数理解析研究所講究録
1172巻 2000年 44-69

Notes on Construction of the Knot Invariant from Quantum Dilogarithm Function

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1 Introduction

Since the discovery of the Jones polynomial [15], the quantum group has been used to construct the invariants of knots and links, and many knot invariants such as HOMFLY polynomial [9], colored Jones polynomial [4], Kauffman polynomial [22], have been proposed. Recently Kashaev constructed a knot invariant by use of the cyclic quantum dilogarithm function [17, 19, 21]. It was shown in Ref. 28 that Kashaev's invariant exactly coincides with the colored Jones polynomial at N-th root of unity, but what is remarkable is that he claimed that the asymptotic value of his knot invariant (or, the colored Jones polynomial) in a limit $N \to \infty$ coincides with the hyperbolic volume of the knot complement. Due to the fact that the hyperbolic knot complement is decomposed into the ideal tetrahedra (see, e.g., Ref. 36), and that the volume of each tetrahedron is given by use of the Lobachevsky function (see, e.g. Ref. 27), it might be natural to Kashaev's invariant is connected with the hyperbolic volume. While Kashaev defined the knot invariant using the quantum dilogarithm function with q being N-th root of unity (cyclic dilogarithm function) and studied the asymptotic behavior $N \to \infty$, our purpose here is rather to use the infinite dimensional representation of the quantum dilogarithm function in a case of $|q| = 1$ and then take a limit $q \to 1$.

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This Note is organized as follows. We first review several relations for the dilogarithm function. See Ref. 25 and references therein for topics of the dilogarithm function. We then define the quantum dilogarithm function as a q-deformation of the dilogarithm function. Depending on a deformation parameter q, we have two definitions of the quantum dilogarithm function; one of them is for q generic, and it is essentially given by the q-exponential function. In the case of $|q| = 1$, we have another definition in an integral form [5]. We show that the quantum dilogarithm function satisfies interesting properties with non-commutative variables. See Ref. 26 for a survey on the special functions and q-commuting variables. At last stage we show that the R-operator as a solution of the constant Yang–Baxter equation can be given from the quantum dilogarithm function. We compute the matrix elements on the infinite dimensional space, and based on this R-operator we construct the knot invariant.

2 Dilogarithm Function

The Euler dilogarithm function $Li_2(x)$ is defined by

$$Li_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} \tag{2.1a}$$

$$= - \int_0^x \frac{\log(1-s)}{s} \, ds, \tag{2.1b}$$

which gives

$$Li_2(0) = 0, \qquad Li_2(1) = \frac{\pi^2}{6}.$$

By the integral representation (2.1b), the Euler dilogarithm $Li_2(x)$ is analytically continued to the complex plane with a cut along the real axis $[1, +\infty]$. We also use the Rogers dilogarithm function $L(x)$, which is given by

$$\begin{aligned} L(x) &= Li_2(x) + \frac{1}{2} \log x \cdot \log(1-x) \\ &= -\frac{1}{2} \int_0^x \left(\frac{\log(1-s)}{s} + \frac{\log s}{1-s} \right) ds. \end{aligned} \tag{2.2}$$

This function satisfies following relations;

$$L(x) + L(1-x) = \frac{\pi^2}{6}, \tag{2.3}$$

$$L(x) + L(y) = L(x\,y) + L(\frac{x\,(1-y)}{1-x\,y}) + L(\frac{y\,(1-x)}{1-x\,y}). \tag{2.4}$$

The second identity is called the pentagon identity. Note that the dilogarithm function often appears in various studies of mathematical physics, such as the computation of the central charge of the conformal field theory [23, 24], where a technique in Ref. 31 has been extensively applied.

The hyperbolic volume of the ideal tetrahedron with face angle α, β, and γ (we have $\alpha+\beta+\gamma=2\pi$) is given by $Л(\alpha)+Л(\beta)+Л(\gamma)$ [27], where the Lobachevsky function $Л(\theta)$ is defined as

$$Л(\theta) = -\int_0^\theta \log|2\sin u|\,du. \tag{2.5}$$

The function $Л(\theta)$ can be written in terms of the dilogarithm function as †

$$Li_2(\mathrm{e}^{2\,\mathrm{i}\,\theta}) = \frac{\pi^2}{6} - \theta\,(\pi-\theta) + 2\,\mathrm{i}\,Л(\theta). \tag{2.6}$$

Further when we parameterize an ideal tetrahedron by a complex parameter z with $\mathrm{Im}z > 0$, the hyperbolic volume is given by the Bloch–Wigner function $D(z)$,

$$D(z) = \arg(1-z)\cdot\log|z| + \mathrm{Im}Li_2(z). \tag{2.7}$$

3 Quantum Dilogarithm Function I

We define the quantum dilogarithm function $S_q(w)$ for $|q|<1$;

$$S_q(w) = \prod_{n=0}^{\infty}(1+q^{2\,n+1}\,w) \tag{3.1a}$$

$$= 1 + \sum_{k=1}^{\infty}\frac{(-1)^k\,q^{\frac{k\,(k-1)}{2}}\,w^k}{(q-q^{-1})\cdots(q^k-q^{-k})} \tag{3.1b}$$

$$= \exp\left(\sum_{k=1}^{\infty}\frac{(-1)^k\,w^k}{k\,(q^k-q^{-k})}\right). \tag{3.1c}$$

† $\mathrm{i} \equiv \sqrt{-1}$

These identities can be proved by the fact that each expression satisfies following difference equation and an initial condition;

$$\frac{S_q(q\,w)}{S_q(q^{-1}w)} = \frac{1}{1+w}, \tag{3.2}$$

$$S_q(0) = 1.$$

This definition shows that the function $S_q(w)$ is merely a q-exponential function. To see that this function is a one-parameter deformation of the dilogarithm, we take an asymptotic behavior $q \to 1$ in eq. (3.1a). Using the Euler–McLaughlin formula, we get

$$S_q(w) = \sqrt{1+w} \cdot \mathrm{e}^{-\frac{1}{\varepsilon} Li_2(-w)} \, (1 + O(\varepsilon^3)), \tag{3.3}$$

where we have set $q = \mathrm{e}^{-\frac{\varepsilon}{2}}$.

