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AN ALGORITHM TO COMPUTE THE b_P-FUNCTIONS VIA GRÖBNER BASES OF INVARIANT DIFFERENTIAL OPERATORS ON PREHOMOGENEOUS VECTOR SPACES.

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ABSTRACT. The calculation of b_P-function via Gröbner basis for an group invarinat differential operator $P(x, \partial)$ on a finite dimensional vector space is considered in this paper. Let (G, V) be a regular prehomogeneous vector space. It is often observed that the space of all G-invariant hyperfunction solutions $u(x)$ to the differential equation $P(x, \partial)u(x) = v(x)$ is determined by its b_P-function, a polynomial associated with the G-invariant differential operator $P(x, \partial)$. We prove in this paper that the b_P-function is computed by an algorithm using Gröbner basis of the Weyl algebra on V for a typical class of prehomogeneous vector spaces.

CONTENTS

INTRODUCTION.

The ultimate purpose of theory of differential equations is to "compute the solutions" of a given differential equations. For example, to give a solution in an explicit form, to write a solution using known special functions and functional relations and to give an algorithm to construct a solution and so on are all trials along this purpose. There never, of course, exists a unified way to solve all differential equations. What we can do is to devise a way of solving the differential equation according to the properties and peculiarities of the given differential equation.

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We consider in this article a linear differential equation

$$P(x,\partial)u(x) = v(x) \tag{1}$$

with one unknown function $u(x)$ for a given differential operator $P(x,\partial)$ and a given function $v(x)$ on a real non-singular algebraic variety $\boldsymbol{X}$. In particular, we suppose that a real algebraic group $\boldsymbol{G}$ acts on $\boldsymbol{X}$ transitively in an algebraic manner. Let $\boldsymbol{H}$ be an algebraic subgroup of $\boldsymbol{G}$. We suppose that $P(x,\partial)$ *and* $u(x), v(x)$ *are all* $\boldsymbol{H}$*-invariant* and call (1) *a* $\boldsymbol{H}$*-invariant differential equation.* The author does not know a universal method to solve the $\boldsymbol{H}$-invariant differential equations in a general form. However, he recently found a natural way to construct $\boldsymbol{H}$-invariant hyperfunction solutions to the $\boldsymbol{H}$-invariant differential equations (1) on a class of regular prehomogeneous vector spaces ([9]) when $\boldsymbol{H}$ is a kernel of the characters of relative invariants which is often denoted by $\boldsymbol{G}^1$. According to the method in [9], when $P(x,\partial)$ is a χ*-invariant* differential operator, it is important to compute a polynomial $b_P(s)$ in a complex parameter $s \in \mathbb{C}$ defined by

$$P(x,\partial)f(x)^s = b_P(s)f(x)^{s+\chi} \tag{2}$$

where $f(x)$ is a relative invariant of the prehomogeneous vector space, $f(x)^s$ is the complex power of $f(x)$ and χ is the "character" of the $\boldsymbol{G}^1$-invariant differential operator $P(x,\partial)$. In fact, if s_0 is a root of the equation $b_P(s) = 0$, then a hyperfunction obtained as a complex power $|f(x)|^{s_0}$ is a $\boldsymbol{G}^1$-invariant kernel of the $\boldsymbol{G}^1$-invariant differential operator. Here $\boldsymbol{G}^1$ is a kernel of the character χ satisfying $f(g\cdot x) = \chi(g)f(x)$. This means that the b_P-function $b_P(s)$ is closely related to the construction of the $\boldsymbol{G}^1$-invariant hyperfunction solutions to (1).

The main issue of this article is the calculation of b_P-function of a given $\boldsymbol{G}^1$-invariant differential operator $P(x,\partial)$ on a given regular prehomogeneous vector space. We shall give an algorithm to compute the b_P-function for a given $P(x,\partial) \in D(V)^{\boldsymbol{G}^1}$ on the space of real symmetric matrices $\boldsymbol{V} := \mathrm{Sym}_n(\mathbb{R})$ with the group action of $\boldsymbol{G}^1 = \mathbf{SL}_n(\mathbb{R})$ (Algorithm 3.1). Instead of the direct computation of the b_P-function, we first compute the b_P-functions for the fundamental invariant differential operators and then write a given $P(x,\partial)$ as a polynomial with variables of the fundamental invariant differential operators. The algorithm we give here is that to express a given $P(x,\partial)$ by using the fundamental invariant differential operators.

1. Invariant differential operators and their b_P-functions on prehomogeneous vector spaces.

Let $\boldsymbol{V}$ be a finite dimensional vector space defined over the real field $\mathbb{R}$ and let $\boldsymbol{G}$ be a closed algebraic subgroup of the linear transformation group $\mathbf{GL}(\boldsymbol{V})$. The complexifications of $\boldsymbol{V}$ and $\boldsymbol{G}$ are denoted by $\boldsymbol{V}_{\mathbb{C}}$ and $\boldsymbol{G}_{\mathbb{C}}$, respectively. We say that the pair $(\boldsymbol{G},\boldsymbol{V})$ a *prehomogeneous vector space* if there exists an open dense $\boldsymbol{G}_{\mathbb{C}}$-orbit in $\boldsymbol{V}_{\mathbb{C}}$ with respect to the Zariski topology. We suppose here it is a *regular* prehomogeneous vector space, i.e., there exists a relative invariant whose Hessian does not vanish identically. Let $f_1(x), \dots, f_l(x)$ be the fundamental system of relative invariants of the prehomogeneous vector space $(\boldsymbol{G}, \boldsymbol{V})$ and let $\chi_1(g), \dots, \chi_l(g)$ be their corresponding characters, i.e., $f_i(g \cdot x) = \chi_i(g) f_i(x)$ for $i = 1, \dots, l$. These polynomials $f_1(x), \dots, f_l(x)$ are defined on $\boldsymbol{V}_{\mathbb{C}}$. We suppose that all $f_i(x)$'s are polynomials with real coefficients on $\boldsymbol{V}$. Then each $\chi_i(g)$ is real valued on $\boldsymbol{G}$. From the general theory of prehomogeneous vector space, we see that $f_1(x), \dots, f_l(x)$ are algebraically independent homogeneous polynomials.

