x_2)\big(u(x_1 + x_2) - u(x_1 - x_2)\big)$$

$$- \big(V'(x_1) + V'(x_2)\big)\big(u(x_1 + x_2) - u(x_1 - x_2)\big)$$

$$= -\partial_1\partial_2\Big(V(x_1)\big(U(x_1 + x_2) + U(x_1 - x_2)\big) + V(x_2)\big(U(x_1 + x_2) - U(x_1 - x_2)\big)\Big).$$

Thus we have

Theorem 3.5. *The operators P and Q satisfy (3.3), (3.4) and (3.5) if and only if there exist odd functions U and V and an even function $H(t)$ such that*

$$(3.14) \quad V(x_1)\big(U(x_1 + x_2) + U(x_1 - x_2)\big) + V(x_2)\big(U(x_1 + x_2) - U(x_1 - x_2)\big)$$

$$= H(x_1) + H(x_2)$$

and that the relation (3.6) holds with $u = U'$.

Hence we will concentrate the functional equation (3.14), which is a special case of (2.3).

Lemma 3.6. *Suppose (U, V, F, G) is a solution of (2.3) such that*

$$\begin{cases} U'(t) \text{ has a period } 2\omega, \\ W(t) := V(t + \omega) - V(t) - V(\omega) \text{ is an odd function.} \end{cases}$$

Then (U, W, H) satisfies (3.14) with an appropriate H.

Proof. Put $U(t + 2\omega) = U(t) + \eta$. Changing the variable (x_1, x_2) into $(x_1 + \omega_1, x_2 + \omega_2)$ in the equation (2.3), we have

$$\big(U(x_1 + x_2) + \eta + U(x_1 - x_2)\big)V(x_1 + \omega) + \big(U(x_1 + x_2) + \eta - U(x_1 - x_2)\big)V(x_2 + \omega)$$

$$= F(x_1 + x_2 + 2\omega) + F(x_1 - x_2) + G(x_1 + \omega) + G(x_2 + \omega).$$

Subtracting the original one from this, we obtain

$$\big(U(x_1 + x_2) + U(x_1 - x_2)\big)W(x_1) + \big(U(x_1 + x_2) - U(x_1 - x_2)\big)W(x_2)$$

$$= \big(F(x_1 + x_2 + 2\omega) - F(x_1 + x_2) - 2V(\omega)U(x_1 + x_2)\big)$$

$$+ \big(G(x_1 + \omega) - G(x_1) - \eta V(x_1 + \omega)\big)$$

$$+ \big(G(x_2 + \omega) - G(x_2) - \eta V(x_2 + \omega)\big).$$

Since the left hand side is $W(B_2)$-invariant, the first term of the right hand side is constant. Then $H(t) = G(t + \omega) - G(t) - \eta V(t + \omega) + C$ with a suitable constant number C and we have the lemma. $\square$

Corollary 3.7. *Suppose the fundamental half periods ω_1 and ω_2 of $\wp(t)$ are finite. Then for odd functions $U(t)$ and $V(t)$ given by*

$$\begin{cases} U'(t) = C_4\wp(t) + C_5, \\ V(t) = \sum_{j=1}^{3} C_j\big(\zeta(t+\omega_j) - \zeta(t) - \zeta(\omega_j)\big) \end{cases}$$

or

$$\begin{cases} U'(t) = C_1\big(\wp(\tfrac{t}{2}+\omega_1) + \wp(\tfrac{t}{2}+\omega_2)\big) + C_2\wp(t) + C_3, \\ V(t) = \zeta(t+\omega_3) - \zeta(t) - \zeta(\omega_3), \end{cases}$$

there exists a function $H(t)$ so that (3.14) holds.