The reason why we call $S_q(w)$ as the quantum dilogarithm function is due to the fact that it also satisfies the pentagon identity [7]. When we use the Weyl operators $\hat{a}$ and $\hat{b}$ satisfying

$$\hat{a}\,\hat{b} = q^2\,\hat{b}\,\hat{a},$$

we have

$$S_q(\hat{a})\,S_q(\hat{b}) = S_q(\hat{a}+\hat{b}). \tag{3.4}$$

This identity first appeared in Ref. 33, and can be proved from eq. (3.1b) with a help of the q-binomial formula;

$$(\hat{a}+\hat{b})^n = \sum_{k=0}^{n} \frac{(q^2;q^2)_n}{(q^2;q^2)_k\,(q^2;q^2)_{n-k}} \hat{b}^{n-k}\,\hat{a}^k.$$

The function $S_q(w)$ further satisfies following identities;

$$S_q(\hat{b})\,S_q(\hat{a}) = S_q(\hat{a} + q^{-1}\hat{a}\,\hat{b})\,S_q(\hat{b}) \tag{3.5a}$$

$$= S_q(\hat{a} + \hat{b} + q^{-1}\hat{a}\,\hat{b}) \tag{3.5b}$$

$$= S_q(\hat{a})\,S_q(q^{-1}\hat{a}\,\hat{b} + \hat{b}) \tag{3.5c}$$

$$= S_q(\hat{a})\,S_q(q^{-1}\hat{a}\,\hat{b})\,S_q(\hat{b}). \tag{3.5d}$$

Proof is as follows. As we have

$$S_q(\hat{b}) \cdot \hat{a} \cdot \left(S_q(\hat{b})\right)^{-1} = \hat{a} \cdot S_q(q^{-2}\,\hat{b}) \cdot \left(S_q(\hat{b})\right)^{-1} = \hat{a} \cdot (1 + q^{-1}\hat{b}),$$

we obtain the first equality;

$$S_q(\hat{b}) \cdot S_q(\hat{a}) \cdot \left(S_q(\hat{b})\right)^{-1} = S_q\left(S_q(\hat{b}) \cdot \hat{a} \cdot \left(S_q(\hat{b})\right)^{-1}\right) = S_q(\hat{a} \cdot (1 + q^{-1}\hat{b})).$$

All other equalities can be derived by repeated use of eq. (3.4). The last equation is the quantum pentagon identity [7]. It was shown in Refs. 2, 7 that it gives the classical pentagon identity (2.4) in $q \to 1$ limit.

We can obtain a braid relation from the quantum dilogarithm function [25]. We define the function $\Theta(w)$ as

$$\begin{aligned} \Theta(w) &= S_q(q\,w)\,S_q(q^{-1}w^{-1}) \\ &= \frac{1}{(q^2;q^2)_\infty} \sum_{n\in\mathbb{Z}} q^{n^2}\,w^n. \end{aligned} \tag{3.6}$$

The second equality is the Jacobi triple product identity. When the operators $\hat{a}$ and $\hat{b}$ satisfy the q-commutation relation, $\hat{a}\,\hat{b} = q^2\,\hat{b}\,\hat{a}$, we obtain that the Θ-function satisfies the braid relation

$$\Theta(\hat{a})\,\Theta(\hat{b})\,\Theta(\hat{a}) = \Theta(\hat{b})\,\Theta(\hat{a})\,\Theta(\hat{b}). \tag{3.7}$$

Proof is straightforward by applying eqs. (3.4) and (3.5) [10, 25], and it can be extend to the sl(N) case [14]. We can give the knot invariant as a q-series by use of this braid relation [11], although this does not seem to be related with Kashaev's invariant. We rather give another solution of the braid relation in terms of the quantum dilogarithm function in the following section.

To close this section, we note that we can solve a non-constant solution of the Yang–Baxter equation in terms of $S_q(w)$ [8, 10, 12, 38], and it was shown [1] that the universal R-matrix with a q-oscillator representation reduces to this solution.

4 Quantum Dilogarithm Function II

In the case of the $|q| = 1$ we should modify the definition (3.1a) of the quantum dilogarithm function. Hereafter we set

$$q = e^{i\gamma}, \tag{4.1}$$

where γ is real,and it corresponds to the Planck constant $\gamma = \hbar/2$. We define $\Phi_\gamma(\varphi)$ by an integral form,

$$\Phi_\gamma(\varphi) = \exp\left(\int_{\mathbb{R}+i0} \frac{e^{-i\varphi x}}{4\,\mathrm{sh}(\gamma x)\,\mathrm{sh}(\pi x)} \frac{dx}{x}\right). \tag{4.2}$$

This integral was first introduced by Faddeev [5, 6]. See also Ref. 32 where a similar integral was studied in a context of the hyperbolic gamma function. The similarity between this integral and the scattering matrix of the Liouville theory is claimed in Ref. 5, and it follows from that the integral (4.2) plays a role of intertwining operator.

For our later convention to study the "volume conjecture", we have interests in the asymptotic behavior in a limit $q \to 1$, *i.e.*, $\gamma \to 0$. Like eq. (3.3), the function $\Phi_\gamma(\varphi)$ behaves in this limit as

$$\Phi_\gamma(\varphi) \sim \exp\left(\frac{1}{2i\gamma} Li_2(-e^{\varphi})\right), \qquad \text{for } \gamma \to 0. \tag{4.3}$$

This behavior indicates that the integral Φ_γ is indeed a deformation of the Euler dilogarithm function. We remark that we have an inversion relation,

$$\Phi_\gamma(\varphi) \cdot \Phi_\gamma(-\varphi) = \exp\left(-\frac{1}{2i\gamma}\left(\frac{\varphi^2}{2} + \frac{\pi^2+\gamma^2}{6}\right)\right), \tag{4.4}$$

which follows from a residue at the origin. From an asymptotic behavior of eq. (4.4) in a limit $\gamma \to 0$, we have a nontrivial identity for the Euler dilogarithm function;

$$Li_2(-e^{\varphi}) + Li_2(-e^{-\varphi}) + \frac{\varphi^2}{2} + \frac{\pi^2}{6} = 0. \tag{4.5}$$

We can check that the integral (4.2) satisfies the difference equations. We find by direct computation that

$$\frac{\Phi_\gamma(\varphi + \mathrm{i}\,\gamma)}{\Phi_\gamma(\varphi - \mathrm{i}\,\gamma)} = \frac{1}{1 + \mathrm{e}^{\varphi}}, \tag{4.6a}$$

$$\frac{\Phi_\gamma(\varphi + \mathrm{i}\,\pi)}{\Phi_\gamma(\varphi - \mathrm{i}\,\pi)} = \frac{1}{1 + \mathrm{e}^{\frac{\pi}{\gamma}\varphi}}. \tag{4.6b}$$

The first equality corresponds to eq. (3.2), and thus the Faddeev integral (4.2) can be regarded as a function $S_q(w)$ (3.1) in a case of $|q| = 1$. Remarkable is that the integral has a kind of "duality"; $\gamma \leftrightarrow \frac{\pi^2}{\gamma}$. In fact by collecting a residue of the integral (4.2) and recalling a definition (3.1c), the integral Φ_γ is represented by

$$\Phi_\gamma(\varphi) = S_q(\mathrm{e}^{\varphi}) \cdot S_Q(\mathrm{e}^{\varphi \frac{\pi}{\gamma}}),$$

where $Q = \mathrm{e}^{\mathrm{i}\,\frac{\pi^2}{\gamma}}$. We therefore have a "factorization" property for the integral.