Let $\mathbb{C}[\boldsymbol{V}]$ and $D(\boldsymbol{V})$ be the polynomial algebra and the algebra of differential operators on $\boldsymbol{V}$, respectively. Here we assume that both of them are with complex coefficients. Let $x = (x_1, \dots, x_n)$ be a linear coordinate of $\boldsymbol{V}$ and let $\partial = (\partial_1, \dots, \partial_n)$ with $\partial_i := \frac{\partial}{\partial x_i}$ be the partial derivatives. Then a polynomial $P(x) \in \mathbb{C}[\boldsymbol{V}]$ is written as $P(x) = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} a_\alpha x^\alpha$ and a linear differential operator $P(x, \partial) \in D(\boldsymbol{V})$ is written as $P(x, \partial) = \sum_{\alpha, \beta \in \mathbb{Z}^n_{\geq 0}} a_{\alpha,\beta} x^\alpha \partial^\beta$, where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$ are multi-powers for the multi-indices $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ and $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n_{\geq 0}$. The coefficients a_α and $a_{\alpha,\beta}$ are taken to be complex numbers $\mathbb{C}$. We denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $|\beta| = \beta_1 + \cdots + \beta_n$ and call them *total index* of α and β. If a polynomial $P(x)$ is written as

$$P(x) = \sum_{|\alpha|=k} a_\alpha x^\alpha$$

for a given non-negative integer k, then $P(x)$ is a *homogeneous* polynomial with homogeneous degree k. On the other hand, if a differential operator $P(x, \partial)$ is written as

$$P(x, \partial) = \sum_{|\alpha|-|\beta|=k} a_{\alpha,\beta} x^\alpha \partial^\beta,$$

then we call it a *homogeneous* differential operator with homogeneous degree k.

Let $\boldsymbol{G}^1$ (resp. $\boldsymbol{G}^1_{\mathbb{C}}$) be a closed algebraic subgroup of $\boldsymbol{G}$ (resp. $\boldsymbol{G}_{\mathbb{C}}$) defined as a kernel of all the characters $\chi_1, \dots, \chi_l$. We denote by $\mathbb{C}[\boldsymbol{V}]^{\boldsymbol{G}^1}$ and $D(\boldsymbol{V})^{\boldsymbol{G}^1}$ the subalgebras of $\boldsymbol{G}^1$-invariant elements of $\mathbb{C}[\boldsymbol{V}]$ and $D(\boldsymbol{V})$, respectively. They are naturally isomorphic to $\mathbb{C}[\boldsymbol{V}_{\mathbb{C}}]^{\boldsymbol{G}^1_{\mathbb{C}}}$ and $D(\boldsymbol{V}_{\mathbb{C}})^{\boldsymbol{G}^1_{\mathbb{C}}}$, the subalgebras of $\boldsymbol{G}^1_{\mathbb{C}}$-invariant elements of $\mathbb{C}[\boldsymbol{V}_{\mathbb{C}}]$ and $D(\boldsymbol{V}_{\mathbb{C}})$, respectively. It is because an element in $\mathbb{C}[\boldsymbol{V}]^{\boldsymbol{G}^1}$ or $D(\boldsymbol{V})^{\boldsymbol{G}^1}$ is naturally extended to $\boldsymbol{V}_{\mathbb{C}}$ as an element in $\mathbb{C}[\boldsymbol{V}_{\mathbb{C}}]$ or $D(\boldsymbol{V}_{\mathbb{C}})$, respectively and an element in $\mathbb{C}[\boldsymbol{V}_{\mathbb{C}}]^{\boldsymbol{G}^1_{\mathbb{C}}}$ or $D(\boldsymbol{V}_{\mathbb{C}})^{\boldsymbol{G}^1_{\mathbb{C}}}$ can be regarded as an element in $\mathbb{C}[\boldsymbol{V}]$ or $D(\boldsymbol{V})$ by restriction to $\boldsymbol{V}$, respectively.

