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Wave front sets and wave packet transforms

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Abstract In this talk, I would like to present some basic methods in the microlocal analysis, which becomes an essential and important tool in the theory of partial differential operators. It is an advanced version of the Fourier analysis in relation to regularities of functions or distribution. The core of microlocal analysis is the notion of wave front sets of distributions defined on an open subset of $\mathbf{R}^n$. We can characterize the wave front sets by use of wave packets transform, introduced by G.B. Folland. The standard FBI transform is its special one.

As an application, we shall study the topic of propagation of singularities of solutions to the Schrödinger equations with magnetic or electric potential. It is intensively developed since the 1980's and there are many approaches to this topic. However, we shall take a new approach based on a microlocal conservation law in terms of the Wigner transformation. We will discuss reconstruction of microlocal singularities and creation of microlocal singularities from oscillatory initial data as well as smoothing effects.

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1 Wave front sets

First of all, we recall the definition of the wave front set of distributions, which can give a precise description of the local smoothness properties of distributions.

Definition 1.1 *Let $\Omega \subset \mathbf{R}^n$ be open and $u \in \mathcal{D}'(\Omega)$. Consider a couple $(x_0, \xi_0) \in \Omega \times \dot{\mathbf{R}}^n$. Here and in what follows, $\dot{\mathbf{R}}^n$ denotes $\mathbf{R}^n \backslash \{0\}$. We will say that (x_0, ξ_0) does not belong to the wave front set of u, denoted by $WF(u)$, if and only if*

$$\exists f \in C_0^\infty(\Omega) \text{ with } f(x_0) \neq 0, \quad \exists \text{ an open cone } \Gamma \ni \xi_0$$

such that

$$\forall N \in \mathbf{N},\ \exists C > 0 \ / \ \forall \xi \in \Gamma, \quad |\widehat{fu}(\xi)| \leq C(1 + |\xi|)^{-N}.$$

In a similar way, we can define the notion of an H^s wave front set.

Definition 1.2 *Let $\Omega \subset \mathbf{R}^n$ be open and $u \in \mathcal{D}'(\Omega)$. Consider a couple $(x_0, \xi_0) \in \Omega \times \dot{\mathbf{R}}^n$. We will say that (x_0, ξ_0) does not belong to the H^s wave front set of u, denoted by $WF_{H^s}(u)$, if and only if*

$$\exists f \in C_0^\infty(\Omega) \text{ with } f(x_0) \neq 0, \quad \exists \text{ an open cone } \Gamma \ni \xi_0$$

such that

$$\widehat{fu}(\xi) \in L^2(\Gamma, (1 + |\xi|^2)^s d\xi).$$

2 Wave packet transforms

In this section, we are going to describe another characterization of wave front sets of tempered distributions in terms of the wave packet transform of u.

The most famous example of the wave packet transform is the standard FBI transformation, introduced by Bros-Iagolnitzer and Sjöstrand. They used the FBI transformation to characterize the analytic wave front set of tempered distributions. It turns out that it is very useful for studying local properties of infinitely differentiability of distributions as well as properties of analyticity of distributions. In fact, P. Gérard gave a characterization of the H^s wave front set.

Let q and p be in $\mathbf{R}^n$, and let u be a measurable function on $\mathbf{R}^n$. i denotes the imaginary unit $\sqrt{-1}$. We define the function $\rho(p,q)f$ on $\mathbf{R}^n$ by

$$(\rho(p,q)f)(x) = e^{iq\cdot x+\frac{i}{2}q\cdot p}f(x+p), \quad x \in \mathbf{R}^n. \tag{2.1}$$

We can write the above function as

$$e^{i(pD+qX)}f(x) = e^{iq\cdot x+\frac{i}{2}q\cdot p}f(x+p), \quad \left(e^{i(pD+qX)}f(x)\right)^{\wedge}(\xi)e^{-iq\cdot x-\frac{i}{2}q\cdot p}\hat{f}(\xi-q).$$

Indeed, the function (2.1) is identically equal to $g(x,1)$. Here, $g(x,t)$ is the solution to the first order partial differential equation

$$\begin{cases} \frac{\partial g}{\partial t} - p\nabla_x g = iqxg \\ g(x,0) = f(x). \end{cases} \tag{2.2}$$

