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Classification by Iwahori subgroup and local densities on hermitian forms

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§1. Introduction

Let k be a nonarchimedean local field of characteristic 0, $\mathcal{O}$ the ring of integers in k, $*$ an involution on k with the fixed field k_0, and q the cardinality of the residue class field of k_0. We assume that the residual characteristic is not 2.

For a matrix $A=(a_{ij})\in M_{m,n}(k)$, we set $A^*=(a_{ji}^*)\in M_{n,m}(k)$. A matrix $A\in M_n(k)$ is called hermitian (with respect to $*$) if it satisfies $A^*=A$. We denote by X_n the set of all nondegenerate hermitian matrices in $GL_n(k)$, and set $X_n(\mathcal{O})=X_n\cap M_n(\mathcal{O})$.

The group $GL_n(k)$ acts on X_n by $g\cdot A=gAg^*\quad (g\in GL_n(k),\ A\in X_n)$.

We choose the prime element ϖ of k for which $\varpi=\pi\in k_0$ if the extension k/k_0 is unramified (Case(U)) or $\varpi^2=\pi\in k_0$ if k/k_0 is ramified (Case(R)).

First, we determine the classification of X_n under the action of Iwahori subgroup

$$\Gamma=\{\gamma=(\gamma_{ij})\in GL_n(\mathcal{O})\mid \gamma_{ij}\in\varpi\mathcal{O}\text{ if } i>j\},$$

by giving a complete set of representatives of $\Gamma\backslash X_n$, which will be denoted by $\mathcal{R}_n$ (**Theorem 1**).

We also give an explicit formula of the volume $\alpha(Y;\Gamma)$ of the stabilizers of each $Y\in\mathcal{R}_n$ in Γ (**Theorem 2**). Here

$$q^{-dn^2}N_d(Y;\Gamma)\quad \left(N_d(Y;\Gamma)=\sharp\left\{\gamma\in\Gamma \bmod (\pi^d)\,\middle|\,\gamma\cdot Y\equiv Y \bmod (\pi^d)\right\}\right)$$

is stable for sufficiently large d, and we define

$$\alpha(Y;\Gamma)=\lim_{d\to\infty}q^{-dn^2}N_d(Y;\Gamma).$$

Next we consider the local density $\mu(B,A)$ of $B\in X_n$ by $A\in X_m$ $(m\geq n)$. Here

$$q^{-dn(2m-n)}N_d(B,A)\quad \left(N_d(B,A)=\sharp\left\{T\in M_{m,n}(\mathcal{O}) \bmod (\pi^d)\,\middle|\,T^*AT\equiv B \bmod (\pi^d)\right\}\right)$$

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is stable for sufficiently large d, and we define

$$\mu(B,A) = \lim_{d\to\infty} q^{-dn(2m-n)} N_d(B,A).$$

It is easy to see that $\mu(B,A)$ depends only on the $GL_n(\mathcal{O})$-orbit containing B and the $GL_m(\mathcal{O})$-orbit containing A. Further, since $\mu(\pi^r B, \pi^r A) = q^{rn^2}\mu(B,A)$ for $r \in \mathbb{N}$, we may assume that A and B are integral.

We give a completely explicit formula for $\mu(B,A)$ in **Theorem 3** for Case(U) and **Theorem 4** for Case(R).

The problem of integral representation of hermitian forms is a classical problem, as is seen in works of Hermite ([He]) or H. Braun ([B]). But few results were known when it is compared with the case of symmetric forms. The classification of $GL_n(\mathcal{O})$-orbits of X_n is a classical result due to Jacobowitz ([Ja]). For an explicit expression of $\mu(A,A)$, Otremba gave some special cases ([O]) and the author gave it in general ([H1, I]).

For unramified case the author has given explicit expressions of local densities $\mu(B,A)$ by two methods including 2-adic case. In both methods the theory of spherical functions on the space of nondegenerate hermitian forms plays an important role, and in the second the theory of zeta functions on the space of hermitian forms is also used([H3], [H4]).

Comparing with above methods, the present one is elementary. The key step for the calculation of the explicit formula is to take the Iwahori subgroup, in stead of $GL_n(\mathcal{O})$, in a reformulation of local densities by using Gaussian sums (**Proposition 3.1**).

By the same method F. Sato and the author have determined a complete explicit formula of local densities of symmetric forms ([SH]). For ramified hermitian case, the situation takes a complicated aspect, which looks like a mixture of symmetric forms and alternating forms. For the classification of $\Gamma\backslash X$, we have to consider both symmetric forms and alternating forms over finite rings. For an explicit expression of $\mu(B,A)$, the situation becomes complicated, since A and B have factors of type

$$\begin{pmatrix} 0 & \varpi^{2e+1} \\ -\varpi^{2e+1} & 0 \end{pmatrix}$$

in general (cf. §**3.3**).

It seems to be many combinatorial identities among our explicit expressions of local densities. In particular, for unramified hermitian case, we have three kinds of explicit expressions for local densities of different appearances. It will be interesting to compare and examine those formulas and draw out combinatorial identities among them, which will be discussed elsewhere. We shall note some examples at the end of §**3**.