The algebra $\mathbb{C}[\boldsymbol{V}]^{\boldsymbol{G}^1}$ is isomorphic to the polynomial algebra $\mathbb{C}[f_1, \dots, f_l]$. Namely, $\mathbb{C}[\boldsymbol{V}]^{\boldsymbol{G}^1}$ is generated by the algebraically independent elements $f_1, \dots, f_l$. In particular, the relative invariant corresponding to the character $\chi = \chi_1^{p_1} \cdots \chi_l^{p_l}$ is given by a constant multiple of $f_1(x)^{p_1} \cdots f_l(x)^{p_l}$. We denote it by $f^\chi(x)$ for abbreviation and we call a relative invariant corresponding to the character χ a χ-*invariant* polynomial. When all the power $p_1, \ldots, p_l$ are non-negative integer, we write it as $\chi \geq 0$. Let $d_1, \ldots, d_l \in \mathbb{Z}_{>0}$ be the homogeneous degrees of $f_1(x), \ldots, f_l(x)$, respectively. Then the homogeneous degree of $f^\chi(x)$ is given by $d_1 p_1 + \cdots + d_l p_l$. We denote it by $|\chi|$. In particular $|\chi_i| = d_i$. A $\boldsymbol{G}^1$-invariant homogeneous polynomial of degree d is written as

$$f(x) = \sum_{|\chi|=d, \chi \geq 0} a_\chi f^\chi(x)$$

with $a_\chi \in \mathbb{C}$ and χ runs through all the characters satisfying $|\chi| = d$ and $\chi \geq 0$.

A $\boldsymbol{G}^1$-invariant homogeneous differential operator $P(x, \partial)$ of degree d is given by

$$P(x, \partial) = \sum_{|\chi|=d} P_\chi(x, \partial).$$

Here $P_\chi(x, \partial)$ is a differential operator satisfying

$$P_\chi(g \cdot x, g^* \cdot \partial) = \chi(g) P_\chi(x, \partial)$$

for all $g \in \boldsymbol{G}$ and we call it a χ-*invariant* differential operator. However either d or χ may not be non-negative in this case. Then the sum $\sum_{|\chi|=d}$ seems to contain infinite number of terms formally. Of course only finite number of them are non-zero in the summand. The differential operator $P_\chi(x, \partial)$ corresponding to the character χ is not determined uniquely up to constant multiples. There may exist infinite number of $\boldsymbol{G}^1$-invariant differential operators corresponding to the character $\chi = \chi_1^{p_1} \cdots \chi_l^{p_l}$ $(p_1, \ldots, p_l \in \mathbb{Z})$.

Next we consider the complex power of the relative invariants. Let $\boldsymbol{S} := \{x \in \boldsymbol{V} | f_1(x) \cdots f_l(x) = 0\}$ and let $\boldsymbol{V}_1 \cup \cdots \cup \boldsymbol{V}_m = \boldsymbol{V} - \boldsymbol{S}$ be the connected component decomposition. We define the complex power function $|f(x)|_j^s$ by

$$|f(x)|_j^s := \begin{cases} |f_1(x)|^{s_1} \cdots |f_l(x)|^{s_l} & x \in \boldsymbol{V}_j \\ 0 & x \notin \boldsymbol{V}_j \end{cases} \tag{3}$$

with $j = 1, \ldots, m$ and $s := (s_1, \ldots, s_l) \in \mathbb{C}^l$. Then each $|f(x)|_j^s$ is well defined as a tempered distribution , and hence a hyperfunction, with holomorphic parameters $s := (s_1, \ldots, s_l) \in \mathbb{C}^l$ on $\boldsymbol{V}$ if the real parts $\Re(s_1), \ldots, \Re(s_l)$ are all sufficiently large since the integral $\int |f(x)|_j^s \phi(x) dx$ is absolutely convergent for any rapidly decreasing C^∞-function $\phi(x)$ on $\boldsymbol{V}$. It is well known that $|f(x)|_j^s$ can be extended to the whole complex numbers $s \in \mathbb{C}^l$ meromorphically. Thus each $|f(x)|_j^s$ is well defined as a hyperfunction on $\boldsymbol{V}$ with meromorphic parameters $s := (s_1, \ldots, s_l)$ on the whole complex parameter space $\mathbb{C}^l$.

Let $P(x,\partial) \in D(\boldsymbol{V})^{\boldsymbol{G}^1}$ be a homogeneous $\boldsymbol{G}^1$-invariant differential operator which is χ-invariant. Then we have

$$P(x,\partial)|f(x)|_j^s = b_P(s)f^{\chi}(x)|f(x)|_j^s \tag{4}$$

where $b_P(s)$ is a polynomial in $s = (s_1,\ldots,s_l) \in \mathbb{C}^l$. We call $b_P(s)$ the *b_P-function* of the relatively invariant differential operator $P(x,\partial)$. By expanding the both hand sides of (4) to the Laurent series[1] and comparing the Laurent expansion coefficients, we can obtain $\boldsymbol{G}^1$-invariant hyperfunction solutions to the differential equation

$$P(x,\partial)u(x) = v(x) \tag{5}$$

with a given $\boldsymbol{G}^1$-invariant hyperfunction $v(x)$ and an unknown $\boldsymbol{G}^1$-invariant hyperfunction $u(x)$, systematically. In particular, if $|f(x)|_j^s$ has a pole at $s = s_0$, then the order of poles of $|f(x)|_j^s$ is often larger than those of $b_P(s)f^{\chi}(x)|f(x)|_j^s$, and hence we see that the Laurent expansion coefficients of $|f(x)|_j^s$ of low orders are annihilated by $P(x,\partial)$.

This method is definitely strong because we can construct all singular hyperfunction solutions[2] in some typical cases like a real symmetric matrix space (see Muro [9]). Singular solutions are known to be difficult to handle since we have little resource to express and compute singular hyperfunctions. However we have to clear the following two obstacles before realizing our method.

1. The explicit computation of the b_P-function $b_P(s)$.
2. The explicit computation of the Laurent expansion of $|f(x)|_j^s$.