This factorization can be realized by use of the quantum canonical operators [6]. We consider the algebra generated by the Heisenberg pair $\hat{p}$ and $\hat{q}$;

$$[\hat{p}\,,\ \hat{q}] = -2\,\mathrm{i}\,\gamma. \tag{4.7}$$

By use of these operators we can realize the Weyl pairs as follows;

$$\hat{u}\,\hat{v} = q^2\,\hat{v}\,\hat{u}, \qquad \hat{U}\,\hat{V} = Q^2\,\hat{V}\,\hat{U},$$

where

$$\hat{u} = \mathrm{e}^{\hat{q}}, \qquad \hat{v} = \mathrm{e}^{\hat{p}}, \qquad \hat{U} = \mathrm{e}^{\frac{\pi}{\gamma}\hat{q}}, \qquad \hat{V} = \mathrm{e}^{\frac{\pi}{\gamma}\hat{p}}.$$

See the commutativity,

$$\hat{U}\,\hat{u} = \hat{u}\,\hat{U}, \qquad \hat{V}\,\hat{v} = \hat{v}\,\hat{V}, \qquad \hat{U}\,\hat{v} = \hat{v}\,\hat{U}, \qquad \hat{u}\,\hat{V} = \hat{V}\,\hat{u}.$$

We can find that the Weyl algebra generated above by $\hat{p}$ and $\hat{q}$ is factored into two algebras $(\hat{u}\,,\ \hat{v})$ and $(\hat{U}\,,\ \hat{V})$. As a result, from the pentagon relation (3.5d) for $S_q(w)$ and a commutativity of two algebras, we also have the pentagon relation for the integral Φ_γ,

$$\Phi_\gamma(\hat{p})\,\Phi_\gamma(\hat{q}) = \Phi_\gamma(\hat{q})\,\Phi_\gamma(\hat{p} + \hat{q})\,\Phi_\gamma(\hat{p}), \tag{4.8}$$

where we have used $e^{\hat{q}}\, e^{\hat{p}} = e^{\hat{p}+\hat{q}+i\gamma}$.

We rewrite the pentagon relation (4.8) in a simple form. We introduce the S-operator as an operator acting on a Hilbert space $\mathbf{V}\otimes\mathbf{V}$,

$$S_{1,2} = e^{\frac{1}{2i\gamma}\hat{q}_1\hat{p}_2}\,\Phi_\gamma(\hat{p}_1+\hat{q}_2-\hat{p}_2), \tag{4.9}$$

where $\hat{p}_1 = \hat{p}\otimes 1$, $\hat{p}_2 = 1\otimes\hat{p}$ and so on. It is easy to see that the S-operator satisfies an identity;

$$S_{2,3}\,S_{1,2} = S_{1,2}\,S_{1,3}\,S_{2,3}. \tag{4.10}$$

Here we have only applied the commutation relation (4.7) to eqs. (4.8). This simple form of the pentagon identity was used to define the $6j$ symbol [16] and to quantize the Teichmüller space [3, 20].

We have the braid generator as in a case of $|q|<1$, and furthermore in the case $|q|=1$ we can construct another type of solution of the braid relation by use of a solution of the pentagon identity (4.10). We define the R-operator on a space $\mathbf{V}^{\otimes 4}$ as [18]

$$R_{12,34} = \left(S_{1,4}^{t_4}\right)^{-1} S_{1,3}\,S_{2,4}^{t_2 t_4}\left(S_{2,3}^{t_2}\right)^{-1}, \tag{4.11}$$

where t_a means a transposition operation on the a-th Hilbert space $\mathbf{V}$. We can see that the R-operator (4.11) satisfies the constant Yang–Baxter relation,

$$R_{11',22'}\,R_{11',33'}\,R_{22',33'} = R_{22',33'}\,R_{11',33'}\,R_{11',22'}. \tag{4.12}$$

Proof follows from recursive use of the pentagon identity (4.10) and its corollary such as

$$S_{1,2}\,(S_{1,3}^{t_3})^{-1}\,(S_{2,3}^{t_3})^{-1} = (S_{2,3}^{t_3})^{-1}\,S_{1,2},$$

$$(S_{1,2}^{t_1})^{-1}\,(S_{1,3}^{t_1})^{-1}\,S_{2,3} = S_{2,3}\,(S_{1,2}^{t_1})^{-1}.$$

When we define the operator $\mathbf{R} : \mathbf{V}^{\otimes 2}\otimes\mathbf{V}^{\otimes 2} \to \mathbf{V}^{\otimes 2}\otimes\mathbf{V}^{\otimes 2}$ by

$$\mathbf{R} = P_{1,2}\,P_{1',2'}\,R_{11',22'}, \tag{4.13}$$

where P is a permutation operator, we find that the operator $\mathbf{R}$ is a solution of the braid relation,

$$(\mathbf{R}\otimes 1)\,(1\otimes\mathbf{R})\,(\mathbf{R}\otimes 1) = (1\otimes\mathbf{R})\,(\mathbf{R}\otimes 1)\,(1\otimes\mathbf{R}). \tag{4.14}$$

For our later convention, we define other operators Y and Z;

$$Y = \exp\Bigl(\frac{C}{2\gamma \mathrm{i}}\,\hat{p}\Bigr), \tag{4.15}$$

$$Z = \exp\Bigl(\frac{C}{2\gamma \mathrm{i}}\,\hat{q}\Bigr), \tag{4.16}$$

where C is an arbitrary parameter. We find simply that these operators satisfy

$$Y_1\,Y_2\,S_{1,2} = S_{1,2}\,Y_1, \tag{4.17a}$$

$$Z_2\,S_{1,2} = S_{1,2}\,Z_1\,Z_2, \tag{4.17b}$$

$$Z_1\,Y_2\,S_{1,2} = S_{1,2}\,Z_1\,Y_2, \tag{4.17c}$$

where $S_{1,2}$ is defined in eq. (4.9). With the operators Y and Z, we define operators $\mathbf{D}$ and $\tilde{\mathbf{D}}$ on $\mathbf{V}^{\otimes 2}$ as

$$\mathbf{D} \equiv D_{1,1'} = Y_1 \left(Y_{1'}^{\mathrm{t}_{1'}}\right)^{-1}, \tag{4.18}$$

$$\tilde{\mathbf{D}} \equiv \tilde{D}_{1,1'} = Z_1 \left(Z_{1'}^{\mathrm{t}_{1'}}\right)^{-1}. \tag{4.19}$$

Using eqs. (4.11) and (4.17), we get

$$\mathbf{D} \otimes \mathbf{D} \cdot \mathbf{R} = \mathbf{R} \cdot \mathbf{D} \otimes \mathbf{D}, \tag{4.20}$$

$$\tilde{\mathbf{D}} \otimes \tilde{\mathbf{D}} \cdot \mathbf{R} = \mathbf{R} \cdot \tilde{\mathbf{D}} \otimes \tilde{\mathbf{D}}. \tag{4.21}$$

5 Representation

5.1 Momentum Space

We consider a matrix representation of the R-matrix given above. Kashaev constructed a finite-dimensional representation for the R-operator (4.11) by taking q as the N-th root of unity [17], though in this section we rather consider the R-operator on the infinite dimensional space.

