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ON THE COLORED JONES POLYNOMIALS OF SOME SIMPLE LINKS

KAZUO HABIRO

葉広和夫

東京大学数理科学研究科

1. INTRODUCTION

This note is a report on my talk given at the workshop "Recent Progress Towards the Volume Conjecture" which was held at International Institute for Advanced Studies in March, 2000, and was supported by Research Institute for Mathematical Sciences, Kyoto University. I wish to thank the participants for conversation and comments.

We describe below a way to calculate the colored Jones polynomials of some "simple" links, such as the trefoil and the figure eight knots, the Whitehead and the Borromean links. Similar calculations have been performed by other authors, see [Le].

In our calculation, we use a basis in the Kauffman bracket skein module of the solid torus, which is a modification of some bases used in [BHMV] and [B]. We give only outlines here; the details will be given in a paper in preparation [H].

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2. Kauffman bracket skein modules

Let us recall the definition of the Kauffman bracket skein module B of a connected oriented 3-manifold.

Let A be an indeterminate and let $\mathbf{Z}[A, A^{-1}]$ be the Laurent polynomial ring. We set $a = A^2$. We use q-integers, q-factorials and q-binomial coefficient in a:

$$[n] = \frac{a^n - a^{-n}}{a - a^{-1}},$$
$$[n]! = [1][2] \dots [n], \quad n \geq 0,$$
$$\begin{bmatrix} n \\ i \end{bmatrix} = \frac{[n]!}{[i]![n-i]!}, \quad n \geq i \geq 0,$$

for integers n, i.

Let M be a connected oriented 3-manifold. Let $\mathcal{L}(M)$ denote the free $\mathbf{Z}[A, A^{-1}]$-module generated by the set of ambient isotopy classes of nonoriented framed links in M. The Kauffman bracket skein module $B(M)$ of M is defined to be the quotient of $\mathcal{L}(M)$ by the following relations.

1. $L \sqcup U = -[2]L$, where U is an unknot and $\sqcup$ denote split union,
2. $\langle \times \rangle = A \langle)(\rangle + A^{-1} \langle \asymp \rangle$.

We have $B(S^3) \cong B(B^3) \cong \mathbf{Z}[A, A^{-1}]$ and we define for a framed link L in S^3 (or in B^3), the Kauffman bracket $\langle L \rangle \in \mathbf{Z}[A, A^{-1}]$ by $L = \langle L \rangle \emptyset$ in $B(S^3)$. In particular, we have $\langle \emptyset \rangle = 1$.

3. Results

Here we list our results of calculations. In the following the links are assumed to be framed with framings zero.

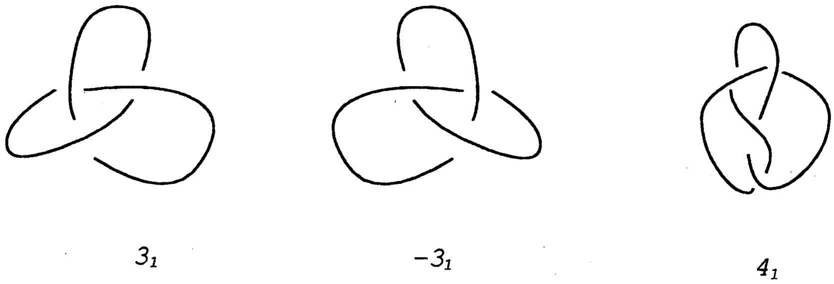

FIGURE 1

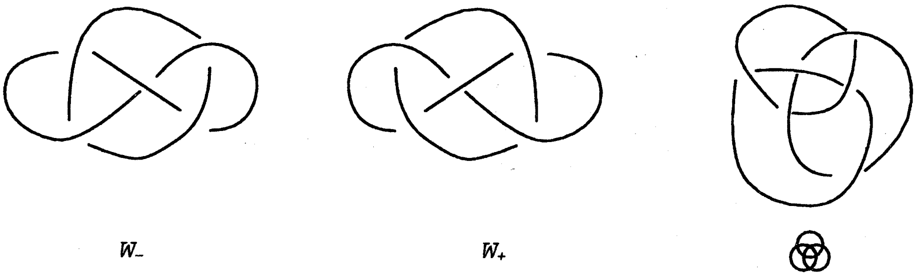

FIGURE 2

Let 3_1 denote the trefoil knots with negative crossings (and hence of positive signature), and let -3_1 denote the mirror image of 3_1. Let 4_1 denote the figure eight knot. See Figure 1. We have

$$
\begin{aligned}
\langle e_n \rangle_{3_1} &= \sum_{i=0}^{n} (-1)^i a^{i(i+3)} (a - a^{-1})^{2i} \frac{[n+i+1]!}{[n-i]!} \\
\langle e_n \rangle_{-3_1} &= \sum_{i=0}^{n} (-1)^i a^{-i(i+3)} (a - a^{-1})^{2i} \frac{[n+i+1]!}{[n-i]!} \qquad (1) \\
\langle e_n \rangle_{4_1} &= \sum_{i=0}^{n} (a - a^{-1})^{2i} \frac{[n+i+1]!}{[n-i]!}
\end{aligned}
$$

The same formula for 4_1 has been obtained by T. Le [Le].

