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FROM CONVEX POLYTOPES TO MULTI-POLYTOPES

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1. Introduction

We introduce the notion of multi-polytopes generalizing that of convex polytopes, and report that Ehrhart polynomials and Khovanskii-Pukhlikov on lattice convex polytopes (i.e., convex polytopes with vertices in the lattice) can be extended to lattice multi-polytopes. This is a joint work with A. Hattori and the detailed argument and a connection with geometry can be found in [7].

Let us briefly explain the idea of multi-polytopes. It comes from geometry. According to the theory of toric varieties, a lattice convex polytope P corresponds to an ample line bundle L over a compact non-singular toric variety M. In fact, P is the image of M by the moment map associated with L. This suggests us to view the convex polytope P as being formed from two combinatorial data corresponding to M and L. We shall explain them for a convex polygon (i.e., two-dimensional convex polytope) P shown in Figure 1(1). We take an (outward) normal vector to each side and form four two-dimensional cones, each of which is spanned by two normal vectors whose corresponding sides intersect at a vertex of P. Then we obtain a complete fan shown in Figure 1(2). This complete fan is the combinatorial datum corresponding to the base space M in the theory of toric varieties. The other datum is an arrangement of lines obtained by extending the sides of P, see Figure 1(3). This arrangement is the information brought by the line bundle L. Note that the arrangement is related to the fan. Namely the lines in the arrangement are perpendicular to the edge vectors in the fan.

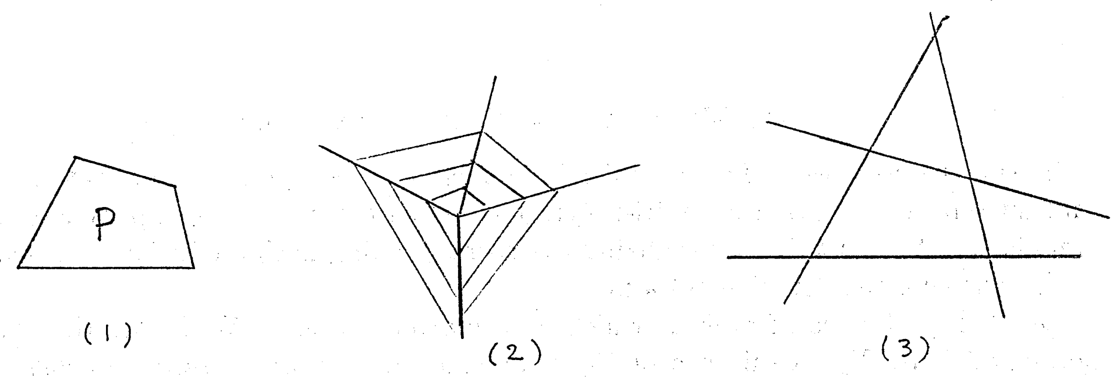

FIGURE 1

The observation above can be applied to n-dimensional convex polytopes. In this case, the associated fan is an n-dimensional complete fan and the arrangement consists of affine hyperplanes in an n-dimensional vector space which are perpendicular to edge vectors in the fan.

Now, let us take the following star shaped figure Q and make the same observation as above.

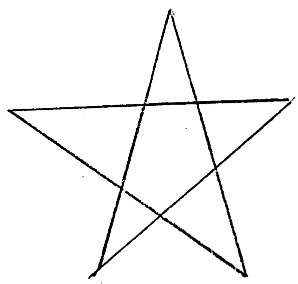

FIGURE 2

Then we obtain five two-dimensional cones, each of which is spanned by two normal vectors whose corresponding sides intersect at a vertex. A notable fact is that the cones have overlap and the degree of overlap is uniformly two. This is an example of almost what we call a complete multi-fan. The multi-polytope associated with Q is a pair of the complete multi-fan and the arrangement of lines obtained by extending the five sides of Q. In general, a multi-polytope is defined to be a pair of a complete multi-fan and an arrangement of affine hyperplanes perpendicular to edge vectors in the multi-fan.

This article is organized as follows. In section 2 we give a precise definition of multi-fan and multi-polytope. We also define the notion of completeness, simpliciality and non-singularity of a multi-fan. The definition of simpliciality and non-singularity is straightforward but the definition of completeness is somewhat complicated and essential in our argument. In section 3 we associate with a simple multi-polytope an interger valued locally constant function (called the Dusitermaat-Heckman function) on the complenent of the hyperplane arrangement. When the multi-polytope arises from a convex polytope P, the function takes 1 on the interior of P and 0 on the other regions divided by the hyperplane arrangement. The generalization of Ehrhart polynomials and Khovanskii-Pukhlikov formula is discussed in sections 4 and 5 respectively.

2. MULTI-FANS AND MULTI-POLYTOPES

In this section, we define a multi-fan which is a complete generalization of a fan and introduce the notion of multi-polytopes. We also define the completeness and non-singularity of a multi-fan generalizing the corresponding notion of a fan. We shall begin with reviewing the definition of a fan.

Let N be a lattice of rank n, which is isomorphic to $\mathbb{Z}^n$. We denote the real vector space $N\otimes\mathbb{R}$ by $N_{\mathbb{R}}$. A subset σ of $N_{\mathbb{R}}$ is called a *strongly convex rational polyhedral cone* (with apex at the origin) if there exits a finite number of vectors $v_1,\dots,v_m$ in N such that

$$\sigma=\{r_1v_1+\cdots+r_mv_m \mid r_i\in\mathbb{R} \text{ and } r_i\geq 0 \text{ for all } i\}$$

and $\sigma\cap(-\sigma)=\{0\}$. Here "rational" means that it is generated by vectors in the lattice N, and "strong" convexity that it contains no line through the origin. We will often call a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$ simply a *cone* in N. The dimension $\dim\sigma$ of a cone σ is the dimension of the linear space spanned by vectors in σ. A subset τ of σ is called a *face* of σ if there is a linear function $\ell\colon N_{\mathbb{R}}\to\mathbb{R}$ such that ℓ takes nonnegative values on σ and $\tau = \ell^{-1}(0) \cap \sigma$. A cone is regarded as a face of itself, while others are called *proper* faces.