§2. Classification of $\Gamma\backslash X_n$

Let

$$\mathfrak{S}_n = \text{ the symmetric group in } n \text{ letters acting on the set } I = \{1,2,\ldots,n\}$$

and we regard elements of $\mathfrak{S}_n$ as matrices, permutation matrices in $GL_n(\mathbb{Z})$.

In Case (U), put

$$\mathcal{R}_n = \left\{ (\sigma, e) \in \mathfrak{S}_n \times \mathbb{Z}^n \,\middle|\, \sigma^2 = 1,\ e_i = e_{\sigma(i)}\ (\forall i) \right\},$$

and for each $(\sigma, e) \in \mathcal{R}_n$, set

$$Y_{\sigma,e} = \sigma \begin{pmatrix} \pi^{e_1} & & 0 \\ & \ddots & \\ 0 & & \pi^{e_n} \end{pmatrix} \in X_n.$$

In case (R), fix a unit $\delta \in k_0$ not contained in the image of the norm map N_{k/k_0} and put

$$\mathcal{R}_n = \left\{ (\sigma, e, \epsilon) \in \mathfrak{S}_n \times \mathbb{Z}^n \times \{1, \delta\}^n \,\middle|\, \begin{array}{l} \sigma^2 = 1, \quad e_i = e_{\sigma(i)}, \epsilon_i = \epsilon_{\sigma(i)}\ (\forall i) \\ 2 \mid e_i \text{ if } \sigma(i) = i,\ \epsilon_i = 1 \text{ if } \sigma(i) \neq i \end{array} \right\},$$

and for each $(\sigma, e, \epsilon) \in \mathcal{R}_n$, set

$$Y_{\sigma,e,\epsilon} = \sigma J_{\sigma,e} \begin{pmatrix} \epsilon_1 \varpi^{e_1} & & 0 \\ & \ddots & \\ 0 & & \epsilon_n \varpi^{e_n} \end{pmatrix} \in X_n,$$

where

$$J_{\sigma,e} = \mathrm{Diag}(j_1, \ldots, j_n) \text{ with } j_i = \begin{cases} -1 & \text{if } i < \sigma(i) \text{ and } 2 \nmid e_i \\ 1 & \text{otherwise} \end{cases},$$

Hereafter we identify each element of $\mathcal{R}_n$ with the corresponding matrix in X_n. Then we have

Theorem 1 *The set $\mathcal{R}_n$ forms a complete set of representatives of $\Gamma \backslash X_n$.*

Some more notation is needed to describe the explicit formula of $\alpha(Y; \Gamma)$ for each $Y \in \mathcal{R}_n$. For each (σ, e) or (σ, e, ϵ) in $\mathcal{R}_n$, let

$$\{ e_i \mid 1 \leq i \leq n \} = \{ \lambda_i \mid 0 \leq i \leq h \} \quad \text{with} \quad \lambda_0 < \lambda_1 < \ldots < \lambda_h\ ,$$

and put

$$\nu_0 = \lambda_0 (\in \mathbb{Z}), \quad \nu_i = \lambda_i - \lambda_{i-1} (\in \mathbb{N}, \quad 1 \leq i \leq h),$$

$$I_i = \{ j \in I \mid e_j = \lambda_i \}, \quad n_i = \sharp(I_i), \quad m_i = n_i + \cdots + n_h, \quad (0 \leq i \leq h).$$

Set

$$\begin{aligned} c_1(\sigma) &= \sharp \{ i \in I \mid \sigma(i) = i \}, \quad c_1(k; \sigma) = \sum_{l=k}^{h} \sharp \{ i \in I_l \mid \sigma(i) = i \}, \\ c_2(\sigma) &= \frac{1}{2}(n - c_1(\sigma)) = \frac{1}{2} \sharp \{ i \in I \mid \sigma(i) \neq i \}, \\ t(\sigma, \{I_i\}) &= \sum_{l=0}^{h} \sharp \{ (i,j) \in I_l \times I_l \mid i < j < \sigma(i),\ \sigma(j) < \sigma(i) \}, \\ \tau(\{I_i\}) &= \sum_{l=1}^{h} \sharp \{ (i,j) \in (I_0 \cup \cdots \cup I_{l-1}) \times I_l \mid i < j \}. \end{aligned}$$

Then we have

Theorem 2 *In Case (U): For $Y = Y_{\sigma,e} \in \mathcal{R}_n$, we have*

$$\alpha(Y;\Gamma) = (q+1)^{c_1(\sigma)}\{q(1-q^{-2})\}^{c_2(\sigma)} q^{-n^2+2d(\sigma,e)}.$$

In Case (R): For $Y = Y_{\sigma,e,\epsilon} \in \mathcal{R}_n$, we have

$$\alpha(Y;\Gamma) = 2^{c_1(\sigma)}(1-q^{-1})^{c_2(\sigma)} q^{-\frac{1}{2}n(n-1)+d(\sigma,e)}.$$

Here

$$d(\sigma,e) = c_2(\sigma) + \tau(\{I_i\}) + t(\sigma,\{I_i\}) + \frac{1}{2}\sum_{l=0}^{h} \nu_l m_l^2.$$

For the proofs we refer to [H6, §2].