For the second problem, micro-local analysis of invariant hyperfunctions is highly efficient (see Muro [8]) and the author believes that this is one of the best way to solve the second problem. But, for the first problem, the author does not know any good solution to compute $b_P(s)$ explicitly. The author believes that it is important to establish the standard way to compute b_P-function and he thinks it will be effective to give an algorithm to compute b_P-function for a given χ-invariant differential operator $P(x,\partial)$.

In the following sections, the author proposes one algorithmic method to compute the b_P-functions on the space of real symmetric matrices. For a given differential operator $P(x,\partial)$, we shall give an algorithm to determine whether $P(x,\partial)$ is χ-invariant or not and to express $P(x,\partial)$ by using "fundamental" invariant differential operators of the algebra of invariant differential operators when $P(x,\partial)$ is an invariant differential operator. Then by computing the b_P-functions for the "fundamental" invariant differential operators, we can compute the b_P-function of the invariant differential operator $P(x,\partial)$.

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[1] However, when s is multivariate, we have to define the *Laurent series expansion* at poles. In our case, since we can prove that f^s is regularized by multiplying some linear polynomials we may expand f^s after regularization.

[2] A hyperfunction is *singular* if its support is contained in a proper algebraic subvarity in $\boldsymbol{V}$.

2. The case of the space of symmetric matrices.

Let $V := \mathrm{Sym}_n(\mathbb{R})$ be the space of $n \times n$ symmetric matrices over the real field $\mathbb{R}$ and let $G := \mathbf{GL}_n(\mathbb{R})$ be the special linear group over $\mathbb{R}$ of degree n. Then the group G acts on the vector space V by

$$\rho(g) : x \longmapsto g \cdot x \cdot {}^t g,$$

with $x \in V$ and $g \in G$. This is a typical prehomogeneous vector space because the complex vector space $V_{\mathbb{C}} = \mathbb{C} \otimes V := \mathrm{Sym}_n(\mathbb{C})$ has an open dense $\mathbf{GL}_n(\mathbb{C})$-orbit consisting of the elements $x \in \mathrm{Sym}_n(\mathbb{C})$ with $\det(x) \neq 0$. The group $G^1 = \{g \in G \mid \det(g \cdot x) = \det(x)\}$ is $\mathbf{SL}_n(\mathbb{R})$ in this case.

From now on, we only consider the case of the prehomogeneous vector space $(\mathbf{GL}_n(\mathbb{R}), V)$. In this section we shall give the generators of the subalgebras $D(V)^G$ and $D(V)^{G^1}$ and compute the b_P-functions for these generators. By giving an algorithm to write a given invariant differential operator $P(x, \partial) \in D(V)^{G^1}$ as a polynomial in the generators of $D(V)^{G^1}$ in §3, we can obtain an algorithm to compute b_P-function for $P(x, \partial)$ automatically.

The polynomial $f(x) = f_1(x) := \det(x)$ is the only one irreducible relative invariant and the corresponding character is $\chi(g) = \chi_1(g) := \det(g)^2$. The subgroup $G_1 := \{g \in G \mid \chi(g) = 1\} = \mathbf{SL}_n(\mathbb{R})$. A complex power function $|f(x)|_j^s$ is defined as follows. Let $V_0 \cup \ldots \cup V_n = V - S$ be the connected component decomposition of the compliment of the set $S := \{f(x) = 0\}$. Here V_j is the set of elements which has j positive elements and $n - j$ negative elements. Then, we can define $|f(x)|_j^s$ $(j = 0, \ldots, n)$ by

$$|f(x)|_j^s := \begin{cases} |f(x)|^s & x \in V_j \\ 0 & x \notin V_j \end{cases} \tag{6}$$

in the same manner as (3). However, in this case, the parameter s is only one complex number.

A homogeneous differential operator of degree $k \in \mathbb{Z}$ is given by

$$P(x, \partial) = \sum_{\substack{\alpha, \beta \in \mathbb{Z}_{\geq 0}^m \\ |\alpha| - |\beta| = k}} a_{\alpha\beta} x^\alpha \partial^\beta \tag{7}$$

where $m = n(n+1)/2$ in the case of symmetric matrix space. In (7), we denote

$$x = (x_{ij})_{n \geq j \geq i \geq 1}, \quad \partial = (\partial_{ij}) = \left(\frac{\partial}{\partial x_{ij}} \right)_{n \geq j \geq i \geq 1}$$

and

$$x^\alpha := \prod_{n \geq j \geq i \geq 1} x_{ij}^{\alpha_{ij}}, \quad \partial^\beta := \prod_{n \geq j \geq i \geq 1} \partial_{ij}^{\beta_{ij}}$$

with $\alpha = (\alpha_{ij}) \in \mathbb{Z}_{\geq 0}^m$, $|\alpha| = \sum_{n \geq j \geq i \geq 1} \alpha_{ij}$ and $\beta = (\beta_{ij}) \in \mathbb{Z}_{\geq 0}^m$, $|\beta| = \sum_{n \geq j \geq i \geq 1} \beta_{ij}$.

Every G^1-invariant homogeneous polynomial of homogeneous degree $k \in \mathbb{Z}_{\geq 0}$ is written by a constant multiple of $f(x)^{k/n} = \det(x)^{k/n}$ and it is a relative invariant with the character $\chi^{k/n}$. On the other hand, G^1-invariant homogeneous differential operator can not been written by using only one differential operator. However, we can prove that every G^1-invariant homogeneous differential operator is automatically relatively invariant differential operator. Therefore, if $P(x, \partial)$ is a G^1-invariant homogeneous differential operator, then there exists an integer $l \in \mathbb{Z}$ satisfying that $P(x, \partial)$ is a relatively invariant differential operator with the character χ^l. Then it is a homogeneous differential operator of homogeneous degree ln.