Definition. A fan Δ in N is a set of a finite number of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ such that

(1) Each face of a cone in Δ is also a cone in Δ;
(2) The intersection of two cones in Δ is a face of each, (so that different cones do not overlap).

Definition. A fan Δ is said to be *complete* if the union of cones in Δ covers the entire space $N_{\mathbb{R}}$.

A cone is called *simplicial*, or a *simplex*, if it is generated by linearly independent vectors. If the generating vectors can be taken as a part of a basis of N, then the cone is called *non-singular.*

Definition. A fan Δ is said to be *simplicial* (resp. *non-singular*) if every cone in Δ is simplicial (resp. non-singular).

The fundamental fact in the theory of toric varieties says that there is a one-to-one correspondence between n-dimensional toric varieties and n-dimensional fans, and a fan is complete (resp. simplicial or non-singular) if and only if the corresponding toric variety is compact (resp. an orbifold or non-singular).

For each $\tau \in \Delta$, we define N^τ to be the quotient lattice of N by the sublattice generated (as a group) by $\tau \cap N$; so the rank of N^τ is $n - \dim \tau$. We consider cones in Δ that contain τ as a face, and project them on $(N^\tau)_{\mathbb{R}}$. These projected cones form a fan in N^τ, which we denote by Δ_τ and call the *projected fan* with respect to τ. The dimensions of the projected cones decrease by $\dim \tau$. The completeness, simpliciality and non-singularity of Δ will be inherited to Δ_τ for any τ.

We now generalize these notions of a fan. Let N be as before. Denote by $C(N)$ the set of all cones in N. An ordinary fan is a subset of $C(N)$. The set $C(N)$ has a (strict) partial ordering $\prec$ defined by: $\tau \prec \sigma$ if and only if τ is a proper face of σ. The cone $\{0\}$ consisting of the origin is the unique minimum element in $C(N)$. On the other hand, let Σ be a partially ordered finite set with a unique minimum element. We denote by the (strict) partial ordering by $<$ and the minimum element by $*$. An example of Σ used later is an abstract simplicial set with an empty set added as a member, which we call an *augmented simplicial set.* In this case the partial ordering is defined by the inclusion relation and the empty set is the unique minimum element which may be considered as a (-1)-simplex. Suppose that there is a map

$$\Lambda\colon \Sigma \to C(N)$$

such that

(1) $\Lambda(*) = \{0\}$;
(2) If $I < J$ for $I, J \in \Sigma$, then $\Lambda(I) \prec \Lambda(J)$;
(3) For any $I, J \in \Sigma$ and $\kappa \in C(N)$ such that $I < J$ and $\Lambda(I) \prec \kappa \prec \Lambda(J)$, there is a unique element $K \in \Sigma$ such that $I < K < J$ and $\Lambda(K) = \kappa$.

For an integer m such that $0 \leq m \leq n$, we set

$$\Sigma^{(m)} := \{I \in \Sigma \mid \dim \Lambda(I) = m\}.$$

One can easily check that $\Sigma^{(m)}$ does not depend on Λ. When Σ is an augmented simplicial set, $I \in \Sigma$ belongs to $\Sigma^{(m)}$ if and only if the cardinality $|I|$ of I is m, namely I is an $(m-1)$-simplex. Therefore, even if Σ is not an augmented simplicial set, we use the notation $|I|$ for m when $I \in \Sigma^{(m)}$.

The image $\Lambda(\Sigma)$ is a finite set of cones in N. We may think of a pair (Σ, Λ) as a set of cones in N labeled by the ordered set Σ. Cones in an ordinary fan intersect only at their faces, but cones in $\Lambda(\Sigma)$ may overlap, even the same cone may appear repeatedly with different labels. The pair (Σ, Λ) is almost what we call a multi-fan, but we incorporate a pair of weight functions on cones in $\Lambda(\Sigma)$ of the highest dimension $n = \operatorname{rank} N$. More precisely, we consider two functions

$$w^{\pm} \colon \Sigma^{(n)} \to \mathbb{Z}_{\geq 0}.$$

These two functions naturally arise from geometry, and their sum corresponds to Euler number while their difference is related to Todd genus (see [10]).

Definition. We call a triple $\Delta := (\Sigma, \Lambda, w^{\pm})$ a *multi-fan* in N. We define the dimension of Δ to be the rank of N (or the dimension of $N_{\mathbb{R}}$).

Since an ordinary fan Δ in N is a subset of $C(N)$, one can view it as a multi-fan by taking $\Sigma = \Delta$, $\Lambda =$ the inclusion map, $w^+ = 1$, and $w^- = 0$. In a similar way as in the case of ordinary fans, we say that a multi-fan $\Delta = (\Sigma, \Lambda, w^{\pm})$ is *simplicial* (resp. *non-singular*) if every cone in $\Lambda(\Sigma)$ is simplicial (resp. non-singular). The following lemma is easy.

Lemma 2.1. *A multi-fan $\Delta = (\Sigma, \Lambda, w^{\pm})$ is simplicial if and only if Σ is isomorphic to an augmented simplicial set as partially ordered sets.*

The definition of completeness of a multi-fan Δ is rather complicated. A naive definition of the completeness would be that the union of cones in $\Lambda(\Sigma)$ covers the entire space $N_{\mathbb{R}}$. But this is not a right definition. Although the two weighted functions $w^{\pm}$ are incorporated in the definition of a multi-fan, only the difference

$$w := w^+ - w^-$$

matters in this article. We shall introduce the following intermediate notion of pre-completeness at first.

Definition. We call a multi-fan $\Delta = (\Sigma, \Lambda, w^{\pm})$ *pre-complete* if the integer

$$\sum_{v \in \Lambda(I)} w(I)$$

is independent of the coice of a generic element v in N. Here the sum above is understood to be zero if there is no such I, and "generic" means that v does not lie on a linear subspace spanned by a cone in $\Lambda(\Sigma)$ of dimension less than n. We call the integer above the *degree* of Δ and denote it by $\deg(\Delta)$.

Remark. For an ordinary fan, pre-completeness is same as completeness.