Remark 1 A complete set of representatives of $GL_n(\mathcal{O})\backslash X_n$ is given in the following way by Jacobowitz([Ja]).

$$\mathrm{Case}(U):\ \{\mathrm{Diag}(\pi^{e_1},\ldots,\pi^{e_n}) \mid e_1 \le \cdots \le e_n\}\ (= \{(1,e) \in \mathcal{R}_n \mid e_1 \le \cdots \le e_n\});$$

$$\mathrm{Case}(R):\ \left\{Y_0 \perp \cdots \perp Y_h \in X_n \;\middle|\; Y_i \in \mathcal{R}(\lambda_i, m_i),\ \lambda_0 < \cdots < \lambda_h,\ \textstyle\sum_{i=0}^{h} m_i = n\right\},$$

where

$$\mathcal{R}(\lambda,m) = \begin{cases} \left\{\mathrm{Diag}(\pi^d,\ldots,\pi^d,\epsilon\pi^d) \;\middle|\; \epsilon = 1,\delta\right\} & \text{if } \lambda = 2d, \\ \left\{\begin{pmatrix} 0 & \varpi^\lambda \\ -\varpi^\lambda & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & \varpi^\lambda \\ -\varpi^\lambda & 0 \end{pmatrix}\right\} & \text{if } 2\not|\lambda,\ 2|m, \\ \emptyset & \text{if } 2\not|\lambda,\ 2\not|m. \end{cases}$$

The explicit formula of $\alpha(Y; GL_n(\mathcal{O})) = \mu(Y,Y)$ is also known ([H1, I, (2.3)]).

Remark 2 For symmetric case ($k = k_0$), the corresponding data is the following (cf. [SH, §2]).

$$\mathcal{R}_n(S) = \left\{(\sigma,e,\epsilon) \in \mathfrak{S}_n \times \mathbb{Z}^n \times \{1,\delta\}^n \;\middle|\; \begin{matrix} \sigma^2 = 1,\ e_i = e_{\sigma(i)}(\forall i), \\ \epsilon_i = 1 \text{ if } i \ne \sigma(i) \end{matrix}\right\}$$

$$Y_{\sigma,e,\epsilon} = \sigma \begin{pmatrix} \epsilon_1\pi^{e_1} & & 0 \\ & \ddots & \\ 0 & & \epsilon_n\pi^{e_n} \end{pmatrix} \in X_n,$$

$$\alpha(Y_{\sigma,e,\epsilon};\Gamma) = 2^{c_1(\sigma)}(1-q^{-1})^{c_2(\sigma)} q^{-\frac{1}{2}n(n-1)+d_S(\sigma,e)},$$

where

$$d_S(\sigma,e) = c_2(\sigma) + \tau(\{I_i\}) + t(\sigma,\{I_i\}) + \frac{1}{2}\sum_{l=0}^{r} \nu_l m_l(m_l+1).$$

§3. Explicit Expressions of local desnsities

§3.1. Reformulation of local densities

Let V_n be the set of matrices Y in $M_n(k)$ satisfying $Y^* = Y$, and ψ be an additive character of k_0 of conductor $\mathcal{O}_{k_0}$. For $X, Y \in V_n$, set $< X, Y >= \mathrm{Tr}(XY)$, which is an element of k_0. For $S \in V_m$ and $X \in M_{m,n}(k)$, we denote $S[X] = X^*SX$ $(\in V_n)$.

Let Δ be a congruence subgroup of $GL_n(\mathcal{O})$. For $Y \in X_n$, we define

$$\alpha(Y;\Delta) = \lim_{d\to\infty} q^{-dn^2} N_d(Y;\Delta),$$

where

$$N_d(Y;\Delta) = \sharp\left\{\gamma \in \Delta \bmod (\pi^d) \;\middle|\; \gamma\cdot Y \equiv Y \bmod (\pi^d)\right\}.$$

Proposition 3.1 *For $A \in X_m$ and $B, Y \in X_n$,*

$$\mu(B,A) = \sum_{Y\in\Delta\backslash X_n} \frac{\mathcal{G}_\Delta(Y,B)\mathcal{G}(Y,A)}{\alpha(Y;\Delta)}.$$

Here

$$\begin{aligned} \mathcal{G}(Y,A) &= \int_{M_{m,n}(\mathcal{O})} \psi(< Y, A[X] >))\, dX, \\ \mathcal{G}_\Delta(Y,B) &= \int_\Delta \psi(< Y, -B[\gamma] >)\, d\gamma, \end{aligned}$$

where $d\gamma$ is the Haar measure on $M_n(\mathcal{O})$ normalized by $\int_{M_n(\mathcal{O})} d\gamma = 1$.