To see this, we set $G(t) = g(x-pt,t)$. Then,

$$G'(t) = iq(x-pt)G(t), \quad G(0) = f(x).$$

Proposition 2.1 *$\rho(p,q)$: $L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ is a unitary operator for all p and q in $\mathbf{R}^n$.*

A simple calculation implies that

$$\|\rho(p,q)u\|_{L^2} = \|u\|_{L^2} \text{ and } \rho(p,q)^{-1} = \rho(-p,-q).$$

Let define a unitary operator on $L^2(\mathbf{R}^n)$ depending on the parameter $(p,q,t) \in \mathbf{R}^{2n+1}$ by

$$\tilde{\rho}(p,q,t)u(x) = e^{it+ipq/2+iqx}u(x+p).$$

It is easily verified that $\tilde{\rho}$ is a unitary representation on the Heisenberg group $\mathbf{H}^n$. Namely,

$$\tilde{\rho}(g)\tilde{\rho}(g') = \tilde{\rho}(g*g'), \quad \forall g,\ g' \in \mathbf{H}^n$$

where, the product of two elements (p,q,t) and (p',q',t') in $\mathbf{H}^n$ is defined as follows.

$$(p,q,t)*(p',q',t') = (p+p', q+q', t+t'+\frac{qp'-pq'}{2}).$$

Given a nonzero function $\phi \in \mathcal{S}(\mathbf{R}^n)$, we set

$$\phi^\lambda(x) = \lambda^{n/4}\phi(\lambda^{1/2}x),$$

and define the wave packet transform of u as

$$\mathcal{P}_\lambda^\phi u(x,\xi) = \int u(y)e^{\frac{i}{2}x\cdot\xi - iy\cdot\xi}\overline{\phi^\lambda(y-x)}dy = \int u(y)\overline{\rho(-x,\xi)\phi^\lambda(y)}dy.$$

Using the unitary transformation (2.1), we rewrite it as

$$\mathcal{P}_\phi^\lambda f(x,\xi) = (f, \rho(-x,\xi)\phi^\lambda).$$

Let us recall the definition of the FBI transformation. Let u be a tempered distribution on $\mathbf{R}^n$. The FBI transformation of u is the function on $\mathbf{C}^n \times [0,+\infty)$ defined by

$$Tu(z,\lambda) = \int e^{-\frac{\lambda}{2}(z-y)^2}u(y)dy,$$

where $(z-y)^2 = \sum_{j=1}^n (z_j - y_j)^2$. It is an entire function of the complex variable z, real analytic with respect to the parameter λ. If u is a compactly supported distribution, it is of finite order. Thus, there exist an integer N and a constant $C > 0$ such that

$$|Tu(z,\lambda)| \le C(1+\lambda+|\mathrm{Im} z|)^N e^{\frac{\lambda}{2}(\mathrm{Im} z)^2}$$

for $z \in \mathbf{C}^n$ and $\lambda \in [0,\infty)$. From the simple identity

$$-\frac{1}{2}(x - i\xi - y)^2 = -\frac{1}{2}|x-y|^2 + i\xi(x-y) + \frac{1}{2}|\xi|^2,$$

it is obvious that

$$e^{-\frac{\lambda|\xi|^2}{2}}Tu(x-i\xi,\lambda) = e^{\frac{i\lambda x\cdot\xi}{2}}P_\phi^\lambda u(x,\lambda\xi)$$

when $\phi(x)$ is a Gaussian function $e^{-|x|^2/2}$.

Now, we can state a characterization of the wave front set of a tempered distribution u.

Theorem 2.2 *Suppose that $\phi \in \mathcal{S}(\mathbf{R}^n)$ satisfies*

$$\int_{\mathbf{R}^n} x^\alpha \phi(x)dx \neq 0.$$

for some $\alpha \in (\mathbf{N} \cup \{0\})^n$. *Let* Ω *be an open subset of* $\mathbf{R}^n$, *let* (x_0, ξ_0) *be a point of* $\Omega \times \dot{\mathbf{R}}^n$ *and let* u *be a compactly supported distribution defined in* Ω. *Then,* (x_0, ξ_0) *does not belong to the wave front set* $WF(u)$ *if and only if there is a conic neighborhood* V *of* (x_0, ξ_0) *such that for all* a, $N \geq 1$,

$$|\mathcal{P}_\phi^\lambda u(x, \lambda\xi)| \leq C_{a,N}\lambda^{-N} \quad \text{for } \lambda \geq 1,\ a^{-1} \leq |\xi| \leq a \ \text{ and } (x, \xi) \in V.$$

Remark 2.1 *This gives a partial answer to an open question by G.B. Folland, who proved this theorem under the restriction that* ϕ *is even function. He proposed an open question whether the same conclusion is still valid without that restriction.*

The proof of Theorem 2.2 is a slightly long, so we omit it (c.f. [18]).