We shall give some examples of G-invariant homogeneous differential operators.

Example 2.1. We define ∂^* by

$$\partial^* = (\partial^*_{ij}) = \left(\epsilon_{ij}\frac{\partial}{\partial x_{ij}}\right), \text{ and } \epsilon_{ij} := \begin{cases} 1 & i = j \\ 1/2 & i \neq j \end{cases} \tag{8}$$

Let h and n be positive integers with $1 \leq h \leq n$. A sequence of increasing integers $p = (p_1, \ldots, p_h) \in \mathbb{Z}^h$ is called an *increasing sequence in* $[1, n]$ *of length* h if it satisfies $1 \leq p_1 < \cdots < p_h \leq n$. We denote by $IncSeq(h, n)$ the set of increasing sequences in $[1, n]$ of length h. For two sequences $p = (p_1, \ldots, p_h)$ and $q = (q_1, \ldots, q_h) \in IncSeq(h, n)$ and for an $n \times n$ symmetric matrix $x = (x_{ij}) \in \mathrm{Sym}_n(\mathbb{R})$, we define an $h \times h$ matrix $x_{(p,q)}$ by

$$x_{(p,q)} := (x_{p_i, q_j})_{1 \leq i \leq j \leq h}.$$

In the same way, for an $n \times n$ symmetric matrix $\partial = (\partial_{ij})$ of differential operators, we define an $h \times h$ matrix $\partial_{(p,q)}$ of differential operators by

$$\partial^*_{(p,q)} := (\partial^*_{p_i, q_j})_{1 \leq i \leq j \leq h}.$$

1. For an integer h with $1 \leq h \leq n$, we define

$$P_h(x, \partial) := \sum_{p,q \in IncSeq(h,n)} \det(x_{(p,q)}) \det(\partial^*_{(p,q)}). \tag{9}$$

   In particular, $P_n(x, \partial) = \det(x)\det(\partial^*)$ and *Euler's differential operator* is given by

$$P_1(x, \partial) = \sum_{n \geq j \geq i \geq 1} x_{ij}\frac{\partial}{\partial x_{ij}} = \mathrm{tr}(x \cdot \partial^*). \tag{10}$$

   These are all homogeneous differential operators of degree 0 and invariant under the action of $\mathbf{GL}(\boldsymbol{V})$, and hence it is also invariant under the action of $\boldsymbol{G}_1 := \mathbf{SL}_n(\mathbb{R}) \subset \mathbf{GL}(\boldsymbol{V})$.

2. $\det(x)$ and $\det(\partial^*)$ are homogeneous differential operators of degree n and $-n$, respectively. They are invariant under the action of $\boldsymbol{G}_1 := \mathbf{SL}_n(\mathbb{R})$, and relatively invariant differential operators with characters χ and χ^{-1}, respectively.

It is an interesting and important problem to find a "good" set of generators of the $\mathbb{C}$-algebra $D(\boldsymbol{V})^{\boldsymbol{G}}$ and $D(\boldsymbol{V})^{\boldsymbol{G}^1}$. One typical set of algebraically independent generators of the algebra of $\mathbf{GL}_n(\mathbb{R})$-invariant differential operators $D(\boldsymbol{V})^{\boldsymbol{G}} = D(\mathrm{Sym}_n(\mathbb{R}))^{\mathbf{GL}_n(\mathbb{R})}$ has been already obtained by Maass [4]. (See also Nomura [11].) It is easily checked that one certain set of algebraically independent generators of the algebra of $\mathbf{SL}_n(\mathbb{R})$-invariant differential operators $D(\boldsymbol{V})^{\boldsymbol{G}^1} = D(\mathrm{Sym}_n(\mathbb{R}))^{\mathbf{SL}_n(\mathbb{R})}$ is obtained by adding $\det(x)$ and $\det(\partial^*)$ to the Maass's generator set of $D(\mathrm{Sym}_n(\mathbb{R}))^{\mathbf{GL}_n(\mathbb{R})}$. Namely we have the following proposition.

Proposition 2.1. *Every $\boldsymbol{G}$-invariant differential operator on $\boldsymbol{V}$ can be expressed as a polynomial in $P_i(x,\partial)(i = 1,\ldots,n)$. Every $\boldsymbol{G}^1$-invariant differential operator on $\boldsymbol{V}$ can be expressed as a polynomial in $P_i(x,\partial)$ $(i = 1,\ldots,n-1)$,* $\det(x)$ *and* $\det(\partial)$.

The b_P-functions for the homogeneous differential operators $P_i(x,\partial)$ $(i = 1,\ldots,n-1)$, $\det(x)$ and $\det(\partial)$ can be computed explicitly by the aid of the theory of prehomogeneous vector space.