We consider the Hilbert space of the quantum canonical operators $\hat{p}$ and $\hat{q}$ (4.7). We call the momentum space and the coordinate space, which are spanned by $|p\rangle$ and $|q\rangle$ with $p, q \in \mathbb{R}$ respectively. They are eigenstates, $\hat{p}\,|p\rangle = p\,|p\rangle$ and $\hat{q}\,|q\rangle = q\,|q\rangle$. We have an orthogonality,

$$\langle q|q'\rangle = \delta(q - q'), \qquad \langle q|p\rangle = \frac{1}{\sqrt{4\pi\gamma}}\,\mathrm{e}^{-\frac{qp}{2\mathrm{i}\gamma}}, \qquad \langle p|p'\rangle = \delta(p - p'), \tag{5.1}$$

and

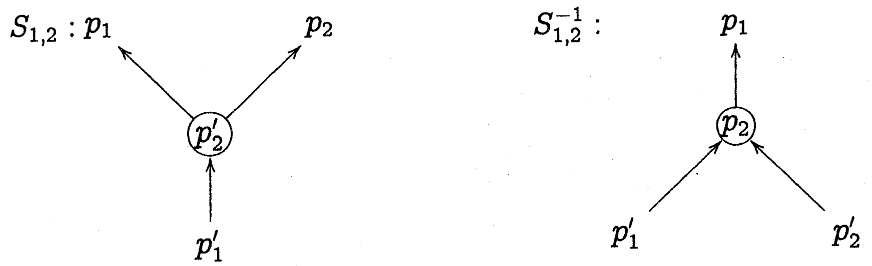
$$1 = \int_{-\infty}^{\infty} \mathrm{d}q\,|q\rangle\,\langle q| = \int_{-\infty}^{\infty} \mathrm{d}p\,|p\rangle\,\langle p|.$$

By use of these identities, matrix elements of the S-operator (4.9) on the momentum space are given by

$$\langle p_1, p_2|S_{1,2}|p_1', p_2'\rangle = \frac{1}{4\pi\gamma}\,\delta(p_1 + p_2 - p_1')\int \mathrm{d}x\,\Phi_\gamma(x + p_1)\,\mathrm{e}^{\frac{1}{2\gamma\mathrm{i}}\left((p_2 - p_2')\,x - \frac{1}{2}(p_2 - p_2')^2\right)}, \tag{5.2a}$$

$$\langle p_1, p_2|S_{1,2}^{-1}|p_1', p_2'\rangle = \frac{1}{4\pi\gamma}\,\delta(p_1 - p_1' - p_2')\int \mathrm{d}x\,\frac{1}{\Phi_\gamma(x + p_1')}\,\mathrm{e}^{\frac{1}{2\gamma\mathrm{i}}\left((p_2 - p_2')\,x + \frac{1}{2}(p_2 - p_2')^2\right)}. \tag{5.2b}$$

See that the δ-function terms are consistent with eqs. (4.17), and that the S-operator is the quantum analogue of the Clebsch–Gordan operator;



Now from the explicit form of the S-operators (5.2), the matrix elements of the R-operator (4.11) are shown to be given by

$$\begin{aligned}\langle p_1, p_2, p_3, p_4|R_{12,34}|p_1', p_2', p_3', p_4'\rangle \\ &= \delta(p_1 - p_4 + p_3 - p_1')\,\delta(p_2' - p_3' - p_2 + p_4') \\ &\qquad \times H(p_2' - p_3', p_1 - p_4', p_1 - p_4, p_2' - p_3),\end{aligned} \tag{5.3a}$$

$$\langle p_1, p_2, p_3, p_4 | (R_{12,34})^{-1} | p_1', p_2', p_3', p_4' \rangle$$
$$= \delta(p_2' - p_2 + p_3 - p_4)\, \delta(p_1 - p_1' - p_3' + p_4')$$
$$\times H(p_2' - p_4, p_1 - p_3, p_1 - p_3', p_2' - p_4'). \quad (5.3b)$$

Here the integral $H(a, b, c, d)$ is defined as

$$H(a, b, c, d) = \frac{1}{(4\pi\gamma)^2} \iint \mathrm{d}x\, \mathrm{d}y\, \frac{\Phi_\gamma(x + a)\, \Phi_\gamma(y + c)}{\Phi_\gamma(x + b)\, \Phi_\gamma(y + d)}\, \mathrm{e}^{\frac{1}{2 \mathrm{i} \gamma}\left(-(b-c)x + (a-d)y - \frac{1}{2}(a-d)^2 - \frac{1}{2}(b-c)^2\right)}. \quad (5.4)$$

We note that the operators $D_{1,2}$ (4.18) and $\tilde{D}_{1,2}$ (4.19) are expressed on the momentum space as

$$\langle p_1, p_2 | D_{1,2} | p_1', p_2' \rangle = \delta(p_1 - p_1') \cdot \delta(p_2 - p_2') \cdot \mathrm{e}^{\frac{C}{2\gamma \mathrm{i}}(p_1 - p_2)}, \quad (5.5)$$

$$\langle p_1, p_2 | \tilde{D}_{1,2} | p_1', p_2' \rangle = \delta(p_1 - p_1' + C) \cdot \delta(p_2 - p_2' + C). \quad (5.6)$$

5.2 Asymptotic Behavior

We shall study a $\gamma \to 0$ limit for the S- and R-matrices by the saddle point method. We first consider the Fourier transform of the Faddeev integral;

$$\tilde{\Phi}_\gamma(p) = \int \mathrm{d}x\, \Phi_\gamma(x)\, \mathrm{e}^{\frac{1}{2 \mathrm{i} \gamma} p x}, \quad (5.7)$$

which owing to eq. (4.3) reduces to

$$\tilde{\Phi}_\gamma(p) \sim \int \mathrm{d}x\, \exp \frac{1}{2 \mathrm{i} \gamma} \left(Li_2(-\mathrm{e}^x) + p x \right), \qquad \text{for } \gamma \to 0.$$

We apply the steepest descent method, and evaluate the integral at the saddle point. The saddle point equation gives $\mathrm{e}^x = \mathrm{e}^p - 1$, and we obtain

$$\tilde{\Phi}_\gamma(p) \sim \exp \frac{1}{2 \mathrm{i} \gamma} \left(\frac{\pi^2}{6} - Li_2(\mathrm{e}^p) + p \pi \mathrm{i} \right). \quad (5.8)$$

Based on this asymptotic behavior and using an analytic continuation of eq. (4.5), we see that the S-operator (5.2) is

$$\langle p_1, p_2 | S_{1,2} | p_1', p_2' \rangle \sim \delta(p_1 + p_2 - p_1') \cdot \exp \frac{1}{2 \mathrm{i} \gamma} \left(-\frac{\pi^2}{6} + Li_2(\mathrm{e}^{p_2' - p_2}) + p_1 (p_2' - p_2) \right). \quad (5.9a)$$

In the same manner we find

$$\langle p_1, p_2 | S_{1,2}^{-1} | p_1', p_2' \rangle$$
$$\sim \delta(p_1 - p_1' - p_2') \cdot \exp \frac{1}{2\,\mathrm{i}\,\gamma} \left(\frac{\pi^2}{6} - Li_2(\mathrm{e}^{p_2 - p_2'}) - p_1' \, (p_2 - p_2') \right). \quad (5.9\mathrm{b})$$

In the next section we shall associate tetrahedra to these S-operators, and in fact we see the exponential factor resembles with the Bloch–Wigner function (2.7).