By Proposition 3.1, the calculation of the local density $\mu(B,A)$ is reduced to the following problems :

(i) Take a suitable Δ and classify $\Delta\backslash X_n$,

(ii) For each representative Y of $\Delta\backslash X_n$, calculate $\alpha(Y;\Delta)$, $\mathcal{G}(Y,A)$, and $\mathcal{G}_\Delta(Y,B)$, and arrange them into a finite sum.

The calculation of $\mathcal{G}(Y,A)$ is easy in general.

When $\Delta = GL_n(\mathcal{O})(= K, \text{say})$, the classification of $K\backslash X_n$ and the value of $\alpha(Y;K) = \mu(Y,Y)$ are known (§2 **Remark 1**). The calculation of $\mathcal{G}_K(Y,B)$ for Case (U) has been done by using spherical functions and functional equations of local zeta functions on the space of unramified hermitian forms, and we have an explicit formula of local densities $\mu(B,A)$ (cf. [H4]). For Case (R), it seems to be difficult to follow a similar line to the unramified case.

Very similar formula to **Proposition 3.1** with $\Delta = K$ has been used to obtain a denominator of the power series

$$\sum_{r=0}^{\infty} \mu(\pi^r B, A)X^r \qquad (A \in X_m(\mathcal{O}),\ B \in X_n(\mathcal{O})),$$

by an suitable estimate of $\mathcal{G}_K(Y,\pi^r B)$ (cf. [H2]).

When we take the Iwahori subgroup Γ for Δ, the classification of $\Gamma\backslash X_n$ and calculation of $\alpha(Y;\Gamma)$ have been done in §**2**, we can calculate $\mathcal{G}_\Gamma(Y,B)$, and we obtain an explicit formula of local densities $\mu(B,A)$ which we shall give below. For details see [H6].

§3.2. Case (U)

We give the explicit formula of $\mu(B,A)$ for Case(U). It suffices to give for A and B in the following form

$$A=(\pi^{A_1})\perp\cdots\perp(\pi^{A_m})\in X_m(\mathcal{O}),\quad B=(\pi^{B_1})\perp\cdots\perp(\pi^{B_n})\in X_n(\mathcal{O}).$$

We set, for $\sigma\in\mathfrak{S}$ with $\sigma^2=1$,

$$\xi_{\sigma,i,k}=\begin{cases}0 & \text{if } k\le i,\ k\le\sigma(i),\\ 1 & \text{if } i<k\le\sigma(i) \text{ or } \sigma(i)<k\le i,\\ 2 & \text{if } i<k,\ \sigma(i)<k.\end{cases}$$

Proposition 3.2 *Let $Y=Y_{\sigma,e}\in\mathcal{R}_n$, and $A\in X_m$ and $B\in X_n$ be as above.*

(i) *We have*

$$\mathcal{G}(Y,A)=(-q)^{a(e,A)}\quad \textit{with}\quad a(e,A)=\sum_{i=1}^{n}\sum_{k=1}^{m}\min\{0,e_i+A_k\}.$$

(ii) *The character sum $\mathcal{G}_\Gamma(Y,B)$ vanishes unless*

$$e_i\ge\begin{cases}-B_i-1 & \textit{if } \sigma(i)\le i\\ -B_i & \textit{if } \sigma(i)>i\end{cases}\qquad(\forall i\in I). \tag{3.1}$$

When the condition (3.1) above is satisfied, we have

$$\mathcal{G}_\Gamma(Y,B)=(1-q^{-2})^{2c_2(\sigma)}q^{-n(n-1)}(-q)^{f(\sigma,e,B)}\prod_{\substack{1\le i\le n\\ \sigma(i)=i}}\widetilde{I^*}(e_i+B_i),$$

where

$$f(\sigma,e,B)=\textstyle\sum_{i=1}^n\sum_{k=1}^n\min\{0,e_i+B_k+\xi_{\sigma,i,k}\},$$

$$\widetilde{I^*}(\lambda)=\begin{cases}1-q^{-2} & \textit{if } \lambda\ge 0\\ 1+q^{-1} & \textit{if } \lambda=-1\end{cases}.$$

For each $\sigma\in\mathfrak{S}_n$ with $\sigma^2=1$ and a partition $I=I_0\cup I_1\cup\cdots\cup I_h$ into disjoint σ-stable subsets, we set

$$b_l(\sigma,B)=\min\Big[\{B_i\mid i\in I_l,\ \sigma(i)>i\}\cup\{B_i+1\mid i\in I_l,\ \sigma(i)\le i\}\Big],$$

$$\Xi_{l,\lambda}(\sigma,A,B)=(-q)^{\rho_{l,\lambda}}\prod_{\substack{i\in I_l\\ \sigma(i)=i}}\theta_{i,\lambda}\quad(0\le l\le h,\ \lambda\in\mathbb{Z}),$$

where

$$\rho_{l,\lambda} = \rho_{l,\lambda}(\sigma, A, B) = n_l \sum_{k=1}^{m} \min\{0, \lambda + A_k\} + \sum_{i \in I_l} \sum_{k=1}^{n} \min\{0, \lambda + B_k + \xi_{\sigma,i,k}\},$$

$$\theta_{i,\lambda} = \theta_{i,\lambda}(B) = \begin{cases} 1 - q^{-2} & \text{if } \lambda + B_i \geq 0 \\ 1 + q^{-1} & \text{if } \lambda + B_i = -1. \end{cases}$$

Then the explicit formula of local density $\mu(B, A)$ in Case (U) is given as follows.