Proposition 2.2. *The b_P-functions for the invariant differential operators $P_i(x,\partial)$ $(i = 1,\ldots,n-1)$,* $\det(x)$ *and* $\det(\partial)$ *are given by the following formulas.*

1. *For the homogeneous differential operator $P_i(x,\partial)$ defined by (9), we have*

$$P_i(x,\partial)|f(x)|_j^s = (const.) \times (s)(s+\frac{1}{2})\cdots(s+\frac{i-1}{2})|f(x)|_j^s$$

   Then the b_P-function for $P_i(x,\partial)$ is

$$b_P(s) = (const.) \times (s)(s+\frac{1}{2})\cdots(s+\frac{i-1}{2}). \tag{11}$$

2. *For the homogeneous differential operator* $\det(x)$ *of homogeneous degree n, we have*

$$\det(x)|f(x)|_j^s = f(x)|f(x)|_j^s$$

   Then the b_P-function for $\det(x)$ *is*

$$b_P(s) = 1. \tag{12}$$

3. *For the homogeneous differential operator* $\det(\partial^*)$ *of homogeneous degree $-n$, we have*

$$\det(\partial^*)|f(x)|_j^s = (const.) \times (s)(s+\frac{1}{2})\cdots(s+\frac{n-1}{2})f(x)|f(x)|_j^{s-2}$$

   Then the b_P-function for $\det(\partial^*)$ *is*

$$b_P(s) = (const.) \times (s)(s+\frac{1}{2})\cdots(s+\frac{n-1}{2}). \tag{13}$$

By combining Proposition 2.1 and Proposition 2.2, we can compute the b_P-function for any G^1-invariant homogeneous differential operator $P(x, \partial)$ if we can find an algorithm to write the operator $P(x, \partial)$ as a polynomial in variables

$$\{P_i(x, \partial)(i = 1, \ldots, n-1), \det(x) \text{ and } \det(\partial)\}.$$

For example, consider the G^1-invariant homogeneous differential operator

$$P(x, \partial) = \det(\partial^*) \det(x) - \det(x) \det(\partial^*).$$

Since we have already computed the b_P-functions for $P(x, \partial) = \det(x)$ and $P(x, \partial) = \det(\partial^*)$ in (12) and (13), the b_P-function for $\det(\partial^*) \det(x) - \det(x) \det(\partial^*)$ is also computed by

$$\begin{aligned} b_P(s) &= (const.) \times \left((s+1)(s+\frac{3}{2}) \cdots (s+\frac{n+1}{2}) - (s)(s+\frac{1}{2}) \cdots (s+\frac{n-1}{2}) \right) \\ &= (const.) \times (s+\frac{n+1}{4})(s+1)(s+\frac{3}{2}) \cdots (s+\frac{n-1}{2}). \end{aligned}$$

3. Algorithm to compute b_P-functions via Gröbner basis.

By the arguments in the preceding section, the computation of b_P-function of $P(x, \partial) \in D(V)^{G^1}$ is reduced to the problem to write $P(x, \partial)$ as a polynomial in $P_1(x, \partial), \ldots, P_{n-1}(x, \partial), \det(x), \det(\partial^*)$. In this section, we shall give an algorithm to compute the expression of a given $P(x, \partial) \in D(V)^{G^1}$ in terms of $P_1(x, \partial), \ldots, P_{n-1}(x, \partial), \det(x), \det(\partial^*)$ as a polynomial.

Let V^* be a dual space of the vector space $V = \mathrm{Sym}_n(\mathbb{R})$. We first give a necessary and sufficient condition for a polynomial $p(x, \xi)$ on $(x, \xi) \in V \times V^*$ to be written as a polynomial in $f_1(x, \xi), \ldots, f_m(x, \xi) \in \mathbb{C}[x, \xi]$, where $f_1(x, \xi), \ldots, f_m(x, \xi)$ are polynomials on $V \times V^*$ which are not necessarily algebraically independent.

Proposition 3.1 (Cox, Little and O'shea [1] Chapter 7). *Suppose* $f_1, \ldots, f_p \in \mathbb{C}[x_1, \ldots, x_m, \xi_1, \ldots, \xi_m]$ *are given. We fix a monomial order in*

$$\mathbb{C}[x_1, \ldots, x_m, \xi_1, \ldots, \xi_m, y_1, \ldots, y_p]$$

where any monomial involving one of $x_1, \ldots, x_m, \xi_1, \ldots, \xi_m$ *is greater than all monomials in* $\mathbb{C}[y_1, \ldots, y_p]$. *Let* Gr *be a Gröbner basis of the ideal*

$$\langle f_1 - y_1, \ldots, f_p - y_p \rangle \subset \mathbb{C}[x_1, \ldots, x_m, \xi_1, \ldots, \xi_m, y_1, \ldots, y_p]$$

Given $f \in \mathbb{C}[x_1, \ldots, x_m, \xi_1, \ldots, \xi_m]$, *let* $g := \overline{f}^{Gr}$ *be the remainder of* f *on division by* Gr. *Then:*

1. $f \in \mathbb{C}[f_1, \ldots, f_p]$ *if and only if* $g \in \mathbb{C}[y_1, \ldots, y_p]$
2. *If* $f \in \mathbb{C}[f_1, \ldots, f_p]$, *then* $f = g(f_1, \ldots, f_p)$ *is an expression of* f *as a polynomial in* $f_1, \ldots, f_p$.

By using Proposition 3.1, we can give Algorithm 3.1.