To define the completeness for a multi-fan Δ, we need to define a projected multi-fan with respect to an element in Σ. We do it as follows. For each $K \in \Sigma$, we set

$$\Sigma_K := \{J \in \Sigma \mid K \leq J\}.$$

It inherits the partial ordering from Σ, and K is the unique minimum element in Σ_K. A map

$$\Lambda_K\colon \Sigma_K \to C(N^{\Lambda(K)})$$

sending $J \in \Sigma_K$ to the cone $\Lambda(J)$ projected on $(N^{\Lambda(K)})_{\mathbb{R}}$ satisfies the three properties above required for Λ. We define two functions

$$w_K{}^{\pm}\colon \Sigma_K^{(n-|K|)} \subset \Sigma^{(n)} \to \mathbb{Z}_{\geq 0}$$

to be the restrictions of $w^{\pm}$ to $\Sigma_K^{(n-|K|)}$. A triple $\Delta_K := (\Sigma_K, \Lambda_K, w_K{}^{\pm})$ is a multi-fan in $N^{\Lambda(K)}$, and this is the desired *projected multi-fan* with respect to $K \in \Sigma$. When Δ is an ordinary fan, this definition agrees with the previous one.

Definition. A pre-complete multi-fan $\Delta = (\Sigma, \Lambda, w^{\pm})$ is said to be *complete* if the projected multi-fan Δ_K is pre-complete for any $K \in \Sigma$.

Example 2.2. Here is an example of a complete non-singular multi-fan of degree two. Let $v_1, \ldots, v_5$ be integral vectors shown in the following figure, where the dots denote lattice points.

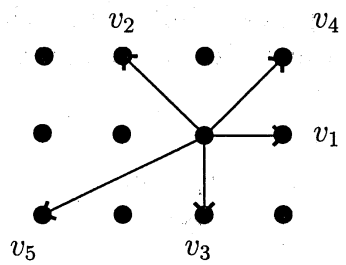

FIGURE 3

The vectors are rotating around the origin twice in counterclockwise. We take

$$\Sigma = \{\phi, \{1\}, \ldots, \{5\}, \{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,1\}\},$$

define $\Lambda\colon \Sigma \to C(N)$ by

$$\begin{aligned} \Lambda(\{i\}) &= \text{ the cone spanned by } v_i, \\ \Lambda(\{i, i+1\}) &= \text{ the cone spanned by } v_i \text{ and } v_{i+1}, \end{aligned}$$

where $i = 1, \ldots, 5$ and 6 is understood to be 1, and take $w^{\pm}$ such that $w = 1$ on every two dimensional cone. Then $\Delta = (\Sigma, \Lambda, w^{\pm})$ is a complete non-singular two-dimensional multi-fan with $\deg(\Delta) = 2$.

Example 2.3. Here is an example of a complete multi-fan "with folds". Let $v_1, \ldots, v_5$ be vectors shown in the following figure.

We define Σ and Λ as in Example 2.2 and take $w^{\pm}$ such that

$$w(\{3,4\}) = -1 \quad \text{and} \quad w(\{i, i+1\}) = 1 \text{ for } i \neq 3.$$

Then $\Delta = (\Sigma, \Lambda, w^{\pm})$ is a complete two-dimensional multi-fan with $\deg(\Delta) = 1$.

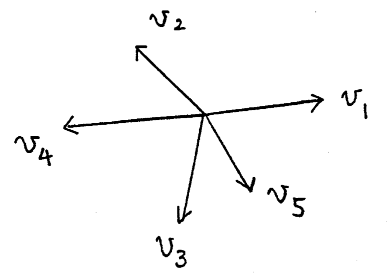

FIGURE 4

A similar example can be constructed for a number of vectors $v_1, \dots, v_d$ $(d \geq 2)$ by defining

$$w(\{i, i+1\}) = 1 \quad \text{if } v_i \text{ and } v_{i+1} \text{ are rotating in counterclockwise,}$$
$$w(\{i, i+1\}) = -1 \quad \text{if } v_i \text{ and } v_{i+1} \text{ are rotating in clockwise,}$$

where $d+1$ is understood to be 1. The degree $\deg(\Delta)$ is the rotation number of the vectors $v_1, \dots, v_d$ around the origin in counterclockwise and may not be one.

Example 2.4. Here is an example of a multi-fan which is pre-complete but not complete. Let $v_1, \dots, v_5$ be vectors shown in the following figure.

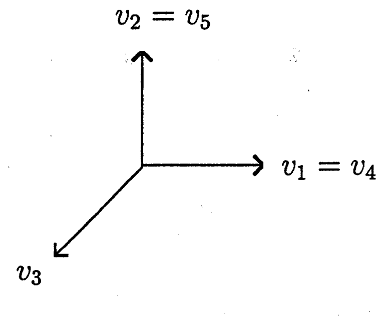

FIGURE 5

We take

$$\Sigma = \{\phi, \{1\}, \dots, \{5\}, \{1,2\}, \{2,3\}, \{3,1\}, \{4,5\}\},$$

define $\Lambda \colon \Sigma \to C(N)$ as in Example 2.2, and take $w^{\pm}$ such that

$$w(\{1,2\}) = 2, \; w(\{2,3\}) = 1, \; w(\{3,1\}) = 1, \; w(\{4,5\}) = -1.$$

Then $\Delta = (\Sigma, \Lambda, w^{\pm})$ is a two-dimensional multi-fan which is pre-complete (in fact, $\deg(\Delta) = 1$) but not complete because the projected multi-fan $\Delta_{\{i\}}$ for $i \neq 3$ is not pre-complete.

So far, we treated *rational* cones that are generated by vectors in the lattice N. But, most of the notions introduced above make sense even if we allow cones generated by vectors in $N_{\mathbb{R}}$ but may not be in N. In fact, the notion of non-singularity requires the lattice N, but others do not. Therefore, one can define a multi-fan and its completeness and simpliciality in this extended category as well. In the following we will denote $N_{\mathbb{R}}$ by V.

As explained in the introduction, a convex polytope or the star shaped figure produces a complete multi-fan and an arrangement of hyperplanes perpendicular to edge vectors in the multi-fan. Taking this observation into account, we reverse a gear. We start with a complete multi-fan $\Delta = (\Sigma, \Lambda, w^{\pm})$. It is convenient to think of the hyperplanes as sitting in the dual space V^* of V. Let $\mathrm{HP}(V^*)$ be the set of all affine hyperplanes in V^*.

