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On Relations between Vector Variational Inequality and Vector Optimization Problem[1]

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1. Introduction and Preliminaries

The concept of vector variational inequality in a finite dimentional Euclidean space was first introduced by Giannessi in 1980. Then several studies have been done on this subject. Vector variational inequalities have shown to be very useful for studying vector optimization problems. Giannessi [3] showed the equivalence between efficient solutions of a differentiable convex vector optimization problem and solutions of a Minty type vector variational inequality for gradients which is a vector version of the classic Minty variational inequality for gradients. Moreover, he proved the equivalence between solutions of weak Minty type and Stampacchia type vector variational inequalities for gradients and weakly efficient solutions of a differentiable convex vector optimization problem. Recently, following the approaches of Giannessi [3], Lee [5] studied equivalent relations between vector variational inequalities for subdifferentials and nondifferentiable convex vector optimization problems.

In this paper, we study the equivalence between solutions of weak Stampacchia type vector variational inequalities for gradients and weakly efficient solutions of vector optimization problem involving pseudoconvex functions, and show the equivalence between solutions of weak Minty type vector variational inequalities and weak Stampacchia type vector variational inequalities for gradients under strict pseudoconvexity assumptions.

Throughout this paper, we consider the following vector optimization problem (VOP):

$$\text{(VOP)} \qquad \begin{array}{ll} \text{minimize} & f(x) := (f_1(x), \cdots, f_p(x)) \\ \text{subject to} & x \in X \end{array}$$

where $f_i : R^n \to R$, $i = 1, \cdots, p$ are functions and X is a nonempty subset of R^n.

Solving (VOP) means to find efficient solutions defined as follows:

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Definition 1.1. A point $\bar{x} \in X$ is said to be an efficient solution for (VOP) if for any $x \in X$,

$$(f_1(x) - f_1(\bar{x}), \cdots, f_p(x) - f_p(\bar{x})) \notin -R^p_+ \setminus \{0\},$$

where R^p_+ is the nonnegative orthant of R^p.

Definition 1.2. A point $\bar{x} \in X$ is said to be a weakly efficient solution for (VOP) if for any $x \in X$,

$$(f_1(x) - f_1(\bar{x}), \cdots, f_p(x) - f_p(\bar{x})) \notin -\mathrm{int}R^p_+,$$

where $\mathrm{int}R^p_+$ is the interior of the set R^p_+.

2. Minty Type Vector Variational inequality

Let $f_i : R^n \to R$, $i = 1, \cdots, p$, be differentiable functions and X a nonempty subset of R^n. We formulate the following Stampacchia type vector variational inequality (SVI) for gradients.

(SVI) Find $\bar{x} \in X$ such that for any $x \in X$,

$$(\nabla f_1(\bar{x})^t(x - \bar{x}), \cdots, \nabla f_p(\bar{x})^t(x - \bar{x})) \notin -R^p_+ \setminus \{0\}.$$

Consider the following Minty type vector variational inequality (MVI) for gradients:

(MVI) Find $\bar{x} \in X$ such that for any $x \in X$,

$$(\nabla f_1(x)^t(x - \bar{x}), \cdots, \nabla f_p(x)^t(x - \bar{x})) \notin -R^p_+ \setminus \{0\}.$$

Giannessi [3] proved the following theorem describing the equivalence between vector optimization problem (VOP) and the Minty type vector variational inequality (MVI) in the convex case.

Theorem 2.1.[3] Let X be convex subset of R^n and f convex and differentiable function. Then $\bar{x} \in X$ is an efficient solution of (P) if and only if it is a solution of (MVI).

Theorem 2.2.[3] Let X be convex and f convex and differentiable function. If $\bar{x} \in X$ is a solution of (SVI), then it is an efficient solution of (MVI).

From Theorem 2.2, (SVI) is a sufficient optimality condition for an efficient solution of (VOP). However, it was shown in Giannessi [3] and Yang [11] that (SVI) is, in general, not a necessary optimality condition for an efficient solution of (VOP). For the completeness, we give this example ([3,5]).

Example. Let $f(x) := (x, x^2)$ and $X := [-1, 0]$. Consider the following differentiable convex vector optimization problem (VOP):

$$\text{(VOP)} \qquad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in X. \end{array}$$

Then $\bar{x} = 0$ is an efficient solution of (VOP) and $\bar{x} = 0$ is a solution of the following Minty type vector variational inequality for gradients: Find $\bar{x} \in X$ such that for any $x \in X$,

$$(\nabla f_1(x)^t(x - \bar{x}), \nabla f_2(x)^t(x - \bar{x})) = (x - \bar{x}, 2x(x - \bar{x})) \notin -R^2_+ \setminus \{0\}.$$

However, $\bar{x} = 0$ is not a solution of the following Stampacchia type vector variational inequality for gradients: Find $\bar{x} \in X$ such that for any $x \in X$,

$$(\nabla f_1(\bar{x})^t(x - \bar{x}), \nabla f_2(\bar{x})^t(x - \bar{x})) = (x - \bar{x}, 2\bar{x}(x - \bar{x})) \notin -R^2_+ \setminus \{0\}.$$

Theorem 2.3. Let X be convex subset of R^n and f_i, $i = 1, \cdots, p$, differentiable and strictly pseudoconvex function. If $\bar{x} \in X$ is a solution of (SVI), then it is an efficient solution of (VOP).