To evaluate the R-matrix, we need the asymptotic behavior of the integral

$$I(a, p) = \int \mathrm{d}x \, \frac{\Phi_\gamma(x)}{\Phi_\gamma(x + p)} \, \mathrm{e}^{\frac{1}{2\,\mathrm{i}\,\gamma} a\, x}. \quad (5.10)$$

From eq. (4.3) we have

$$I(a, p) \sim \int \mathrm{d}x \, \exp \frac{1}{2\,\mathrm{i}\,\gamma} \left(Li_2(-\mathrm{e}^x) - Li_2(-\mathrm{e}^{x+p}) + a\, x \right).$$

The saddle point equation for this integral is fixed by

$$\log \left(\frac{1 + \mathrm{e}^x}{1 + \mathrm{e}^{x+p}} \right) = a,$$

which gives

$$I(a, p) \sim \exp \frac{1}{2\,\mathrm{i}\,\gamma} \left(Li_2 \left(1 - \frac{\mathrm{e}^a (1 - \mathrm{e}^p)}{1 - \mathrm{e}^{a+p}} \right) - Li_2 \left(\frac{\mathrm{e}^p \, (1 - \mathrm{e}^a)}{1 - \mathrm{e}^{a+p}} \right) + a \log \left(- \frac{1 - \mathrm{e}^a}{1 - \mathrm{e}^{a+p}} \right) \right).$$

By applying eq. (2.3) and the pentagon identity (2.4), we finally obtain

$$I(a, p) \sim \exp \frac{1}{2\,\mathrm{i}\,\gamma} \left(Li_2(\mathrm{e}^{a+p}) - Li_2(\mathrm{e}^a) - Li_2(\mathrm{e}^p) + a\,\mathrm{i}\,\pi + \frac{\pi^2}{6} \right). \quad (5.11)$$

Using this asymptotic behavior and the inversion relation (4.5), the integral (5.4) has a form,

$$H(a, b, c, d)$$
$$\sim \exp \frac{1}{2\,\mathrm{i}\,\gamma} \left(Li_2(\mathrm{e}^{b-c}) + Li_2(\mathrm{e}^{c-d}) - Li_2(\mathrm{e}^{a-d}) - Li_2(\mathrm{e}^{b-a}) - a\,(a - b + c - d) \right). \quad (5.12)$$

At last we find that the R-matrix has an asymptotic form,

$$\begin{aligned}
&\langle p_1, p_2, p_3, p_4 | R_{12,34} | p_1', p_2', p_3', p_4' \rangle \\
&\qquad\qquad \sim \delta(p_1 + p_3 - p_4 - p_1') \cdot \delta(p_2' - p_3' + p_4' - p_2) \\
&\qquad \times \exp \frac{1}{2\,\mathrm{i}\,\gamma} \Bigl(Li_2(\mathrm{e}^{p_1' - p_2'}) + Li_2(\mathrm{e}^{p_4 - p_4'}) - Li_2(\mathrm{e}^{p_1 - p_2}) - Li_2(\mathrm{e}^{p_3 - p_3'}) \\
&\qquad\qquad + (p_2' - p_3')\,(p_1 - p_2 - p_1' + p_2') \Bigr),
\end{aligned} \tag{5.13a}$$

$$\begin{aligned}
&\langle p_1, p_2, p_3, p_4 | (R_{12,34})^{-1} | p_1', p_2', p_3', p_4' \rangle \\
&\qquad\qquad \sim \delta(p_2' - p_2 + p_3 - p_4)\, \delta(p_1 - p_1' - p_3' + p_4') \\
&\qquad \times \exp \frac{1}{2\,\mathrm{i}\,\gamma} \Bigl(Li_2(\mathrm{e}^{p_3' - p_3}) + Li_2(\mathrm{e}^{p_1' - p_2'}) - Li_2(\mathrm{e}^{p_4' - p_4}) - Li_2(\mathrm{e}^{p_1 - p_2}) \\
&\qquad\qquad + (p_2 - p_3)\,(p_1 - p_2 - p_1' + p_2') \Bigr).
\end{aligned} \tag{5.13b}$$

This form suggests that the R-matrix (5.3a) has 4 tetrahedra because 4 dilogarithm function terms have appeared.

6 Knot Invariant

6.1 Braid Group

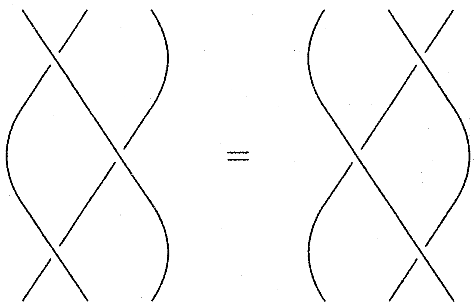
The knot invariant can be constructed by use of solutions of the Yang–Baxter equation [30, 37]. We suppose that we have the enhanced Yang–Baxter operators $(\mathbf{R}, \mu, \alpha, \beta)$ satisfying

$$(\mathbf{R} \otimes 1)\,(1 \otimes \mathbf{R})\,(\mathbf{R} \otimes 1) = (1 \otimes \mathbf{R})\,(\mathbf{R} \otimes 1)\,(1 \otimes \mathbf{R}), \tag{6.1a}$$

$$(\mu \otimes \mu)\,\mathbf{R} = \mathbf{R}\,(\mu \otimes \mu), \tag{6.1b}$$

$$\mathrm{Tr}_2 \left(\mathbf{R}^{\pm 1}\,(1 \otimes \mu) \right) = \alpha^{\pm 1}\,\beta. \tag{6.1c}$$

The first one is called the braid relation (constant Yang–Baxter equation),



and the other two are necessary to be invariant under the Markov moves. When the knot K is given as the closure of a braid ξ with n strands, the knot invariant $\tau(K)$ is defined as

$$\tau(K) = \alpha^{-w(\xi)}\,\beta^{-n}\ \mathrm{Tr}_{1,\dots,n}\Big(b_R(\xi)\,\mu^{\otimes n}\Big). \tag{6.2}$$