Theorem 3 *Let $m \geq n$ and $A \in X_m(\mathcal{O})$ and $B \in X_n(\mathcal{O})$ be as above. Then we have*

$$\mu(B, A)$$

$$= \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma^2 = 1}} \cdot (1 + q^{-1})^{-c_1(\sigma)} \left(q^{-1}(1 - q^{-2})\right)^{c_2(\sigma)} \times \sum_{I = I_0 \cup \cdots \cup I_h} q^{-2\tau(\{I_i\}) - 2t(\sigma, \{I_i\})}$$

$$\times \sum_{k=0}^{h+1} \frac{(1 - q^{-2})^{c_1(k;\sigma)} q^{-\sum_{l=k+1}^{h} m_l^2}}{\prod_{l=k}^{h} (1 - q^{-m_l^2})} \times \sum_{\{\nu\}_k} q^{\sum_{l=0}^{k-1} \nu_l (m_k^2 - m_l^2)} \times \prod_{l=0}^{k-1} \Xi_{l, \nu_0 + \cdots + \nu_l}(\sigma, A, B).$$

Here the summation with respect to $I = I_0 \cup \cdots \cup I_h$ is taken over all partitions of I into disjoint σ-stable subsets, the summation with respect to $\{\nu\}_k$ for $k \geq 1$ is taken over the finite set

$$\left\{ (\nu_0, \nu_1, \ldots, \nu_{k-1}) \in \mathbb{Z} \times \mathbb{N}^{k-1} \;\middle|\; -b_l(\sigma, B) \leq \nu_0 + \nu_1 + \cdots + \nu_l \leq -1 \quad (0 \leq l \leq k-1) \right\},$$

and if $k = 0$, we understand the summation with respect to $\{\nu\}_k$ to be equal to 1.

§3.3. Case (R)

We give the explicit formula $\mu(B, A)$ for Case(R). It suffices to give for A and B in the following form

$$A = \left(u_1 \pi^{a_1}\right) \perp \cdots \perp \left(u_r \pi^{a_r}\right) \perp \begin{pmatrix} 0 & \varpi^{2b_1+1} \\ -\varpi^{2b_1+1} & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & \varpi^{2b_s+1} \\ -\varpi^{2b_1+s} & 0 \end{pmatrix} \in X_m(\mathcal{O}),$$

$$B = \left(v_1 \pi^{c_1}\right) \perp \cdots \perp \left(v_t \pi^{c_t}\right) \perp \begin{pmatrix} 0 & \varpi^{2d_1+1} \\ -\varpi^{2d_1+1} & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & \varpi^{2d_w+1} \\ -\varpi^{2d_w+1} & 0 \end{pmatrix} \in X_n(\mathcal{O}),$$

where $u_i, v_j \in \mathcal{O}_{k_0}^{\times}$ $(1 \leq i \leq r,\ 1 \leq j \leq t)$. Set

$$A_k = \begin{cases} 2a_k & \text{if } k \leq r \\ 2b_j + 1 & \text{if } k = r + 2j \text{ or } k = t + 2j - 1, \end{cases}$$

$$B_k = \begin{cases} 2c_k & \text{if } k \leq t \\ 2d_j + 1 & \text{if } k = t + 2j \text{ or } k = t + 2j - 1. \end{cases}$$

We set

$$\begin{aligned} \alpha(\lambda) &= \alpha(\lambda, A) = \{k \mid 1 \le k \le r,\ \lambda + A_k < 0\}, \\ \beta_i(\lambda) &= \beta_i(\lambda, B) \\ &= \left\{ k \,\middle|\, \begin{array}{l} 1 \le k \le \min\{i-1, t\} \\ \lambda + B_k < 0 \end{array} \right\} \cup \left\{ k \,\middle|\, \begin{array}{l} \min\{i, t\} < k \le t \\ \lambda + B_k < -2 \end{array} \right\}. \end{aligned}$$

For $\sigma \in \mathfrak{S}_n$ with $\sigma^2 = 1$, we set

$$\begin{aligned} c_1''(\sigma) &= c_1''(\sigma, B) = \sharp\{i \in I \mid \sigma(i) = i \ge t\}, \\ \xi_{\sigma,i,k} &= \begin{cases} 1 & \text{if } k \le i, k \le \sigma(i) \\ 2 & \text{if } i < k \le \sigma(i), \text{ or } \sigma(i) < k \le i \\ 3 & \text{if } i < k, \sigma(i) < k \end{cases}. \end{aligned}$$