Let W_- and W_+ denote the Whitehead links and let denote the Borromean rings as shown in Figure 2. We have

$$\langle e_m, e_n\rangle_{W_+} = \sum_{i=0}^{\min(m,n)} (-1)^i a^{\frac{1}{2}i(i+3)}(a-a^{-1})^{3i}\frac{[m+i+1]![n+i+1]![i]!}{[m-i]![n-i]![2p+1]!}$$

$$\langle e_m, e_n\rangle_{W_-} = \sum_{i=0}^{\min(m,n)} a^{-\frac{1}{2}i(i+3)}(a-a^{-1})^{3i}\frac{[m+i+1]![n+i+1]![i]!}{[m-i]![n-i]![2p+1]!}$$

$$\langle e_l, e_m, e_n\rangle_{\circledast} = \sum_{i=0}^{\min(l,m,n)} (a-a^{-1})^{4i}\frac{[l+i+1]![m+i+1]![n+i+1]!([i]!)^2}{[l-i]![m-i]![n-i]!([2p+1]!)^2}$$

Note that the six sums above can be regarded as the sums of the form $\sum_{i=0}^{\infty}$, with the convention that $1/[p]! = 0$ if $p < 0$. A systematic study on similar formulae for more general links will be performed in [H].

4. The Kauffman bracket skein module of the solid torus

We use the Kauffman bracket skein module $B = B(V)$ of the solid torus $V = S^1 \times D^2$. For details, see [BHMV].

We use the familiar commutative $\mathbf{Z}[A, A^{-1}]$-algebra structure of B. B is a polynomial algebra $\mathbf{Z}[A, A^{-1}][z]$, where z is represented by the framed knot K in V which winds just once along the core of V.

Let B^{even} denote the $\mathbf{Z}[A, A^{-1}]$-subalgebra of B generated by z^2, and set $B^{\text{odd}} = zB^{\text{even}}$. Then the direct sum decomposition $B = B^{\text{even}} \oplus B^{\text{odd}}$ gives a $\mathbf{Z}/2$-graded algebra structure on B.

In [BHMV] the *natural basis* $\{e_i | i = 0, 1, 2, \dots\}$ of B is defined inductively by $e_0 = 1$, $e_1 = z$ and $ze_j = e_{j-1} + e_{j+1}$ for $j \geq 1$. We have $e_j \in B^{\text{even}}$ if j is even and, $e_j \in B^{\text{odd}}$ otherwise. Each e_i corresponds to the $(i+1)$-dimensional irreducible representation of the quantum group $U_q(sl_2)$.

5. The pairing

Let $L = L_1 \cup \cdots \cup L_m$ be an ordered nonoriented framed link in a 3-manifold M. A $\mathbf{Z}[A, A^{-1}]$-linear map

$$B \otimes_{\mathbf{Z}[A,A^{-1}]} \cdots \otimes_{\mathbf{Z}[A,A^{-1}]} B \to B(M), u_1 \otimes \ldots \otimes u_m \mapsto \langle u_1, \ldots, u_m \rangle_L,$$

introduced in [BHMV], is defined by setting $\langle z^{l_1}, \ldots, z^{l_m} \rangle_L$ to be the element of $B(M)$ obtained from L by replacing each component L_i with l_i parallel copies of L_i. In particular, if M is S^3 or B^3, then the above map is regarded as $\mathbf{Z}[A, A^{-1}]$-valued.

Let $\mathrm{Hopf} = \mathrm{Hopf}_1 \cup \mathrm{Hopf}_2$ be the nonoriented Hopf link with framings zero. The element $\langle u_1, u_2 \rangle_{\mathrm{Hopf}}$ is denoted simply by $\langle u_1, u_2 \rangle$. Thus we have a $\mathbf{Z}[A, A^{-1}]$-linear map

$$\langle , \rangle \colon B \otimes_{\mathbf{Z}[A,A^{-1}]} B \to \mathbf{Z}[A, A^{-1}].$$

Clearly $\langle , \rangle$ is symmetric: $\langle u, v \rangle = \langle v, u \rangle$.