Algorithm 3.1 (Writing in $D(\boldsymbol{V})^{G^1}$). *The following is an algorithm to compute a polynomial $Q(y_1, \ldots, y_{n+1})$ satisfying*

$$Q(P_1(x,\partial), \ldots, P_{n-1}(x,\partial), \det(x), \det(\partial^*)) = P(x,\partial)$$

for a given $P(x,\partial) \in D(\boldsymbol{V})^{G^1}$. Here Gr is the Gröbner basis of the ideal

$$I := \langle y_1 - P_1(x,\xi), \ldots, y_{n-1} - P_{n-1}(x,\xi), y_n - \det(x), y_{n+1} - \det(\partial^*) \rangle$$

in the polynomial algebra $\mathbb{C}[x_1, \ldots, x_m, \xi_1, \ldots, \xi_m, y_1, \ldots, y_{n+1}]$ and $\overline{\sigma(P)(x,\xi)}^{Gr}$ is the remainder(or normal form) of the polynomial $\sigma(P)(x,\xi)$ (= the principal symbol of $P(x,\partial)$) on division by the Gröbner basis Gr

{*(Input and Output)*}
Input: $P(x,\partial) \in D(\boldsymbol{V})$

$$\textit{Output: } \begin{cases} Q(y_1, \ldots, y_{n+1}) & \textit{if } P(x,\partial) \in D(\boldsymbol{V})^{G^1} \\ \text{“}P(x,\partial) \textit{ is not } \boldsymbol{G}^1\textit{-invariant.”} & \textit{if } P(x,\partial) \notin D(\boldsymbol{V})^{G^1} \end{cases}$$

```
{(initialization)}
        q := the order of P(x,∂);
        Q := 0;
        P := P(x,∂);
{(iteration)}
        WHILE q > 0 DO
            R := σ(P)(x,ξ)^Gr ;
            IF R ∈ C[y_1, ..., y_{n+1}]
            THEN
                P := P−R(P_1(x,∂), ..., P_{n−1}(x,∂), det(x), det(∂*));
                Q := Q(y_1, ..., y_{n+1}) + R(y_1, ..., y_{n+1});
                q := the order of P(x,∂) ;
            ELSE
                q:=-1;
            FI;
        OD;
{(result)}
        IF q = 0
        THEN
            Q:=Q+P;
            RETURN(Q);
        ELSE
            "P(x,∂) is not G^1-invariant."
        FI;
```

Algorithm 3.1 works well. We shall show the correctness of the program below.

Let $x = (x_1, \dots, x_m)$ and $\xi = (\xi_1, \dots, \xi_m)$ be the coordinates on $\boldsymbol{V}$ and $\boldsymbol{V}^*$, respectively. Here, $m = \frac{n(n+1)}{2}$ and $(x_1, \dots, x_m)$ is a suitable arrangement of the entries of the matrix x. Let $\partial = (\partial_1, \dots, \partial_m)$ be the partial differential operators with respect to the coordinate $x = (x_1, \dots, x_m)$. The differential operators ∂ have the commutation relations

$$\partial_j x_i - x_i \partial_j = \delta_{ij}$$

where δ_{ij} is the Kronecker's delta.

For a given $P(x, \partial) \in D(V)^{\boldsymbol{G}^1}$, we suppose that it is written as

$$P(x, \partial) = \sum_{\alpha, \beta \in \mathbb{Z}^m_{\geq 0}} a_{\alpha\beta} x^\alpha \partial^\beta.$$

It is also written as

$$P(x, \partial) = \sum_{k=0}^{q} P(x, \partial)_k$$

with

$$P(x, \partial)_k = \sum_{\alpha, \beta \in \mathbb{Z}^m_{\geq 0}, |\beta| = k} a_{\alpha\beta} x^\alpha \partial^\beta.$$

We call the non-negative integer q the *order* of $P(x, \partial)$, $P(x, \partial)_q$ the *principal part* of $P(x, \partial)$ and the polynomial $P(x, \xi)_q$ on $\boldsymbol{V} \times \boldsymbol{V}^*$ obtained by exchanging ξ and ∂ the *principal symbol* of $P(x, \partial)$. We often denote by $\sigma(P(x, \partial))$ or by $\sigma(P)(x, \xi)$ the principal symbol $P(x, \xi)_q$.

We see below which result we obtain after carrying out the program for an input $P(x, \partial) \in D(\boldsymbol{V})$.

First, we substitute given $P(x, \partial)$ for P, the order of $P(x, \partial)$ for q and 0 for Q in the initialization process. Then suppose first that $P(x, \partial) \in D(\boldsymbol{V})^{\boldsymbol{G}^1}$. We repeat the following operations in the iteration process. Since $\sigma(P(x, \partial))$ is a $\boldsymbol{G}^1$-invariant polynomial on $\boldsymbol{V} \times \boldsymbol{V}^*$,

$$R := \overline{\sigma(P(x, \partial))}^{Gr}$$

is a polynomial in $\mathbb{C}[y_1, \dots, y_{n+1}]$ and

$$\sigma(P(x, \partial)) = R(P_1(x, \xi), \dots, P_{n-1}(x, \xi), \det(x), \det(\xi^*))$$

by Proposition 3.1[3]. Then we substitute

$$P(x, \partial) - R(P_1(x, \partial), \dots, P_{n-1}(x, \partial), \det(x), \det(\partial^*)) \tag{14}$$

for P and

$$Q(y_1, \dots, y_{n+1}) + R(y_1, \dots, y_{n+1}) \tag{15}$$

for Q and then we substitute the order of $P(x, \partial)$ for q. The order of (14) is strictly less than that of $P(x, \partial)$ since the principal parts of $P(x, \partial)$ and $r(P_1(x, \partial), \dots, P_{n-1}(x, \partial), \det(x), \det(\partial^*))$ coincide with each other[4]. Then

[3]This is the first essential point of the algrithm.