Proposition 3.3 *Let $Y = Y_{\sigma,e,\epsilon} \in \mathcal{R}_n$, and $A \in X_m$ and $B \in X_n$ be as above.*
(i) *We have*

$$\mathcal{G}(Y, A) = q^{a(e,A)} \prod_{\substack{1 \le i \le n \\ \sigma(i) = i}} \prod_{k \in \alpha(e_i, A)} \left(\frac{-1}{\mathfrak{p}}\right)^{\frac{e_i + A_k}{2}} \left(\frac{-\epsilon_i u_k}{\mathfrak{p}}\right)\omega,$$

where

$$a(e, A) = \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{m} \min\{0, e_i + A_k + 1\}.$$

(ii) *The character sum $\mathcal{G}_\Gamma(Y, B)$ vanishes unless*

$$e_i \ge \begin{cases} -B_i - 1 & \text{if } i < \sigma(i),\ 2 | i - t \text{ when } i = \sigma(i) - 1 > t \\ & \text{or } i = \sigma(i) > t,\ 2 | i - t \\ -B_i - 2 & \text{if } \sigma(i) \le i,\ 2 \not| i - t \text{ when } i = \sigma(i) > t \\ & \text{or } i = \sigma(i) - 1 > t,\ 2 \not| i - t \end{cases} \qquad (\forall i \in I). \tag{3.2}$$

When the condition (3.2) above is satisfied, we have

$$\begin{aligned} \mathcal{G}_\Gamma(Y, B) &= (1 - q^{-1})^{2c_2(\sigma) + c_1''(\sigma)} \cdot (-1 + q^{-1})^{-\delta(\sigma,e,B)} \cdot q^{-\frac{n(n-1)}{2} + f(\sigma,e,B)} \\ &\quad \times \prod_{\substack{1 \le i \le t \\ \sigma(i) = i}} I^*(\frac{1}{2}(e_i + B_i); -\epsilon_i v_i) \cdot \prod_{\substack{1 \le i \le n \\ \sigma(i) = i}} \prod_{k \in \beta_i(e_i, B)} \left(\frac{-1}{\mathfrak{p}}\right)^{\frac{e_i + B_k}{2}} \left(\frac{\epsilon_i v_k}{\mathfrak{p}}\right)\omega, \end{aligned}$$

where

$$\delta(\sigma, e, B) = \sharp\{i \mid t < i = \sigma(i) - 1,\ 2 \not| i - t,\ e_i + B_i = -2\},$$

$$f(\sigma, e, B) = \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\min\{0, e_i + B_k + \xi_{\sigma,i,k}\} - \frac{1}{2}\sum_{\substack{1\le i\le t\\ \sigma(i)=i}}\min\{0, e_i + B_i + 1\},$$

$$I^*(\lambda;\eta) = \begin{cases} 1 - q^{-1} & \textit{if } \lambda \ge 0 \\ q^{-\frac{1}{2}}\left(\frac{\eta}{\mathfrak{p}}\right)\omega - q^{-1} & \textit{if } \lambda = -1 \\ 0 & \textit{if } \lambda \le -2 \end{cases}.$$

For each $\sigma \in \mathfrak{S}_n$ with $\sigma^2 = 1$ and a partition $I = I_0 \cup I_1 \cup \cdots \cup I_h$ into disjoint σ-stable subsets, we set

$$\begin{aligned}
c'_1(k;\sigma) &= \sum_{l=k}^{h} \sharp\{i \in I_l \mid \sigma(i) = i < t\}, \\
b_l(\sigma, B) &= \min\Big[\{B_i + 1 \mid i \in I_l,\ i < \sigma(i),\ 2|i - t \text{ if } i = \sigma(i) - 1 > t\} \\
&\qquad \cup \{B_i + 1 \mid i \in I_l,\ i = \sigma(i) > t,\ 2|i - t\} \\
&\qquad \cup \{B_i + 2 \mid i \in I_l,\ \sigma(i) \le i,\ 2\not| i - t \text{ if } i = \sigma(i) > t\} \\
&\qquad \cup \{B_i + 2 \mid i \in I_l,\ i = \sigma(i) - 1,\ 2\not| i - t\}\Big], \\
\Xi_{l,\lambda}(\sigma, A, B) &= (-1 + q^{-1})^{-\delta_{l,\lambda}} \cdot q^{\rho_{l,\lambda}} \cdot \prod_{\substack{i\in I_l\\ \sigma(i)=i}} \theta_{i,\lambda}.
\end{aligned}$$