Definition. Let $\Delta = (\Sigma, \Lambda, w^{\pm})$ be a complete multi-fan and let $\mathcal{F}\colon \Sigma^{(1)} \to \mathrm{HP}(V^*)$ be a map such that the affine hyperplane $\mathcal{F}(J)$ is 'perpendicular' to the half line $\Lambda(J)$ for each $J \in \Sigma^{(1)}$, i.e., an element in $\Lambda(J)$ takes a constant on $\mathcal{F}(J)$. We call a pair $(\Delta, \mathcal{F})$ a *multi-polytope* and denote it by $\mathcal{P}$. The dimension of a multi-polytope $\mathcal{P}$ is defined to be the dimension of the multi-fan Δ. We say that a multi-polytope $\mathcal{P}$ is *simple* if Δ is simplicial. When $\mathcal{P}$ is simple, $\cap_{i\in I}\mathcal{F}(\{i\})$ for $I \in \Sigma^{(n)}$ is called a vertex of $\mathcal{P}$, and if all vertices of $\mathcal{P}$ are lattice points, then we say that $\mathcal{P}$ is a simple *lattice* multi-polytope.

Remark. There are two notions similar to that of multi-polytopes, which were introduced by Karshon-Tolman [8] and Khovanskii-Pukhlikov [9] when Δ is an ordinary fan. They use the terminology *twisted polytope* and *virtual polytope* respectively. The notion of multi-polytopes is a direct generalization of that of twisted polytopes, and also generalizes that of virtual polytopes, see [11].

3. Duistermaat-Heckman functions

A multi-polytope $\mathcal{P} = (\Delta, \mathcal{F})$ defines an arrangement of affine hyperplanes in V^*. In this section, we associate with $\mathcal{P}$ a function on V^* minus the affine hyperplanes when $\mathcal{P}$ is simple. This function is locally constant and Guillemin-Lerman-Sternberg formula ([5] [6]) tells us that it agrees with the density function of a Duistermaat-Heckman measure when $\mathcal{P}$ arises from a moment map.

Hereafter our multi-polytope $\mathcal{P}$ is assumed to be simple, so that the multi-fan $\Delta = (\Sigma, \Lambda, w^{\pm})$ is complete and simplicial unless otherwise stated. We may assume that Σ consists of subsets of $\{1, \dots, d\}$ and $\Sigma^{(1)} = \{\{1\}, \dots, \{d\}\}$. Denote by v_i a nonzero vector in the one-dimensional cone $\Lambda(\{i\})$. To simplify notation, we denote $\mathcal{F}(\{i\})$ by F_i and set

$$F_I := \cap_{i\in I} F_i \quad \text{for } I \in \Sigma.$$

F_I is an affine space of dimension $n - |I|$. In particular, if $|I| = n$ (i.e., $I \in \Sigma^{(n)}$), then F_I is a point, denoted by u_I.

Suppose $I \in \Sigma^{(n)}$. Then the set $\{v_i \mid i \in I\}$ forms a basis of V. Denote its dual basis of V^* by $\{u_i^I \mid i \in I\}$, i.e., $\langle u_i^I, v_j\rangle = \delta_{ij}$ where δ_{ij} denotes the Kronecker delta. Take a generic vector $v \in V$ such that $\langle u_i^I, v\rangle \neq 0$ for all $I \in \Sigma^{(n)}$ and $i \in I$, and set

$$(-1)^I := (-1)^{\sharp\{i\in I \mid \langle u_i^I, v\rangle > 0\}} \quad \text{and} \quad (u_i^I)^+ := \begin{cases} u_i^I & \text{if } \langle u_i^I, v\rangle > 0 \\ -u_i^I & \text{if } \langle u_i^I, v\rangle < 0. \end{cases}$$

We denote by $\Lambda(I)^+$ the cone in V^* spanned by $(u_i^I)^+$'s ($i \in I$) with apex at u_I, and by ϕ_I its characteristic function.

Definition. We define a function $\mathrm{DH}_{\mathcal{P}}$ on $V^* \setminus \cup_{i=1}^d F_i$ by

$$\mathrm{DH}_{\mathcal{P}} := \sum_{I\in\Sigma^{(n)}} (-1)^I w(I)\phi_I$$

and call it the *Duistermaat-Heckman function* associated with $\mathcal{P}$.

Apparently, the function $\sum_{I\in\Sigma^{(n)}}(-1)^I w(I)\phi_I$ is defined on the whole space V^* and depends on the choice of the generic vector $v \in V$, but it turns out that it restricted to $V \setminus \cup F_i$ is independent of v. This is the reason why we restricted the domain of the function to $V \setminus \cup F_i$. On can also prove that the support of the function $\mathrm{DH}_{\mathcal{P}}$ is bounded.

Remark. There is a completely different way to define the Duistermaat-Heckman function, see [7].

Example 3.1. When $\mathcal{P}$ is a multi-polytope associated with the following rectangle P and the vector v is taken as is shown,

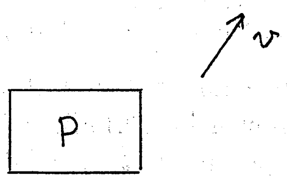

FIGURE 6

the Duistermaat-Heckman function $\mathrm{DH}_{\mathcal{P}}$ is the sum (or difference) of the following characteristic functions of the four shaded domains:

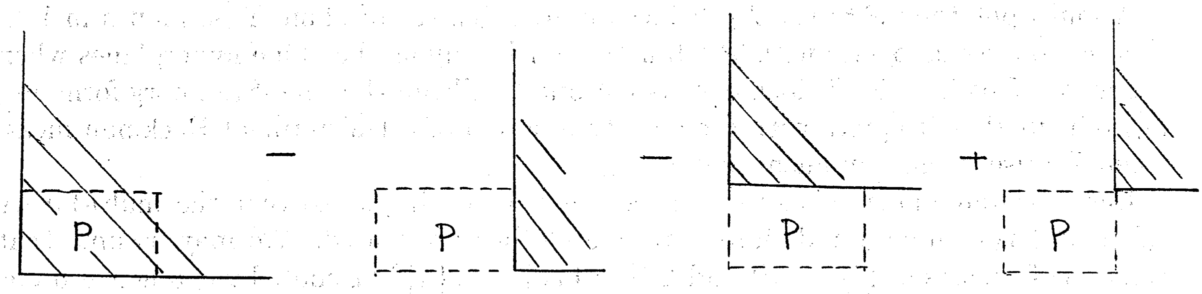

FIGURE 7

Therefore, $\mathrm{DH}_{\mathcal{P}}$ takes 1 on the interior of P and 0 on the other regions divided by the arrangement of $\mathcal{P}$. This is the case for any (simple) convex polytope P.