We also define a $\mathbf{Z}[A, A^{-1}]$-linear map $\langle\rangle = \langle\rangle_{\mathrm{unknot}} \colon B \to \mathbf{Z}[A, A^{-1}]$. We have, for each $x \in B$, $\langle x \rangle = \langle x, 1 \rangle$.

In [BHMV] a basis $\{Q_n\}_{n \geq 0}$ of B is defined by

$$Q_n = \prod_{i=0}^{n-1} (z + a^{i+1} + a^{-i-1}).$$

In particular, $Q_0 = 1$. It satisfies the following properties:

1. $\langle Q_n, e_i \rangle = \langle Q_n, z^i \rangle = \langle Q_n, Q_i \rangle = 0$ if $0 \leq i < n$.
2. $\langle Q_n, Q_n \rangle (= \langle Q_n, z^n \rangle = \langle Q_n, e_n \rangle) \neq 0$.

Namely the Q_i form an orthogonal basis of B. The property (1) above comes from the fact that $\langle z + a^{i+1} + a^{-i-1}, e_i \rangle = 0$.

We modify the definition of Q_n above as follows.

$$(2)\qquad \begin{aligned} R_n &= \prod_{i=0}^{n-1}(z + a^{2i+1} + a^{-2i-1}) \\ S_n &= \prod_{i=0}^{n-1}(z^2 - (-a^{i+1} - a^{-i-1})^2) \\ T_n &= \prod_{i=0}^{n-1}(z^2 - (-a^{2i+1} - a^{-2i-1})^2) \end{aligned}$$

Note that $\{R_n\}_{n\geq 0}$ is a basis of B and that $\{S_n\}_{n\geq 0}$ and $\{T_n\}_{n\geq 0}$ are bases of B^{even}. The R_n and S_n satisfy the following properties.

1. $\langle R_n, e_{2i}\rangle = \langle R_n, z^{2i}\rangle = \langle R_n, S_i\rangle = 0$ if $0 \leq i < n$.
2. $\langle e_n, S_i\rangle = \langle z^n, S_i\rangle = \langle R_n, S_i\rangle = 0$ if $0 \leq n < i$.
3. $\langle R_n, S_n\rangle = (-1)^n(a - a^{-1})^{2n}[2n+1]!$.

The above properties implies that the basis $\{R_n\}_n$ of B and the basis $\{S_n\}_n$ of B^{even} are dual to each other up to multiplication by nonzero factors. Similarly, $\{T_n\}_n$ is an orthogonal basis of B^{even}. Blanchet has defined similar but different bases of B^{even} and B^{odd} which are orthogonal to each other [B].

In what follows we use only the R_n basis.

Proposition 5.1. *For each $n \geq 0$,*

$$e_n = \sum_{i=0}^{n}(-1)^i \begin{bmatrix} n+i+1 \\ n-i \end{bmatrix} R_i.$$

6. The twisting element

Let $k = \mathbf{Q}(A)$ denote the quotient field of $\mathbf{Z}[A, A^{-1}]$. Let $B_k = B \otimes_{\mathbf{Z}[A,A^{-1}]} k$. We set $R'_n = R_n/[n]! \in B_k$. Let B' denote the $\mathbf{Z}[A, A^{-1}]$-submodule of B_k generated by $\{R'_n\}_{n\geq 0}$. It is actually a $\mathbf{Z}[A, A^{-1}]$-subalgebra of B_k.

Set $I_n = R'_n B' \subset B'$. Let $\hat{B}'$ denote the projective limit $\varprojlim_{n\to\infty} B'/I_n$ of

$$\dots \xrightarrow{\text{proj}} B'/I_2 \xrightarrow{\text{proj}} B'/I_1 \xrightarrow{\text{proj}} B'/I_0.$$

Proposition 6.1. *For $n \geq 0$ and $a \in B^{\text{even}}$ we have $\langle R'_n, a\rangle \in \mathbf{Z}[A, A^{-1}]$. Hence there is the natural well-defined pairing*

$$\langle , \rangle \colon \hat{B}' \otimes_{\mathbf{Z}[A,A^{-1}]} B^{\text{even}} \to \mathbf{Z}[A, A^{-1}].$$

We define an element $\omega \in \hat{B}'$ as follows.

$$\omega = \sum_{n=0}^{\infty} (-1)^n a^{\frac{1}{2}n(n+3)} R'_n. \tag{3}$$

Let $t\colon B \to B$ the $\mathbf{Z}[A, A^{-1}]$-linear map defined as one positive full twist, see [BHMV]. Alternatively, t is the unique $\mathbf{Z}[A, A^{-1}]$-linear map defined by $t(e_n) = (-1)^n A^{n(n+2)}$ for $n \geq 0$.