Here

$$\begin{aligned}
\delta_{l,\lambda} &= \delta_{l,\lambda}(\sigma, B) = \sharp\{i \in I_l \mid i = \sigma(i) - 1 > t,\ 2\not| i - t,\ \lambda + B_i = -2\}, \\
\rho_{l,\lambda} &= \rho_{l,\lambda}(\sigma, A, B) = \frac{n_l}{2}\sum_{k=1}^{r}\min\{0, \lambda + A_k + 1\} + \frac{1}{2}\sum_{i\in I_l}\sum_{k=1}^{n}\min\{0, \lambda + B_k + \xi_{\sigma,i,k}\} \\
&\qquad - \frac{1}{2}\sum_{\substack{i\in I_l\\ i=\sigma(i)<t}}\min\{0, \lambda + B_i + 1\}, \\
\theta_{i,\lambda} &= \theta_{i,\lambda}(A, B) \\
&= 2 \cdot \prod_{k\in\alpha(\lambda)}\left(\frac{-1}{\mathfrak{p}}\right)^{\frac{A_k}{2}}\left(\frac{-u_k}{\mathfrak{p}}\right) \cdot \prod_{k\in\beta_i(\lambda)}\left(\frac{-1}{\mathfrak{p}}\right)^{\frac{B_k}{2}}\left(\frac{v_k}{\mathfrak{p}}\right) \cdot \left(\frac{-1}{\mathfrak{p}}\right)^{\left[\frac{\sharp\alpha(\lambda)+\sharp\beta_i(\lambda)+1}{2}\right]} \\
&\qquad \times \begin{pmatrix} 0 & \text{if } 2\not|\sharp\alpha(\lambda) + \sharp\beta_i(\lambda) \text{ and } i > t, \text{ or } i \le t,\ \lambda + B_i \ge 0 \\ 1 & \text{if } 2|\sharp\alpha(\lambda) + \sharp\beta_i(\lambda) \text{ and } i > t \\ 1 - q^{-1} & \text{if } 2|\sharp\alpha(\lambda) + \sharp\beta_i(\lambda) \text{ and } i \le t,\ \lambda + B_i \ge 0 \\ -q^{-1} & \text{if } 2|\sharp\alpha(\lambda) + \sharp\beta_i(\lambda) \text{ and } i \le t,\ \lambda + B_i = -2 \\ q^{-\frac{1}{2}}\left(\frac{-v_i}{\mathfrak{p}}\right) & \text{if } 2\not|\sharp\alpha(\lambda) + \sharp\beta_i(\lambda) \text{ and } i \le t,\ \lambda + B_i = -2 \end{pmatrix},
\end{aligned}$$

where [ ] is the Gaussian symbol. Then the explicit formula of local density $\mu(B,A)$ in Case (R) is given as follows.

Theorem 4 *Let $m \geq n$ and $A \in X_m(\mathcal{O})$ and $B \in X_n(\mathcal{O})$ be as above. Then we have*

$$\begin{aligned}
&\mu(B,A)\\
&= \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma^2=1}} 2^{-c_1(\sigma)}\cdot(1-q^{-1})^{c_2(\sigma)+c_1''(\sigma)}\cdot q^{-c_2(\sigma)} \times \sum_{I=I_0\cup\cdots\cup I_h} q^{-\tau(\{I_i\})-t(\sigma,\{I_i\})}\\
&\times\sum_{k=0}^{h+1} \frac{2^{c_1(k;\sigma)}\cdot(1-q^{-1})^{c_1'(k;\sigma)}\cdot q^{m_k^2-\frac{1}{2}\sum_{l=k}^{h} m_l^2}}{\prod_{l=k}^{h}(1-q^{-m_l^2})} \times \sum_{\{\nu\}_k} q^{\sum_{l=0}^{k-1}\nu_l(m_k^2-m_l^2)} \times \prod_{l=0}^{k-1} \Xi_{l,\nu_0+\cdots+\nu_l}(\sigma,A,B).
\end{aligned}$$

Here the summation with respect to $I = I_0\cup\cdots\cup I_h$ is taken over all partitions of I into disjoint σ-stable subsets, the summation with respect to $\{\nu\}_k$ for $k \geq 1$ is taken over the finite set

$$\left\{(\nu_0,\nu_1,\ldots,\nu_{k-1}) \in \mathbb{Z}\times\mathbb{N}^{k-1} \;\middle|\; -b_l(\sigma,B) \leq \nu_0+\nu_1+\cdots+\nu_l \leq -1 \quad (0\leq l\leq k-1)\right\},$$

and if $k=0$, we understand the summation with respect to $\{\nu\}_k$ to be equal to 1.

§3.4. An application

As an application, we consider the following polynomial in X:

$$\mu(X;B,A) = \mu(B,A(g)),$$

where

$$A(g) = A \perp \begin{pmatrix} 0 & 1_g \\ 1_g & 0 \end{pmatrix} \quad (g\geq 0), \qquad \text{and} \qquad X = \sharp\,(\mathcal{O}/\varpi)^{-g}.$$

In the case of symmetric forms, a similar polynomial has been introduced by Kudla and plays an important role in arithmetic of Eisenstein series ([Ku]).