4. Pick's formula and Ehrhart polynomials

In this section we explain how to define the number of lattice points in a lattice multi-polytope and state a generaliztion of a Ehrhart's theorem on lattice convex polytopes to lattice simple multi-polytopes.

Let P be a convex lattice polytope of dimension n in V^*, where "lattice polytope" means that each vertex of P lies in the lattice $N^* = \mathrm{Hom}(N, \mathbb{Z})$ of $V^* = \mathrm{Hom}(V, \mathbb{R})$. We denote by $\sharp(P)$ (resp. $\sharp(P^\circ)$) the number of lattice points in P (resp. in the interior of

P). The following formula called Pick's formula asserts that when $\dim P = 2$, the area $\mathrm{Area}(P)$ of P can be found by counting lattice points in P and in the boundary ∂P of P.

Theorem 4.1 (Pick's formula). (see [4] or [12] for example.) *If P is a lattice (convex) polygon, then*

$$\mathrm{Area}(P) = \sharp(P^\circ) + \frac{1}{2}\sharp(\partial P) - 1.$$

Example 4.2. In the following lattice polygon $\mathrm{Area}(P) = 17/2, \sharp(P) = 13$ and $\sharp(\partial P) = 7$.

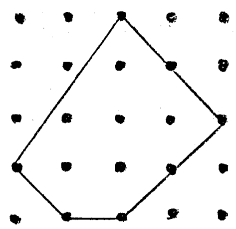

FIGURE 8

Remark. (1) The convexity of P is unnecessary in Pick's formula as is seen in the following non-convex polygon:

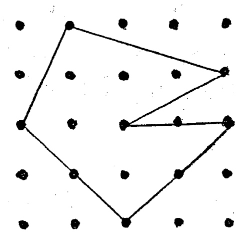

FIGURE 9

(2) There are many generalizations of Pick's formula. For instance, it is generalized in [10] to any piecewise linear closed curve with vertices in the lattice which may have self-intersections such as the star shaped figure in the introduction. In this case, we have to define the terms $\mathrm{Area}(P), \sharp(P^\circ)$ and $\sharp(\partial P)$ in an appropriate way. An interesting fact is that the constant term, that is 1 in Pick's formula, is not necessarily 1 any more.

Pick's formula can be rewritten as

$$\sharp(P^\circ) = \mathrm{Area}(P) - \frac{1}{2}\sharp(\partial P) + 1 \qquad \text{or} \qquad \sharp(P) = \mathrm{Area}(P) + \frac{1}{2}\sharp(\partial P) + 1$$

because $\sharp(P^\circ) = \sharp(P) - \sharp(\partial P)$.

For a positive integer ν, let $\nu P := \{\nu u \mid u \in P\}$. It is again a convex lattice polytope in V^*. Since $\mathrm{Area}(\nu P) = \mathrm{Area}(P)\nu^2$ and $\sharp(\partial(\nu P)) = \sharp(\partial P)\nu$, the above two identities imply

(1) $\sharp(\nu P^\circ)$ and $\sharp(\nu P)$ are both polynomials in ν of degree 2,
(2) $\sharp(\nu P^\circ) = (-1)^n \sharp(-\nu P)$, where $\sharp(-\nu P)$ denotes the polynomial $\sharp(\nu P)$ with ν replaced by $-\nu$.
(3) The coefficient of ν^2 in $\sharp(\nu P)$ is $\mathrm{Area}(P)$ and the constant term in $\sharp(\nu P)$ is 1.

Ehrhart shows that these statements hold for higher dimensional convex lattice polytopes. The lattice N^* determines a volume element on V^* by requiring that the volume of the unit cube determined by a basis of N^* is 1. Thus the volume of P, denoted by $\mathrm{vol}(P)$, is defined.

Theorem 4.3 (Ehrhart). (See [4], [12] for example.) *Let P be an n-dimensional convex lattice polytope.*

(1) *$\sharp(\nu P)$ and $\sharp(\nu P^\circ)$ are polynomials in ν of degree n.*
(2) *$\sharp(\nu P^\circ) = (-1)^n \sharp(-\nu P)$, where $\sharp(-\nu P)$ denotes the polynomial $\sharp(\nu P)$ with ν replaced by $-\nu$.*
(3) *The coefficient of ν^n in $\sharp(\nu P)$ is $\mathrm{vol}(P)$ and the constant term in $\sharp(\nu P)$ is 1.*

The polynomial $\sharp(\nu P)$ in ν is called the *Ehrhart polynomial* of P. The fan Δ associated with P may not be simplicial, but if we subdivide Δ, then we can always take a simplicial fan that is compatible with P. We claim that the theorem above can be extended to *simple* lattice multi-polytopes $\mathcal{P} = (\Delta, \mathcal{F})$. For that, we need to define $\sharp(\mathcal{P})$ and $\sharp(\mathcal{P}^\circ)$. This is done as follows. Let v_i $(i = 1, \dots, d)$ be a primitive integral vector in the half line $\Lambda(\{i\})$. In our convention, v_i is chosen "outward normal" to the face $\mathcal{F}(\{i\})$ when $\mathcal{P}$ arises from a convex polytope. We slightly move $\mathcal{F}(\{i\})$ in the direction of v_i (resp. $-v_i$) for each i, so that we obtain a map $\mathcal{F}_+$ (resp. $\mathcal{F}_-$) $: \Sigma^{(1)} \to \mathrm{HP}(V^*)$. We denote the multi-polytopes $(\Delta, \mathcal{F}_+)$ and $(\Delta, \mathcal{F}_-)$ by $\mathcal{P}_+$ and $\mathcal{P}_-$ respectively. Since the affine hyperplanes $\mathcal{F}_\pm(\{i\})$'s miss the lattice N^*, the functions $\mathrm{DH}_{\mathcal{P}_\pm}$ are defined on N^*.