Proof. Let $\bar{x} \in X$ be a solution of (SVI). Suppose to the contrary that $\bar{x} \in X$ is not efficient solution of (VOP). Then there exists $y \in X$ such that

$$(f_1(y) - f_1(\bar{x}), \cdots, f_p(y) - f_p(\bar{x})) \in -R^p_+ \setminus \{0\}.$$

By the strict pseudoconvexity of f_i at $\bar{x}$, we have

$$(\nabla f_1(\bar{x})^t(y - \bar{x}), \cdots, \nabla f_p(\bar{x})^t(y - \bar{x})) \in -\text{int}R^p_+.$$

Thus $\bar{x} \in X$ is not a solution of (SVI).

Theorem 2.4. Let X be convex subset of R^n and f_i Differentiable strictly pseudoconvex functions. If $\bar{x} \in X$ is a weakly efficient solution of (VOP), then it is a solution of (MVI).

Proof. Let $\bar{x} \in X$ be a weakly efficient solution of (VOP). Suppose to the contrary that $\bar{x} \in X$ is not solution of (MVI). Then there exists $y \in X$ such that

$$(\nabla f_1(y)^t(y - \bar{x}), \cdots, \nabla f_p(y)^t(y - \bar{x})) \in -R^p_+ \setminus \{0\}.$$

Since f_i, $i = 1, \cdots, p$, are strictly pseudoconvex, we have

$$(f_1(y) - f_1(\bar{x}), \cdots, f_p(y) - f_p(\bar{x})) \in \text{int}R^p_+,$$

which contradicts to the fact that $\bar{x} \in X$ is a weakly efficient solution of (VOP).

3. Weak Vector Variational Inequality

We consider the following weak Stampacchia type vector variational inequality (WSVI) and weak Minty type vector variational inequality (WMVI):

(WSVI) Find $\bar{x} \in X$ such that for any $x \in X$,

$$(\nabla f_1(\bar{x})^t(x - \bar{x}), \cdots, \nabla f_p(\bar{x})^t(x - \bar{x})) \notin -\text{int}R^p_+.$$

(WMVI) Find $\bar{x} \in X$ such that for any $x \in X$,

$$(\nabla f_1(x)^t(x - \bar{x}), \cdots, \nabla f_p(x)^t(x - \bar{x})) \notin -\text{int}R^p_+.$$

Giannessi [3] proved the equivalence between solutions of weak Minty type and Stampacchia type vector variational inequalities for gradients and weakly efficient solutions of a differentiable convex vector optimization problem.

Now we show the equivalence between solutions of (WSVI) and weakly efficient solutions of (VOP) involving pseudoconvex functions.

Theorem 3.1. Let X is nonempty convex subset of R^n and f_i, $i = 1, \cdots, p$, continuousely differentiable and pseudoconvex functions. Then $\bar{x} \in X$ is a weakly efficient solution of (VOP) if and only if $\bar{x} \in X$ is a solution of (WSVI).

Proof. Let $\bar{x} \in X$ be a weakly efficient solution of (VOP). Because of the convexity of X, we have $\bar{x} + \alpha(x - \bar{x}) \in X$ for all $x \in X$ and all $\alpha \in (0, 1)$. Hence

$$\frac{f(\bar{x} + \alpha(x - \bar{x})) - f(\bar{x})}{\alpha} \notin -\text{int}R^p_+.$$

Since f_i, $i = 1, \cdots, p$, are differentiable, we have

$$(\nabla f_1(\bar{x})^t(x - \bar{x}), \cdots, \nabla f_p(\bar{x})^t(x - \bar{x})) \notin -\text{int}R^p_+, \quad \forall y \in X.$$

Conversely, let $\bar{x} \in X$ be a solution of (WSVI). Suppose to the contrary that $\bar{x} \in X$ is not weakly efficient solution of (VOP). Then there exists $y \in X$ such that

$$(f_1(y) < f_1(\bar{x}), \cdots, f_p(y) < f_p(\bar{x})) \in -\text{int}R^p_+.$$

By the pseudoconvexity of f_i at $\bar{x}$ for each $i = 1, \cdots, p$, we have

$$(\nabla f_1(\bar{x})^t(y-\bar{x}), \cdots, \nabla f_p(\bar{x})^t(y-\bar{x})) \in -\text{int}R^p_+.$$

Thus $\bar{x} \in X$ is not a solution of (WSVI). This completes the proof.