[4]This is the second essential point of the algrithm.

q will be 0 sometime after several repetition of the iteration process and then escape from the process. Therefore the iteration process must stop after finite steps of iteration. While we are carrying out the iteration process, $P(x, \partial)$ is always in $D(\boldsymbol{V})^{\boldsymbol{G}^1}$ and becomes the operator of order 0 after all.

$$\begin{gathered}Q(P_1(x,\partial),\dots,P_{n-1}(x,\partial),\det(x),\det(\partial^*))\\+P(P_1(x,\partial),\dots,P_{n-1}(x,\partial),\det(x),\det(\partial^*))\end{gathered}$$

is invariant through the iteration process. In the result process, by substituting $Q := Q + P$,

$$Q(P_1(x,\partial),\dots,P_{n-1}(x,\partial),\det(x),\det(\partial^*))$$

coincides with the originally given $P(x, \partial)$. This means that $Q(y_1, \dots, y_{n+1}) \in \mathbb{C}[y_1, \dots, y_{n+1}]$ is what we are seeking for.

Next we suppose first that $P(x, \partial) \notin D(\boldsymbol{V})^{\boldsymbol{G}^1}$. Then while we are repeating the operations in the iteration process, we will sometime encounter the result

$$R := \overline{\sigma(P(x,\partial))}^{Gr} \notin \mathbb{C}[y_1,\dots,y_{n+1}].$$

Then the program escapes from the loop and stops by outputting the message "$P(x, \partial)$ is not $\boldsymbol{G}^1$-invariant".

Thus we have proved that the algorithm works well. Algorithm 3.1 gives one expression of $P(x, \partial)$ as an polynomial in $P_1(x,\partial),\dots,P_{n-1}(x,\partial),\det(x),\det(\partial^*)$. However, this expression is not unique. This is because the generators

$$P_1(x,\partial),\dots,P_{n-1}(x,\partial),\det(x),\det(\partial^*)$$

of $D(\boldsymbol{V})^{\boldsymbol{G}^1}$ are not algebraically independent.

As a special case of Algorithm 3.1, we can give the same algorithm to compute the polynomial expression of $P(x, \partial) \in D(\boldsymbol{V})^{\boldsymbol{G}}$ in terms of $P_1(x,\partial),\dots,P_n(x,\partial)$. However since

$$D(\boldsymbol{V})^{\boldsymbol{G}} = \mathbb{C}[P_1(x,\partial),\dots,P_n(x,\partial)]$$

the polynomial expression of $P(x, \partial)$ is unique.

Algorithm 3.2 (Writing in $D(\boldsymbol{V})^{\boldsymbol{G}}$). *The following is an algorithm to compute a polynomial $Q(y_1, \dots, y_n)$ satisfying*

$$Q(P_1(x,\partial),\dots,P_n(x,\partial)) = P(x,\partial)$$

for a given $P(x, \partial) \in D(\boldsymbol{V})^{\boldsymbol{G}}$. Here Gr is the Gröbner basis of the ideal

$$I := \langle y_1 - P_1(x,\xi),\dots,y_n - P_n(x,\xi)\rangle$$

in the polynomial algebra $\mathbb{C}[x_1,\dots,x_m,\xi_1,\dots,\xi_m,y_1,\dots,y_n]$ and $\overline{\sigma(P)(x,\xi)}^{Gr}$ is the remainder of the polynomial $\sigma(P)(x,\xi)$ on division by the Gröbner basis Gr

{(Input and Output)}
Input: $P(x,\partial) \in D(\boldsymbol{V})$

$$\textit{Output: } \begin{cases} Q(y_1,\dots,y_n) & \textit{if } P(x,\partial) \in D(\boldsymbol{V})^{\boldsymbol{G}} \\ \text{“}P(x,\partial) \textit{ is not } \boldsymbol{G}\textit{-invariant.”} & \textit{if } P(x,\partial) \notin D(\boldsymbol{V})^{\boldsymbol{G}} \end{cases}$$

{(initialization)}
 $q :=$ *the order of* $P(x,\partial)$;
 $Q := 0$;
 $P := P(x,\partial)$;
{(iteration)}
 WHILE $q > 0$ *DO*
  $R := \overline{\sigma(P)(x,\xi)}^{Gr}$;
  IF $R \in \mathbb{C}[y_1,\dots,y_n]$
  THEN
   $P := P - R(P_1(x,\partial),\dots,P_n(x,\partial))$;
   $Q := Q(y_1,\dots,y_n) + R(y_1,\dots,y_n)$;
   $q :=$ *the order of* $P(x,\partial)$;
  ELSE
   q:=-1;
  FI;
 OD;
{(result)}
 IF $q = 0$
 THEN
  Q:=Q+P;
  RETURN(Q);
 ELSE
  “$P(x,\partial)$ *is not* $\boldsymbol{G}$*-invariant.*”
 FI;

We can prove that Algorithm 3.2 works well in the same way as the proof of Algorithm 3.1. Algorithm 3.2 gives one expression of $P(x,\partial)$ as an polynomial in $P_1(x,\partial),\dots,P_n(x,\partial)$. The expression is unique in this case. This is because $P_1(x,\partial),\dots,P_n(x,\partial)$ are algebraically independent generators of $D(\boldsymbol{V})^{\boldsymbol{G}}$ and $D(\boldsymbol{V})^{\boldsymbol{G}}$ is isomorphic to the polynomial algebra $\mathbb{C}[P_1(x,\partial),\dots,P_n(x,\partial)]$.

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