Here we have associated the homomorphism $b_R(B)$ by replacing $\sigma_i^{\pm 1}$ in ξ with $\mathbf{R}^{\pm 1}$, and $w(\xi)$ is a writhe, a sum of the exponents. We also use an invariant,

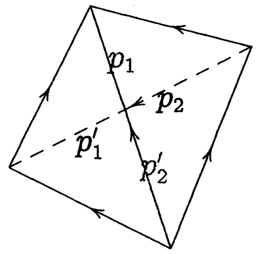
$$\tau_1(K) = \alpha^{-w(\xi)}\,\beta^{-n}\ \mathrm{Tr}_{2,\dots,n}\Big(b_R(\xi)\,\big(1\otimes\mu^{\otimes(n-1)}\big)\Big), \tag{6.3}$$

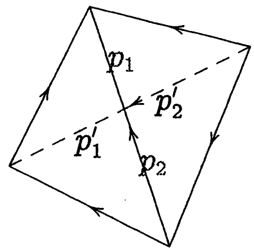
which is associated for $(1,1)$-tangle.

We have a representation (5.3a) (and its asymptotic form (5.13)) for the braid relation (6.1a). The relation (6.1b) is also fulfilled by the operator either $\mathbf{D}$ (4.18) or $\tilde{\mathbf{D}}$ (4.19). We can check that in a case of $C \to 0$ (*i.e.*, we set $\mu = 1$) the asymptotic expression (5.13) satisfies the third equation (6.1c) with $\alpha = \beta = 1$. As a result we have a knot invariant (6.3) with the R-matrix (5.13) in the limit $\gamma \to 0$.

6.2 Three Dimensional Picture

Following Ref. 16, we give three dimensional picture for our knot invariant which was defined by eq. (6.3) in a limit $\gamma \to 0$ with the R-matrix (5.13). A key point is that the meaning of the pentagon identity (4.10) becomes much clearer when we associate the tetrahedron for the S-operators (5.9) as follows;

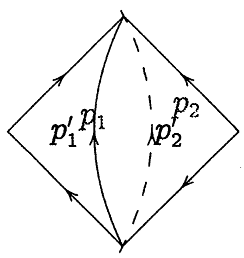
$$\langle p_1, p_2 | S | p_1', p_2' \rangle = $$  (6.4a)

$$\langle p_1, p_2 | S^{-1} | p_1', p_2' \rangle = $$  (6.4b)

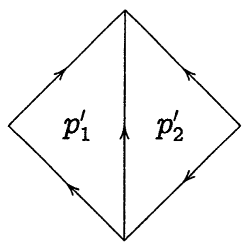
Each triangular face has a momentum p, and the momenta p_i and p_i' respectively denote the out-going and in-coming states. We have added arrows on edges to specify the orientation of the tetrahedron. See that the orientation of the tetrahedra is different from each other for the S- and S^{-1}-operators. With these identification, we regard the integration of the momentum means the glueing of the triangular faces. Each triangle face has an orientation, and how to glue these two faces can be fixed. In this view, the inversion relation,

$$\iint \mathrm{d}x\,\mathrm{d}y\,\langle p_1, p_2 | S | x, y \rangle \langle x, y | S^{-1} | p_1', p_2' \rangle = \delta(p_1 - p_1')\,\delta(p_2 - p_2'), \tag{6.5}$$

simply denotes the collapse of two tetrahedra into a plane, when the two triangles thereof are glued to each other;

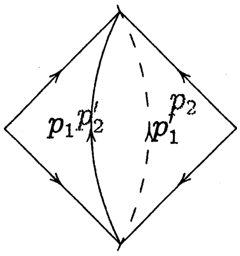


$$= \delta(p_1 - p_1')\,\delta(p_2 - p_2') \times$$

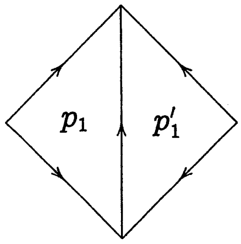


In the same way we have

$$\iint \mathrm{d}x\,\mathrm{d}y\,\langle p_1, x|S|p_1', y\rangle\,\langle p_2, y|S^{-1}|p_2', x\rangle \sim \delta(p_1 - p_2')\,\delta(p_2 - p_1'). \tag{6.6}$$

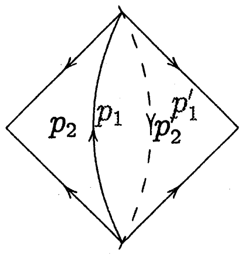


$$\sim \quad \delta(p_1 - p_2')\,\delta(p_1' - p_2) \times$$



Note that another type of glueing of two tetrahedra by two faces does not collapse into a plane but a "suspension";

$$\iint \mathrm{d}x\,\mathrm{d}y\,\langle x, p_1|S|y, p_1'\rangle\,\langle y, p_2|S|x, p_2'\rangle$$
$$\sim \delta(p_1 + p_2)\,\delta(p_1' + p_2')\,\exp\frac{1}{2\,\mathrm{i}\,\gamma}\Big(\mathrm{i}\,\pi\,(p_1 - p_1') + \frac{1}{2}(p_1^2 - (p_1')^2)\Big).$$