Proof. Note that the equation (3.14) is bilinear for (U,V). In the lemma put $\big(U(t),V(t)\big) = \big(\zeta(t),\zeta(t)\big)$ and $\omega = \omega_j$ or put $\big(U(t),V(t)\big) = \big(\zeta(\tfrac{t}{2}+\omega_1) + \zeta(\tfrac{t}{2}+\omega_2) - \zeta(\omega_1) - \zeta(\omega_2),\zeta(t)\big)$ and $\omega = \omega_3$, we have the corollary. $\square$

Now we will continue the proof of Theorem 3.2. Since

$$\zeta(t+\omega_j) - \zeta(t) - \zeta(\omega_j) = \frac{1}{2}\frac{\wp'(t)}{\wp(t) - e_j},$$

the last statement in Theorem 3.2 follows from Corollary 3.7 with the holomorphic continuation of the parameters e_1 and e_2 of $\wp(t)$ and from the following Lemma 3.8 i). Thus we have proved that the operators given in Remark 3.4 satisfy (3.5).

Lemma 3.8. i) *If $U(t) = Ct$ ($C \in \mathbb{C}$), then for any V, $H(t) := 2CtV(t)$ satisfies (3.14). If $V(t) = 0$, then $H(t) = 0$ satisfies (3.14). We call these (U,V) trivial solutions of (3.14), which correspond to (3.9) and (3.10).*

ii) *If (U,V,H) is a solution of (3.14), then $(U+Ct, V, H+2tV)$ is also a solution of (3.14)*

iii) *If $V'(t)$ has a period ω, then $U'(t)$ has a period 2ω.*

iv) *If $V(t+\omega) = V(t) + \eta$ with $\eta \in \mathbb{C}$, then $\eta = 0$.*

Proof. The claims i) and ii) are clear and the claim iii) is also clear from the result in §2. Suppose $V(t+\omega) = V(t) + \eta$. Then $U(t+2\omega) = U(t) + \eta'$ with some $\eta' \in \mathbb{C}$. By the change of variable (x_1, x_2) into $(x_1 + \omega, x_2 + \omega)$ in (3.14)

$$\big(U(x_1+x_2)+\eta'+U(x_1-x_2)\big)\big(V(x_1)+\eta\big)+\big(U(x_1+x_2)+\eta'-U(x_1-x_2)\big)\big(V(x_2)+\eta\big)$$
$$= H(x_1+\omega) + H(x_2+\omega).$$

Subtracting (3.14) from the above,

$$2\eta U(x_1+x_2)$$
$$= \big(H(x_1+\omega) - H(x_1) - \eta'V(x_1+\omega)\big) + \big(H(x_2+\omega) - H(x_2) - \eta'V(x_2+\omega)\big).$$

Since we have assumed $U''(x_1+x_2) \neq 0$, we can conclude $\eta = 0$. $\square$

Finally we will prove that there is no more solutions than we have already described in Remark 3.4 and Lemma 3.8 i).

From now on we consider only non-trivial solutions unless otherwise stated. Let (U, V, H) be a non-trivial solution of (3.14). Owing to Theorem 2.9 we see that $V(t)$ is expressed by $\wp(t)$.

Suppose the fundamental half periods ω_1 and ω_2 of $\wp$ are finite. Then

$$V(t) = \sum_{j=1}^{4} C_j \zeta(t + \omega_j) + C_0 t$$

with $C_j \in \mathbb{C}$. By Lemma 3.8, we have

$$0 = V(t + 2\omega_i) - V(\omega_i) = \sum_{j=1}^{4} 2C_j \eta_i + 2C_0 \omega_i$$

for $i = 1$ and 2. Since $\eta_2 \omega_1 - \eta_1 \omega_2 = \pm \frac{\pi\sqrt{-1}}{2} \neq 0$, $C_0 = \sum_{j=1}^{4} C_j = 0$ and we have the theorem.

When $\omega_1 = \omega_2 = \infty$, the theorem follows from

Lemma 3.9. *The rational solution of* (3.14) *is of the form in Remark 3.4 vi) or vii).*

Proof. Let (U, V, H) is a rational solution of (3.14). If $U(t) = t^{-1}$, the left hand side of (3.14) equals

$$2\left(\frac{x_1 V(x_1) - x_2 V(x_2)}{x_1^2 - x_2^2}\right) = 2\sum_{n \geq 0} a_{2n} \frac{x_1^{2n} - x_2^{2n}}{x_1^2 - x_2^2}$$

for $tV(t) = \sum_{n \geq 0} a_{2n} t^{2n}$. Hence if $u(t) = C_4 t^{-2} + C_5$ with $C_4 \neq 0$, the solution is of the form in Remark 3.4 vi).