Proposition 6.2. *The element ω is invertible in $\hat{B}'$ with the inverse*

$$\omega^{-1} = \sum_{n=0}^{\infty} a^{-\frac{1}{2}n(n+3)} R'_n.$$

We have for every $x \in B^{\text{even}}$

$$\langle \omega^{\pm 1}, x\rangle = \langle t^{\pm 1}(x)\rangle.$$

Proposition 6.2 may be rephrased that the element ω gives a positive full twist to even numbers of strings linking with ω.

An element similar to ω but based on the Q_n basis is defined in [BHMV].

7. Borromean rings

Let $\circledast$ denote the Borromean rings.

Proposition 7.1. *We have*

$$\langle R'_l, R'_m, R'_n \rangle_{\circledast} = \begin{cases} (a - a^{-1})^n \frac{[2n+1]!}{[n]!}, & \text{if } l = m = n, \\ 0, & \text{otherwise.} \end{cases}$$

8. Surgery descriptions

In what follows we sometimes refer to framed links with zero framings simply as 'links'.

Definition 8.1. Let L be an m-component *algebraically split* link in S^3, i.e., a link in S^3 with all the linking numbers of components are zero. In particular, a knot is algebraically split. A *surgery description* of an algebraically split link L is a pair (L_0, J) of L_0 and a framed link J such that

1. J is disjoint from L_0,
2. J is an unlink,
3. $L_0 \cup J$ is algebraically split,
4. each component of J has framing ± 1,
5. the result from L_0 of surgery on J is ambient isotopic to L.

Proposition 8.2. *Let L be an m-component algebraically split link and let (L_0, J) be a surgery description of L. Let l denote the number of components of J and let $f_p \in \{1, -1\}$ $(p = 1, \dots, l)$ denote the framing of the pth component of J. Then we have, for $a_1, \dots, a_m \in B$,*

$$\langle a_1, \dots, a_m \rangle_L = \langle a_1, \dots, a_m, \omega^{-f_1}, \dots, \omega^{-f_l} \rangle_{L_0 \cup J} \in \mathbf{Z}[A, A^{-1}]$$

(and the right hand side is well defined).

Remark 8.3. Proposition 8.2 can be stated in a stronger form. Details will appear in [H].

9. Calculation of the colored Jones polynomial of the trefoil knot

Here we illustrate how to calculate the colored Jones polynomial $\langle e_n \rangle_{3_1}$ of the trefoil knot 3_1 with negative crossings. The polynomials for other links listed in Section 3 can be calculated similarly.

The trefoil knot 3_1 has a surgery description $(U, J_1 \cup J_2)$ such that

1. $U \cup J_1 \cup J_2$ is the Borromean rings,
2. the framings of J_1 and J_2 are both -1.

Hence we have by Propositions 8.2 and 7.1

$$\begin{aligned}
\langle R_n \rangle_{3_1} &= \langle R_n, \omega, \omega \rangle_{\circledast} \\
&= (a - a^{-1})^n [n]! \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a^{\frac{1}{2}i(i+3) + \frac{1}{2}j(j+3)} \langle R'_n, R'_i, R'_j \rangle_{\circledast} \\
&= (a - a^{-1})^n [n]! a^{n(n+3)} \cdot (a - a^{-1})^n \frac{[2n+1]!}{[n]!} \\
&= a^{n(n+3)} (a - a^{-1})^{2n} [2n+1]!.
\end{aligned}$$

By Proposition 5.1, we have

$$\begin{aligned}\langle e_n\rangle_{3_1} &= \sum_{i=0}^{n}(-1)^i \begin{bmatrix} n+i+1 \\ n-i \end{bmatrix} \langle R_i\rangle_{3_1} \\ &= \sum_{i=0}^{n}(-1)^i a^{i(i+3)}(a-a^{-1})^{2i}[2i+1]! \begin{bmatrix} n+i+1 \\ n-i \end{bmatrix} \\ &= \sum_{i=0}^{n}(-1)^i a^{i(i+3)}(a-a^{-1})^{2i}\frac{[n+i+1]!}{[n-i]!}.\end{aligned}$$

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Graduate School of Mathematical Sciences, University of Tokyo
E-mail address: `habiro@ms.u-tokyo.ac.jp`