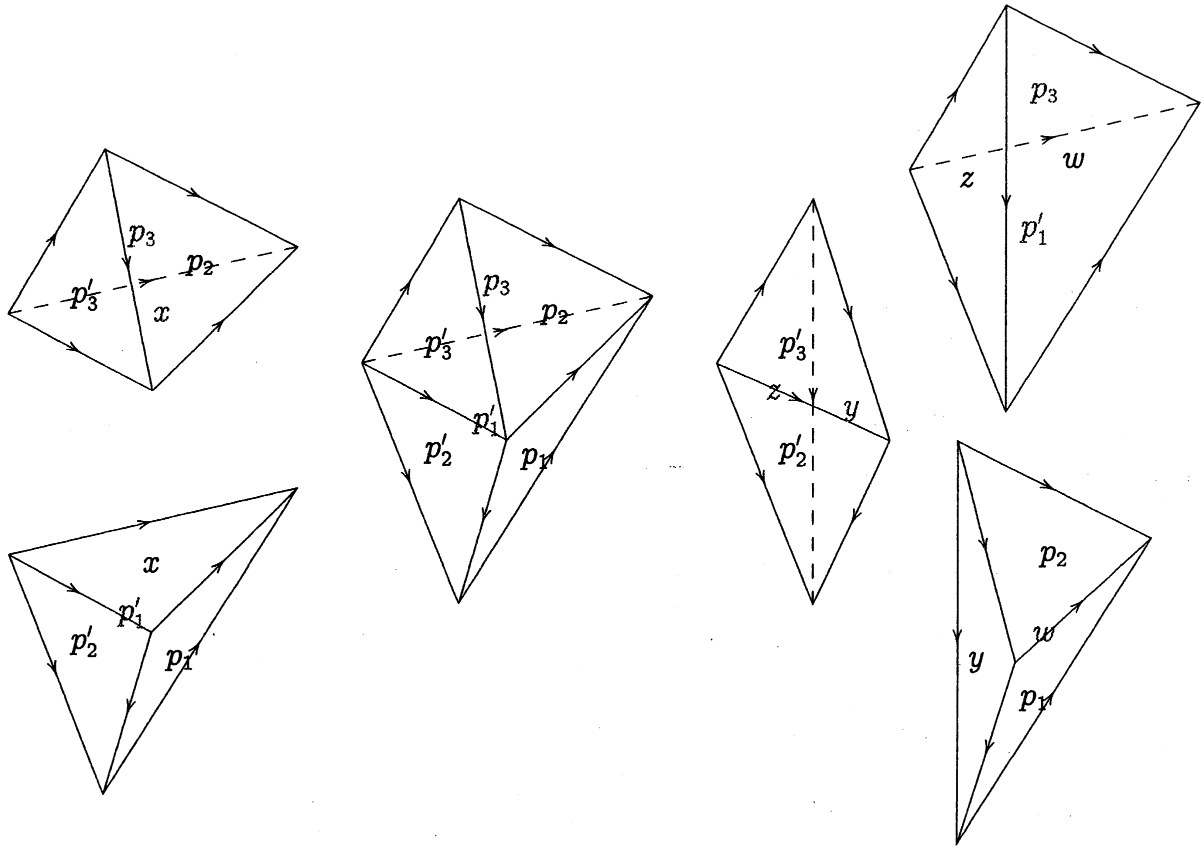


See that the two tetrahedra (6.4) can be transformed to each other by glueing this suspension.

We then find that the pentagon identity (4.10), which is explicitly rewritten as

$$\int \mathrm{d}x\,\langle p_2, p_3|S|x, p_3'\rangle\,\langle p_1, x|S|p_1', p_2'\rangle$$
$$= \iiint \mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w\,\langle p_1, p_2|S|y, z\rangle\,\langle y, p_3|S|p_1', w\rangle\,\langle z, w|S|p_2', p_3'\rangle, \tag{6.7}$$

can be viewed in a three dimensional picture as dividing a polytope in two ways;

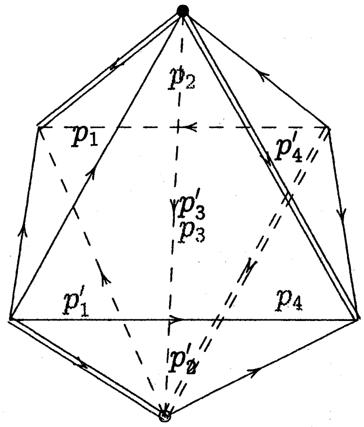


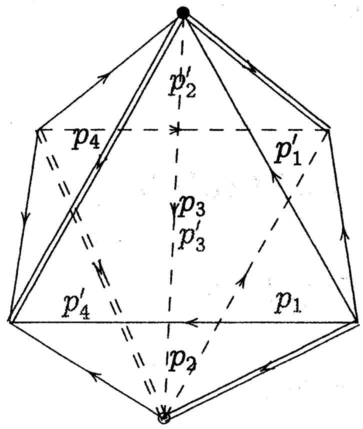
Corollaries (next to eq. (4.12)) can be geometrically checked in the same manner.

Once we have identified the asymptotic S-operators with the oriented tetrahedra, we can construct the isotopic invariant of the manifold M. Here to relate with the knot invariant (6.3) we suppose that M is a finite triangulation of the oriented 3-dimensional manifold without boundary. We can associate operators $S^{\pm 1}$ (5.9) to the oriented tetrahedra, and have the partition function by

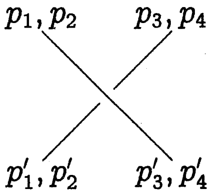
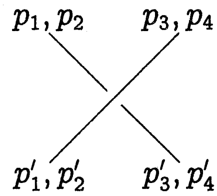
$$Z(M) = \int \mathrm{d}p \prod \langle \vec{p_{a_i}} | S^{\pm 1} | \vec{p_{a_j}} \rangle . \tag{6.8}$$

This is an invariant of M; if M' can be transformed from M by the operations (6.5) and (6.7), we have $Z(M) = Z(M')$. To relate this partition function with the invariant of a link L, we suppose that any 0-simplex in M belongs to exactly two 1-simplexes in L. Then the invariant $Z(M)$ is associated to the link L, and furthermore becomes a knot invariant [17]. This can be checked by showing that every 0-simplexes of the octahedron are on the link L as follows. Using above three dimensional picture, we can see that the braid generators $\mathbf{R}^{\pm 1}$, which are defined by eqs. (4.11) and (4.13), can be seen as an octahedron, which includes 4 tetrahedra;

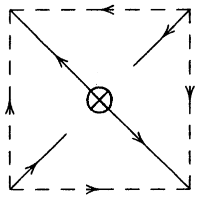
$\langle \vec{p}|\mathbf{R}|\vec{p'}\rangle =$  (6.9a)

$\langle \vec{p}|\mathbf{R}^{-1}|\vec{p'}\rangle =$  (6.9b)

This identification of the $\mathbf{R}^{\pm 1}$-matrices with an oriented octahedron essentially coincides with a description in Ref. 35. Though both operators $\mathbf{R}^{\pm 1}$ are represented by the similar octahedra, the difference becomes clearer when we recall that the momenta p_i and p_i' respectively denote the out-going and in-coming states. To see explicitly a property of the $\mathbf{R}$-matrices as the braid generators (4.14), we view the octahedra from the top (above a point • in each octahedron), and we have a following projection of tangle;

$\langle \vec{p}|\mathbf{R}|\vec{p'}\rangle =$  $\langle \vec{p}|\mathbf{R}^{-1}|\vec{p'}\rangle =$  (6.10)

The link corresponds to the double lines in the octahedra (6.9) (important is that the 0-simplexes are on the link), and the crossing point denotes a line from • to ∘. Note that both crossings indicate that there are 4 oriented tetrahedra, which are projected as follows;

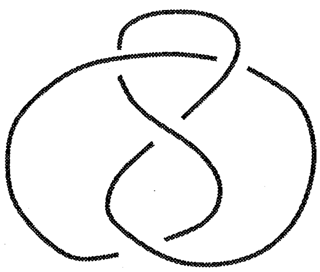


where ⊗ denotes a vector pointing downwards. This projection clarifies the meaning of both the braid relation (4.14) and the inversion relation, $\mathbf{R}\mathbf{R}^{-1} = 1$. Therefore we can find that every 0-simplexes on the octahedron are also on the link L, and that any 0-simplex belongs to exactly two 1-simplexes in L by construction of the knot invariant from the braid generators. In conclusion the partition function $Z(M)$ becomes a knot invariant.