The following theorem give the equivalence of (WSVI) and (WMVI).

Theorem 3.2. Let X be a nonempty convex subset of R^n and $f_i,\ i = 1, \cdots, p$, continuousely differentiable and strictly pseudoconvex functions. Then $x \in X$ is a solution of (WSVI) if and only if $x \in X$ is a solution of (WMVI).

Proof. Let $\bar{x} \in X$ be a solution of (WSVI). Suppose to the contrary that $\bar{x} \in X$ is not solution of (WMVI). Then there exists $y \in X$ such that

$$(\nabla f_1(y)^t(y-\bar{x}), \cdots, \nabla f_p(y)^t(y-\bar{x})) \in -\text{int}R^p_+.$$

Since f_i, $i = 1, \cdots, p$, are strictly pseudoconvex functions, we have

$$(f_1(y) - f_1(\bar{x}), \cdots, f_p(y) - f_p(\bar{x})) \in -\text{int}R^p_+.$$

So $\bar{x} \in X$ is not weakly efficient solution of (VOP). By Theorem 3.1, $\bar{x} \in X$ is not solution of (WSVI). This is a contradiction. Hence $\bar{x} \in X$ is a solution of (WMVI).

Conversely, let $\bar{x} \in X$ be a solution of (WMVI). Because of the convexity of X, we have $\bar{x} + \alpha(y - \bar{x}) \in X$ for all $y \in X$ and all $\alpha \in (0, 1)$ and

$$(\nabla f_1(\bar{x} + \alpha(y-\bar{x}))^t\alpha(y-\bar{x}), \cdots, \nabla f_p(\bar{x} + \alpha(y-\bar{x}))^t\alpha(y-\bar{x})) \notin -\text{int}R^p_+.$$

Since each f_i is continuously differentiable, for each $i = 1, \cdots, p$,

$$\nabla f_i(\bar{x} + \alpha(y-\bar{x})) \to \nabla f_i(\bar{x}) \ \text{ as } \alpha \to 0^+.$$

Then we have

$$(\nabla f_1(\bar{x})^t(y-\bar{x}), \cdots, \nabla f_p(\bar{x})^t(y-\bar{x})) \notin -\text{int}R^p_+, \quad \forall y \in X$$

Hence $\bar{x} \in X$ is a solution of (WSVI).

4. Vector Variational-Like Inequality

Let X be a nonempty subset of R^n and $\eta : X \times X \to R^p$ a vector-valued function.

Definition 4.1. A set X is said to be an η-invex if, for any $x, y \in X$, $\alpha \in [0, 1]$

$$y + \alpha\eta(x, y) \in X.$$

Definition 4.2. A function $f : X \to R^p$ is said to be invex with respect to η if, for each $x, y \in X$ and each $i = 1, \cdots, p$

$$f_i(x) - f_i(y) \geq \nabla f_i(y)^t \eta(x, y).$$

Now we consider the following vector variational-like inequality (VV-LI):

(VV-LI) Find $\bar{x} \in X$ such that for any $x \in X$,

$$(\nabla f_1(\bar{x})^t \eta(x, \bar{x}), \cdots, \nabla f_p(\bar{x})^t \eta(x, \bar{x})) \notin -\text{int}R^p_+.$$

The following theorem give the equivalence of (VOP) and (VV-LI).

Theorem 4.1. Let X be η-invex, and let f be differentiable and invex with respect to η. Then $\bar{x}$ is a weakly efficient solution of (VOP) if and only if $\bar{x}$ is a solution of (VV-LI).

Proof. Let $\bar{x}$ be a weakly efficient solution of (VOP). If $x \in X$ and $\alpha \in (0, 1]$, then $\bar{x} + \alpha\eta(x, \bar{x}) \in X$ since X is η-invex. Hence

$$\frac{f(\bar{x} + \alpha\eta(x, \bar{x})) - f(\bar{x})}{\alpha} \notin -\text{int}R^p_+, \quad \forall \alpha \in (0, 1].$$

Since f is differentiable, it follows that

$$(\nabla f_1(\bar{x})^t \eta(x, \bar{x}), \cdots, \nabla f_p(\bar{x})^t \eta(x, \bar{x})) \notin -\text{int}R^p_+, \quad \forall x \in X.$$

Hence $\bar{x}$ is a solution of (VV-LI).

Conversely, let $\bar{x} \in X$ be a solution of (VV-LI). Since f is invex, we have

$$f_i(x) - f_i(\bar{x}) \geq \nabla f_i(\bar{x})^t \eta(x, \bar{x}), \quad i = 1, \cdots, p.$$

Hence we obtain

$$(f_1(x) - f_1(\bar{x}), \cdots, f_p(x) - f_p(\bar{x})) \notin -\text{int}R^p_+.$$

This complete the proof.

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