Definition. We define

$$\sharp(\mathcal{P}) := \sum_{u \in N^*} \mathrm{DH}_{\mathcal{P}_+}(u), \qquad \sharp(\mathcal{P}^\circ) := \sum_{u \in N^*} \mathrm{DH}_{\mathcal{P}_-}(u).$$

When $\mathcal{P}$ arises from a convex polytope P, $\mathrm{DH}_{\mathcal{P}_+}$ (resp. $\mathrm{DH}_{\mathcal{P}_-}$) takes 1 on $u \in N^*$ in P (resp. in the interior of P) and 0 otherwise. Therefore, $\sharp(\mathcal{P})$ (resp. $\sharp(\mathcal{P}^\circ)$) agrees with the number of latice points in P (resp. in the interior of P) in this case.

Denote the volume element on V^* by dV^*, and define the volume $\mathrm{vol}(\mathcal{P})$ of $\mathcal{P}$ by

$$\mathrm{vol}(\mathcal{P}) := \int_{V^*} \mathrm{DH}_{\mathcal{P}}\, dV^*.$$

Needless to say, when $\mathcal{P}$ arises from a convex polytope P, $\mathrm{vol}(\mathcal{P})$ agrees with the actual volume of P, but otherwise it can be zero or negative.

For a (not necessarily positive) integer ν, we denote $(\Delta, \nu\mathcal{F})$ by $\nu\mathcal{P}$, where

$$(\nu\mathcal{F})(\{i\}) := \{u \in V^* \mid \langle u, v_i\rangle = \nu c_i\}$$

when $\mathcal{F}(\{i\}) = \{u \in V^* \mid \langle u, v_i\rangle = c_i\}$ for a constant c_i.

Theorem 4.4. *Let $\mathcal{P} = (\Delta, \mathcal{F})$ be a simple lattice multi-polytope of dimension n.*

(1) *$\sharp(\nu\mathcal{P})$ and $\sharp(\nu\mathcal{P}^\circ)$ are polynomials in ν of degree (at most) n.*
(2) *$\sharp(\nu\mathcal{P}^\circ) = (-1)^n\sharp(-\nu\mathcal{P})$ for any integer ν.*
(3) *The coefficient of ν^n in $\sharp(\nu\mathcal{P})$ is $\mathrm{vol}(\mathcal{P})$ and the constant term in $\sharp(\nu\mathcal{P})$ is $\deg(\Delta)$. (See Section 2 for $\deg(\Delta)$.)*

Let us state a key identity used to prove the theorem above. For $I \in \Sigma^{(n)}$, we define G_I to be the projection image of

$$\{(a_1, \ldots, a_d) \in \mathbb{R}^d \mid \sum_{i=1}^{d} a_i v_i \in N \text{ and } a_j = 0 \text{ for } j \notin I\}$$

on $\mathbb{R}^d/\mathbb{Z}^d$. Since vectors v_i's for $i \in I$ are linearly independent and belong to N, G_I is a finite subgroup of $\mathbb{R}^d/\mathbb{Z}^d$. It is trivial if the set of the vectors v_i for $i \in I$ is a basis of the lattice N, in particular, all G_I for $I \in \Sigma^{(n)}$ are trivial if Δ is non-singular.

On the other hand, for $i = 1, \ldots, d$, we define

$$\rho_i \colon \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{C}^*$$

to be the homomorphism induced from a homomorphism : $\mathbb{R}^d \to \mathbb{C}^*$ mapping $(a_1, \ldots, a_d) \to \exp(2\pi\sqrt{-1}a_i)$.

Let N_Δ^* be the lattice of $N_\mathbb{R}^*$ generated by all u_i^I's for $I \in \Sigma^{(n)}$ and $i \in I$ (see Section 3 for u_i^I's). If Δ is non-singular, then $N_\Delta^* = N^*$. The group ring $\mathbb{C}[N_\Delta^*]$ is a commutative $\mathbb{C}$-algebra, and it has a basis t^u $(u \in N_\Delta^*)$ as a complex vector space with multiplication determined by the addition in N_Δ^*:

$$t^u \cdot t^{u'} := t^{u+u'}.$$

The following is the key identity used in the proof of Theorem 4.4.

Lemma 4.5. *Let the notation be as above. Then*

$$\sum_{I\in\Sigma^{(n)}} \frac{w(I)t^{u_I}}{|G_I|} \sum_{g\in G_I} \frac{1}{\prod_{i\in I}(1-\rho_i(g)t^{-u_i^I})} = \sum_{u\in N^*} \mathrm{DH}_{\mathcal{P}_+}(u)t^u$$

as elements in the quotient ring of $\mathbb{C}[N_\Delta^*]$. In particular, if the multi-fan Δ is non-singular, then $N_\Delta^* = N^*$ and

$$\sum_{I\in\Sigma^{(n)}} \frac{w(I)t^{u_I}}{\prod_{i\in I}(1-t^{-u_i^I})} = \sum_{u\in N^*} \mathrm{DH}_{\mathcal{P}_+}(u)t^u.$$

5. Cohomological formula for $\sharp(\mathcal{P})$

In the theory of toric varieties, a fan corresponds to a toric variety and a lattice convex polytope corresponds to an ample line bundle over a toric variety. Therefore, one can view the cohomology of a toric variety as that of the corresponding fan and the first Chern class of an ample line bundle as that of the corresponding lattice convex polytope. This viewpoint leads us to define the "(equivariant) cohomology" of a complete simplicial multi-fan and the "(equivariant) first Chern class" of a multi-polytope. We then define an index map "in cohomology" and establish a "cohomological formula" describing $\sharp(\mathcal{P})$ for a lattice multi-polytope. As an application of the cohomological formula, we show that the Khovanskii-Pukhlikov formula for a simple lattice convex polytope ([2] [3]) can be generalized to a simple lattice multi-polytope.