Corollary 3.4 (i) *Case (U) : With the same notation as in Theorem 3, we have*

$$\begin{aligned}
\mu(X;B,A) \;=\; &\sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma^2=1}} (1+q^{-1})^{-c_1(\sigma)}\cdot(1-q^{-2})^{c_2(\sigma)}\cdot q^{-c_2(\sigma)} \sum_{I=I_0\cup\cdots\cup I_h} q^{-2\tau(\{I_i\})-2t(\sigma,\{I_i\})}\\
&\times\sum_{k=0}^{h+1} \frac{(1-q^{-2})^{c_1(k;\sigma)}\cdot q^{-\sum_{l=k+1}^{h} m_l^2}}{\prod_{l=k}^{h}\left(1-q^{-m_l^2}\right)} \sum_{\{\nu\}_k}\Bigg(q^{\sum_{l=0}^{k-1}\nu_l\left(m_k^2-m_l^2\right)}\\
&\times\prod_{l=0}^{k-1}\Xi_{l,\nu_0+\cdots+\nu_l}(\sigma,B,A)\Bigg)\, X^{\sum_{l=0}^{k-1}|\nu_0+\cdots+\nu_l|n_l}.
\end{aligned}$$

In particular, the degree of $\mu(X;B,A)$ in X is equal to $n+\mathrm{ord}_\pi(\det B)$. When $\{B_i\}$ has distinct values $c_0 > c_1 > \cdots > c_h$ with multiplicity n_i $(0\leq i\leq h)$, the leading coefficient is

$$q^{-n^2-\sum_{l=0}^{h} n_l^2+\nu_l m_l^2} \times (-q)^{\sum_{l=0}^{h}\left(-\frac{1}{2}n_l(n_l+1)+\sum_{k=1}^{m} n_l\min\{0,A_k-c_l-1\}+\sum_{j=l+1}^{h} n_l n_j(c_j-c_l+1)\right)},$$

where $\nu_0 = -c_0 - 1$ *and* $\nu_l = c_{l-1} - c_l$ *for* $l \geq 1$.

(ii) *Case* (R) : *With the same notation as in Theorem 4, we have*

$$\begin{aligned}\mu(X;B,A) \;=\; & \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma^2=1}} 2^{-c_1(\sigma)}\cdot(1-q^{-1})^{c_2(\sigma)+c_1''(\sigma)}\cdot q^{-c_2(\sigma)} \sum_{I=I_0\cup\cdots\cup I_h} q^{-\tau(\{I_i\})-t(\sigma,\{I_i\})}\\ & \times\sum_{k=0}^{h+1}\frac{2^{c_1(k;\sigma)}\cdot(1-q^{-1})^{c_1'(k;\sigma)}\cdot q^{m_k^2-\frac{1}{2}\sum_{l=k+1}^{h}m_l^2}}{\prod_{l=k}^{h}\left(1-q^{-m_l^2}\right)}\sum_{\{\nu\}_k}\Bigg(q^{\frac{1}{2}\sum_{l=0}^{k-1}\nu_l\left(m_k^2-m_l^2\right)}\\ & \times\prod_{l=0}^{k-1}\Xi_{l,\nu_0+\cdots+\nu_l}(\sigma;B,A)\Bigg)\left(\frac{-1}{\mathfrak{p}}\right)^{g(c_1(\sigma)-c_1(k;\sigma))}\cdot X^{\sum_{l=0}^{k-1}|\nu_0+\cdots+\nu_l|n_l},\end{aligned}$$

in particular, the degree of $\mu(X;B,A)$ *in* X *is equal to* $2n+\mathrm{ord}_{\varpi}(\det B)$.

§3.5. Some identities

It seems to be many conbinatorial identities among our formulas of local densities. Here we give some examples.

For Case (U), by the explicit formula in [H3], we have

$$\mu(1_n, A) = \prod_{i=0}^{n-1}(1-(-q^{-1})^{l_0-i}),$$

where $l_0 = \sharp\{i \mid 1\leq i\leq m,\ A_i = 0\}$. Comparing it with the formula induced from Theorem 3, we obtain the following identities with indeterminate X

$$\sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma^2=1}}(1-X)^{c_1(\sigma)}\cdot(X(1-X))^{c_2(\sigma)}\sum_{\substack{I=I_0\cup\cdots\cup I_h\\ \sigma-\text{stable}}}\frac{X^{2\tau(\{I_i\})+2t(\sigma,\{I_i\})+\sum_{l=1}^{h}m_l^2}}{\prod_{l=0}^{h}(1-X^{m_l^2})}=1,$$

and for a with $0\leq a\leq n-1$,

$$\sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma^2=1}}(1-X)^{c_1(\sigma)}\cdot(X(1-X))^{c_2(\sigma)}\sum_{\substack{I=I_0\cup\cdots\cup I_h\\ \sigma-\text{stable},\sigma|_{I_0}=id}}\frac{X^{2\tau(\{I_i\})+2t(\sigma,\{I_i\})+\sum_{l=1}^{h}m_l^2}}{\prod_{l=1}^{h}(1-X^{m_l^2})}\cdot\frac{(-X)^{n(I_0)+n_0a}}{(1-X)^{n_0}}=-X^{n^2},$$

where $n_0=\sharp I_0$ and $n(I_0)=\sum_{i\in I_0}i$.

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