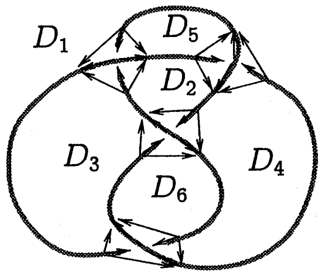
7 Simple Examples

7.1 Figure-Eight Knot

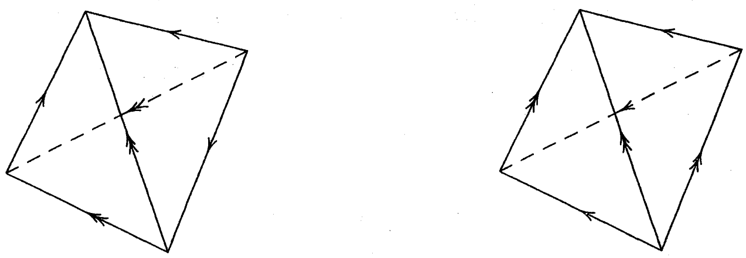
The figure-eight knot is;



This knot is represented as $\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}$ by use of the braid generators. We then associate the tetrahedra for each crossing as



In regions $D_1, \cdots, D_4$, the three tetrahedra are glued, and due to the pentagon identity (6.7) they reduce to two tetrahedra. By glueing these tetrahedra with suspensions which follow from the regions D_5 and D_6, we finally obtain the 2 tetrahedra;



See that every triangle face corresponds to a surface $D_1, \ldots, D_4$. It is a well known result [36] that the complement of the figure-eight knot is constructed from above 2 tetrahedra. Following our construction of the triangulations, we have

$$Z(4_1) = \int \mathrm{d}p \, \langle p_1 = 0, p_2 | S | p_3, p_4 \rangle \, \langle p_4, p_3 | S^{-1} | p_2, p_1 = 0 \rangle$$
$$\sim \int \mathrm{d}p \, \exp \frac{1}{2\,\mathrm{i}\,\gamma} \left(Li_2(\mathrm{e}^{-p}) - Li_2(\mathrm{e}^{p}) \right).$$

Here we have introduced a restriction $p_1 = 0$ which comes from an invariant for a (1,1)-tangle. The integral can be evaluated by the saddle point equation,

$$(1 - \mathrm{e}^{p})\,(1 - \mathrm{e}^{-p}) = 1,$$

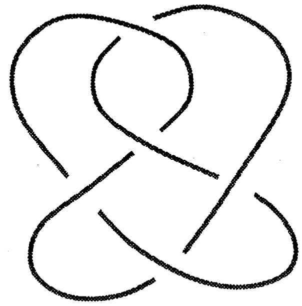
which with a root of $\omega^2 - \omega + 1 = 0$ gives

$$\lim_{\gamma \to 0} \left(2\,\mathrm{i}\,\gamma \log Z(4_1) \right) = 2.02988\,\mathrm{i}. \tag{7.1}$$

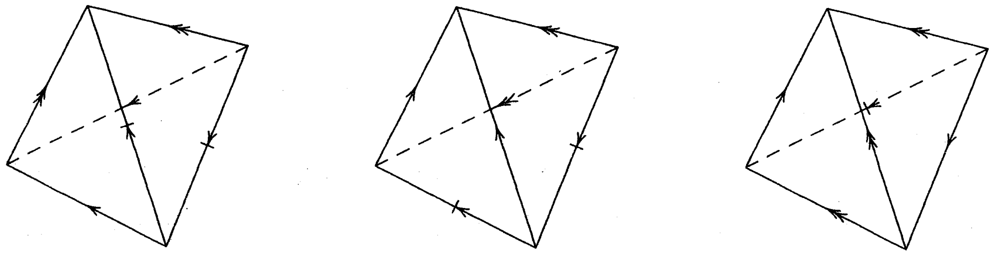
One sees that the imaginary part is nothing but the hyperbolic volume of the complement of knot 4_1 [13, 39].

7.2 5_2 Knot

The 5_2 knot is generated by the braid generators as ${\sigma_2}^2\,{\sigma_1}^{-1}\,\sigma_2\,{\sigma_1}^2$, and has the following projection;



By associating 4 tetrahedra for each crossing, we find that, after glueing and transforming these tetrahedra following rules in previous section, the complement is triangulated into as follows (see also Ref. 34 for another method of triangulation);



With these oriented tetrahedra, we get the partition function as

$$\begin{aligned} Z(5_2) &= \int \mathrm{d}p\, \langle p_1 = 0, p_2 | S^{-1} | p_3, p_4 \rangle\, \langle p_5, p_4 | S^{-1} | p_2, p_6 \rangle\, \langle p_6, p_3 | S^{-1} | p_5, p_1 = 0 \rangle \\ &\sim \iint \mathrm{d}x\, \mathrm{d}y\, \exp \frac{1}{2\,\mathrm{i}\,\gamma} \Big(-\frac{\pi^2}{2} - Li_2(\mathrm{e}^{-x}) - 2\, Li_2(\mathrm{e}^{-y}) + x\, y \Big), \end{aligned}$$

whose saddle point equations are

$$\mathrm{e}^{y} = 1 - \mathrm{e}^{-x}, \qquad \qquad \mathrm{e}^{x} = (1 - \mathrm{e}^{-y})^2 .$$

We finally obtain

$$\lim_{\gamma \to 0} \Big(2\,\mathrm{i}\,\gamma \log Z(5_2) \Big) = -6.84548 + 2.82812\,\mathrm{i}. \tag{7.2}$$

One finds again the imaginary part coincides with the hyperbolic volume of the complement of knot 5_2 [13, 39].

8 Concluding Remarks

In this note we have studied an invariant which are defined from the quantum dilogarithm function. We have shown that it satisfies the pentagon identity, and by use of the quantum dilogarithm function, the solution of the Yang–Baxter equation has been constructed. Considering the quantum dilogarithm function on the momentum space in a limit $\gamma \to 0$, we have given the three dimensional picture for the quantum dilogarithm function. A three dimensional meaning of the momenta in our representation (5.9) is unclear for us. Furthermore it was proposed that $\mathrm{Vol}(K) + \mathrm{i}\,\mathrm{CS}(K)$ has good analytic properties [29] where $\mathrm{CS}(K)$ and $\mathrm{Vol}(K)$ respectively denotes the Chern–Simons invariant and the hyperbolic volume of the knot K. As we have studied the knot invariant in an integral form, we hope that this Note would be helpful to understand a relationship with the Chern–Simons invariant and to define a "simplicial" invariant of the 3-dimensional manifold.

Acknowledgement

The author would like to thank Hitoshi Murakami for useful discussions.

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