Let T be a compact toral group of dimension $n = \operatorname{rank}_{\mathbb{Z}} N$ and let BT be the classifying space of T. Then $H_2(BT)$ is canonically isomorphic to $\operatorname{Hom}(S^1, T)$ the group consisting of homomorphisms from S^1 to T. In fact, a homomorphism $f\colon S^1 \to T$ induces a continuous map $Bf\colon BS^1 \to BT$ and once we fix a generator α of $H_2(BS^1) \cong \mathbb{Z}$, $(Bf)_*\alpha$ defines an element of $H_2(BT)$. The correspondence : $f \to (Bf)_*\alpha$ is known to be an isomorphism from $\operatorname{Hom}(S^1,T)$ to $H_2(BT)$. In the following we assume $N = H_2(BT)$ and identify it with $\operatorname{Hom}(S^1,T)$. Then $N^* = H^2(BT)$ is identified with $\operatorname{Hom}(T, S^1)$ and the group ring $\mathbb{C}[N^*]$ can be identified with the representation ring of T.

Let $\Delta = (\Sigma, \Lambda, w^{\pm})$ be a complete simplicial multi-fan in N. Let $v_i \in H_2(BT)$ be a unique primitive vector in $\Lambda(\{i\})$ for each $i = 1, \dots, d$ as before. Motivated by the description of the equivariant cohomology of a compact non-singular toric variety (see [10]), we define $H_T^*(\Delta)$ to be the face ring of the augmented simplicial set Σ, i.e.,

$$H_T^*(\Delta) := \mathbb{Z}[x_1, \dots, x_d]/(x_I \mid I \notin \Sigma),$$

where $x_I = \prod_{i\in I} x_i$ and the degree of x_i is two, and call $H_T^*(\Delta)$ the *equivariant cohomology* of Δ. We also define a homomorphism $\pi^*\colon H^2(BT) \to H_T^2(\Delta)$ by

$$\pi^*(u) = \sum_{i=1}^{d} \langle u, v_i \rangle x_i,$$

where $\langle\ ,\ \rangle$ denotes the natural pairing between cohomology and homology. It extends to an algebra homomorphism $H^*(BT) \to H_T^*(\Delta)$, which we also denote by π^*. One can think of $H_T^*(\Delta)$ as a module (or more generally an algebra) over $H^*(BT)$ through π^*.

For $I \in \Sigma^{(n)}$, let $\{u_i^I \mid i \in I\}$ be the dual basis of $\{v_i \mid i \in I\}$ as before. We define a ring homomorphism $\iota_I^*\colon H_T^*(\Delta) \otimes \mathbb{Q} \to H^*(BT; \mathbb{Q})$ by

$$\iota_I^*(x_i) := \begin{cases} u_i^I & \text{if } i \in I, \\ 0 & \text{otherwise.} \end{cases}$$

This map is well-defined because x_J for $J \notin \Sigma$, which is zero in $H_T^*(\Delta) \otimes \mathbb{Q}$, maps to zero through ι_I^*. One checks that ι_I^* is an $H^*(BT;\mathbb{Q})$-module map.

A multi-polytope $\mathcal{P} = (\Delta, \mathcal{F})$ is associated with real numbers c_i's by

$$\mathcal{F}(\{i\}) = \{u \in H^2(BT; \mathbb{R}) \mid \langle u, v_i \rangle = c_i\},$$

and these numbers determine an element $c_1^T(\mathcal{P}) := \sum_{i=1}^d c_i x_i$ of $H_T^2(\Delta) \otimes \mathbb{R}$, which we call the *equivariant first Chern class* of $\mathcal{P}$. This gives a bijective correspondence between the set of multi-polytopes defined on Δ and $H_T^2(\Delta) \otimes \mathbb{R}$. Note that $\iota_I^*(c_1^T(\mathcal{P}))$ agrees with the vertex $\cap_{i\in I}\mathcal{F}(\{i\})$. If $\mathcal{P}$ is a lattice multi-polytope, then c_i's are integers and the u_I in Corollary 4.5 agrees with $\iota_I^*(c_1^T(\mathcal{P}))$.

Let S be the multiplicative set consisting of nonzero homogeneous elements of positive degree in $H^*(BT;\mathbb{Q})$. Since $H^*(BT;\mathbb{Q})$ is a polynomial ring (hence an integral domain), $H^*(BT;\mathbb{Q})$ can be thought of as a subring of the localized ring $S^{-1}H^*(BT;\mathbb{Q})$. We define the index map

$$\pi_!\colon H_T^*(\Delta) \otimes \mathbb{Q} \to S^{-1}H^*(BT;\mathbb{Q})$$

"in cohomology" by

$$\pi_!(x) := \sum_{I \in \Sigma^{(n)}} \frac{w(I)\iota_I^*(x)}{|G_I| \prod_{i \in I} u_i^I}$$

(cf. [1, (3.8)]). This map decreases degrees by $2n$ and is an $H^*(BT;\mathbb{Q})$-module map. It turns out that the image of $\pi_!$ lies in $H^*(BT;\mathbb{Q})$.

Now, motivated by the description of the cohomology ring of a compact non-singular toric variety (see p.106 in [4]), we define $H^*(\Delta)$ to be the quotient ring of $H_T^*(\Delta)$ by the ideal generated by $\pi^*(H^2(BT))$, in other words,

$$H^*(\Delta) := \mathbb{Z}[x_1, \dots, x_d]/\mathfrak{A},$$

where $\mathfrak{A}$ is the ideal generated by all

(1) x_I for $I \notin \Sigma$,

(2) $\sum_{i=1}^d \langle u, v_i \rangle x_i$ for $u \in N$.

Since $\pi_!$ is an $H^*(BT;\mathbb{Q})$-module map and $H^*(BT;\mathbb{Q})/(H^2(BT;\mathbb{Q}))$ is isomorphic to $H^0(BT;\mathbb{Q}) = \mathbb{Q}$, $\pi_!$ induces a homomorphism

$$\int_\Delta : H^*(\Delta) \otimes \mathbb{Q} \to \mathbb{Q},$$

where only elements of degree $2n$ in $H^*(\Delta) \otimes \mathbb{Q}$ survive through the map $\int_\Delta$.