Suppose $U(t) = C_1 t^{-1} + C_2 t + C_3 t^3$ and $V(t) = C_4 t^{-1} + C_5 t + C_6 t^3$ with $C_3 \neq 0$. We may assume $C_1 = C_2 = C_4 = 0$. Then the left hand side of (3.14) equals $6C_3(x_1 V(x_1) x_2^2 + x_1^2 V(x_2) x_2)$ and hence $C_5 = C_6 = 0$. This is the case in Remark 3.4 vii).

Suppose $V(t) = C_1 t^{-1} + C_2 t$ and $U(t) = C_3 t^{-1} + C_4 t + C_5 t^3 + C_6 t^5 + C_7 t^7$. We may assume $C_3 = C_4 = 0$. If $C_1 \neq 0$ and $C_2 = 0$, the left hand side of (3.14) equals

$$C_1 \sum_{j=2}^{4} C_{n+3} \frac{(x_1 + x_2)^{2n} - (x_1 - x_2)^{2n}}{x_1 x_2}$$

and therefore $C_6 = C_7 = 0$, which also corresponds to Remark 3.4 vii). Hence suppose $C_2 \neq 0$. Since $(U(t), V(t)) = (t, t^n)$ does not satisfy (3.14) for $n = 7, 5$ and 3, we have $C_7 = C_6 = C_5 = 0$ by considering the homogeneous parts of degree 8, 6 and 4, successively. This is the case in Remark 3.4 vi). $\square$

Lastly suppose $\omega_1 = \infty$ and ω_2 is finite. We may assume

$$V(t) = C_1 \coth \lambda t + C_2 \tanh \lambda t + C_3 \sinh 2\lambda t + C_4 \sinh 4\lambda t + C_0 t.$$

Applying Lemma 3.9 to this, we have $C_0 = 0$.

If the dimension of the space $\{V(t)|\,(\lambda\coth\lambda t, V(t))$ is a solution of $(3.14)\}$ is larger than 3 for $\lambda\neq 0$, the dimension of the space $\{V(t)|\,(t^{-1}, V(t))$ is a solution of $(3.14)\}$ is proved to be larger than 3 by considering the limit to $\lambda=0$ (cf. [OSj] Proposition 2.21), which contradicts to Lemma 3.9. Hence we have Theorem 3.2 if $u(t)=C\sinh^{-2}\lambda t+C'$.

In the same way we can prove that the space $\{U(t)|\,(U(t), \lambda\coth\lambda t)$ is a solution of $(3.14)\}$ is of dimension 2 for $\lambda\neq 0$, which implies the theorem in the case $V(t)=C_1\coth\lambda t$.

Thus we may assume that

$$(3.15)\qquad \begin{cases} U(t)=C_1\coth\lambda t+C_2 t+C_3\coth\tfrac{\lambda}{2}t,\\ V(t)=C_4\coth\lambda t+C_5\sinh 2\lambda t. \end{cases}$$

or

$$(3.16)\qquad \begin{cases} U(t)=C_1\coth\lambda t+C_2 t+C_3\sinh 2\lambda t,\\ V(t)=C_4\sinh^{-1}2\lambda t+C_5\coth\lambda t. \end{cases}$$

The pairs (U,V) corresponding to $C_3C_5=0$ have been proved to be solutions. Suppose there exists a solution with $C_3C_5\neq 0$ in (3.15) or (3.16). Then the bilinearity of the equation (3.14) implies that the pair (U,V) given by (3.15) or (3.16) is a solution for any complex numbers $C_1,\ldots,C_5$, which similarly contradicts to Lemma 3.9 by taking the limit to $\lambda=0$.

Thus we have completed the proof of Theorem 3.2.

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