Remember that G_I is a finite subgroup of $\mathbb{R}^d/\mathbb{Z}^d$. We denote by G_Δ the union of G_I over all $I \in \Sigma^{(n)}$. Since ρ_i is defined on $\mathbb{R}^d/\mathbb{Z}^d$, $\rho_i(g)$ makes sense for $g \in G_\Delta$. It follows from the definition of G_I and ρ_i that if $g \in G_I$, then $\rho_i(g) = 1$ for $i \notin I$.

We define the *Todd class* $\mathcal{T}(\Delta)$ of the complete simplicial multi-fan Δ by

$$\mathcal{T}(\Delta) := \sum_{g \in G_\Delta} \prod_{i=1}^d \frac{\bar{x}_i}{1 - \rho_i(g) e^{-\bar{x}_i}} \in H^{**}(\Delta) \otimes \mathbb{Q},$$

where $\bar{x}_i$ denotes the image of $x_i \in H_T^*(\Delta)$ in $H^*(\Delta)$. We also define the *first Chern class* $c_1(\mathcal{P})$ of a multi-polytope $\mathcal{P}$ defined on Δ to be the image of $c_1^T(\mathcal{P}) \in H_T^2(\Delta) \otimes \mathbb{R}$ in $H^2(\Delta) \otimes \mathbb{R}$.

Theorem 5.1. *If $\mathcal{P}$ is a simple lattice multi-polytope, then* $\int_\Delta e^{c_1(\mathcal{P})} \mathcal{T}(\Delta) = \sharp(\mathcal{P})$.

As an application of the theorem above, we shall show that Khovanskii-Pukhlikov formula, which relates a certain variation of the volume of a simple convex lattice polytope to the number of lattice points in it, can be generalized to simple multi-polytopes. We begin with

Lemma 5.2. $\mathrm{vol}(\mathcal{P}) = \frac{1}{n!} \int_\Delta c_1(\mathcal{P})^n = \int_\Delta e^{c_1(\mathcal{P})}$ *for a simple multi-polytope $\mathcal{P}$.*

Multi-polytopes defined on Δ form a vector space isomorphic to $H_T^2(\Delta) \otimes \mathbb{R}$ and Lemma 5.2 implies that the volume function is a homogeneous polynomial function of degree n. In fact, if one writes $c_1^T(\mathcal{P}) = \sum_{i=1}^d c_i x_i$, then $\mathrm{vol}(\mathcal{P})$ is a homogeneous polynomial in $c_1, \dots, c_d$ of total degree n.

For $h = (h_1, \dots, h_d) \in \mathbb{R}^d$, we denote by $\mathcal{P}_h$ a multi-polytope with $c_1^T(\mathcal{P}_h) = \sum_{i=1}^d (c_i + h_i)x_i$. Since $c_1(\mathcal{P}_h) = \sum_{i=1}^d (c_i + h_i)\bar{x}_i$, Lemma 5.2 applied to $\mathcal{P}_h$ implies that $\operatorname{vol}(\mathcal{P}_h)$ is a polynomial in $h_1, \dots, h_d$ (of total degree n). We define the *Todd operator* as follows:

$$\mathcal{T}(\partial/\partial h) := \sum_{g \in G_\Delta} \prod_{i=1}^d \frac{\partial/\partial h_i}{1 - \rho_i(g) e^{-\partial/\partial h_i}}.$$

Although the Todd operator is of infinite order, its operation on $\operatorname{vol}(\mathcal{P}_h)$ converges because $\operatorname{vol}(\mathcal{P}_h)$ is a polynomial in $h_1, \dots, h_d$. The following theorem extends the Khovanskii-Pukhlikov formula ([9] [2] [3]) to simple lattice multi-polytopes.

Theorem 5.3. *If $\mathcal{P}$ is a simple lattice multi-polytope, then*

$$\mathcal{T}(\partial/\partial h) \operatorname{vol}(\mathcal{P}_h)|_{h=0} = \sharp(\mathcal{P}).$$

Proof. An elementary computation shows that

$$\frac{\partial/\partial h_i}{1 - \rho_i(g) e^{-\partial/\partial h_i}} e^{(c_i + h_i)\bar{x}_i}|_{h_i=0} = e^{c_i \bar{x}_i} \frac{\bar{x}_i}{1 - \rho_i(g) e^{-\bar{x}_i}}.$$

Therefore, it follows from Lemma 5.2 and Theorem 5.1 that

$$\begin{aligned}
\mathcal{T}(\partial/\partial h) \operatorname{vol}(\mathcal{P}_h)|_{h=0} &= \mathcal{T}(\partial/\partial h) \int_\Delta e^{c_1(\mathcal{P}_h)}|_{h=0} \\
&= \sum_{g \in G_\Delta} \prod_{i=1}^d \frac{\partial/\partial h_i}{1 - \rho_i(g) e^{-\partial/\partial h_i}} \int_\Delta e^{(c_i + h_i)\bar{x}_i}|_{h_i=0} \\
&= \int_\Delta \sum_{g \in G_\Delta} \prod_{i=1}^d e^{c_i \bar{x}_i} \frac{\bar{x}_i}{1 - \rho_i(g) e^{-\bar{x}_i}} \\
&= \int_\Delta e^{c_1(\mathcal{P})} \mathcal{T}(\Delta) = \sharp(\mathcal{P}),
\end{aligned}$$

proving the theorem. □

Remark. One can reformulate the Khovanskii-Pukhlikov formula as follows. As remarked above, the volume function vol is a polynomial in $c_1, \dots, c_d$, so one can apply the Todd operator $\mathcal{T}(\partial/\partial c)$ (with the variables $c = (c_1, \dots, c_d)$ instead of $h = (h_1, \dots, h_d)$) to the volume function vol and evaluate at a simple lattice multi-polytope $\mathcal{P}$. The same argument as in the proof of Theorem 5.3 shows that the evaluated value agrees with $\sharp(\mathcal{P})$.

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