3 The Wigner transformation

The Gaussian function is the most useful function in $\mathcal{S}(\mathbf{R}^n)$ in terms of the Fourier transformation. In fact, we have the following well-known fact.

Lemma 3.1 *The Fourier transformation of the Gaussian function* $e^{-\frac{|x|^2}{2}}$ *is equal to* $(2\pi)^{n/2}e^{-\frac{|\xi|^2}{2}}$.

We are going to define the Wigner transformation of two functions f and g in $\mathcal{S}(\mathbf{R}^n)$. It is a function on $\mathbf{R}^{2n}$, defined by

$$W(f,g)(x,\xi) = \int_{\mathbf{R}^n} e^{-i\xi\cdot p} f(x + \frac{p}{2})\overline{g(x - \frac{p}{2})}dp, \quad x,\ \xi \in \mathbf{R}^n$$

We can easily verify the following useful properties of the Wigner transform.

Lemma 3.2

(3.1) $$W(g,f) = \overline{W(f,g)}, \quad f,\ g \in \mathcal{S}(\mathbf{R}^n)$$

(3.2) $$\int W(u)(x,\xi)dx = |\hat{u}(\xi)|^2, \quad \int W(u)(x,\xi)d\xi = (2\pi)^n|u(x)|^2,$$

(3.3) $$W(\hat{u})(x,\xi) = (2\pi)^n W(u)(\xi, -x),$$

(3.4) $$\|u\|^2_{L^2(\mathbf{R}^n)} = \int W(u)(x,\xi)dxd\xi.$$

Here and in what follows, we use the notation $W(f) = W(f, f)$. Then, from (3.1), it follows that $W(f)(x, \xi)$ is real-valued function on $\mathbf{R}^{2n}$ for $f \in L^2(\mathbf{R}^n)$. Furthermore,

Lemma 3.3 *Let a, b, c, d be in $\mathbf{R}^n$ and let f and g be in $\mathcal{S}(\mathbf{R}^n)$. Then*

$$(3.5) \quad W(\rho(b,a)f, \rho(d,c)g)(x,\xi) = e^{i\{(a-c)\cdot x+(b-d)\cdot\xi\}} e^{\frac{i}{2}(a\cdot d-b\cdot c)} W(f,g)(x+\frac{b+d}{2}, \xi - \frac{a+c}{2})$$

and

$$(3.6) \qquad W(f^\lambda, g^\lambda)(x,\xi) = W(f,g)(\lambda^{1/2}x, \lambda^{-1/2}\xi)$$

for all x and ξ in $\mathbf{R}^n$.

In particular,

$$W(\rho(a,b)f)(x,\xi) = W(f)(x+a, \xi - b), \quad a, b, x, \xi \in \mathbf{R}^n.$$

The next is important identity, called the Moyal identity.

Lemma 3.4 *For all f_j, g_j, $j = 1, 2$, in $\mathcal{S}(\mathbf{R}^n)$, we have*

$$(W(f_1, g_1), W(f_2, g_2)) = (2\pi)^n (f_1, f_2)\overline{(g_1, g_2)}.$$

Proof: Define $\tilde{W} : \mathcal{S}(\mathbf{R}^{2n}) \to \mathcal{S}(\mathbf{R}^{2n})$ by

$$\tilde{W}F(x,\xi) = \int e^{-i\xi\cdot p} F(x + \frac{p}{2}, x - \frac{p}{2}) dp.$$

According to the Plancherel theorem,

$$(3.7) \quad (\tilde{W}F_1, \tilde{W}F_2)_{L^2} = (2\pi)^n \int \left\{ \int F_1(x+\frac{p}{2}, x-\frac{p}{2}) \overline{F_2(x+\frac{p}{2}, x-\frac{p}{2}) dp} \right\} dx$$
$$= (2\pi)^n \iint F_1(u,v)\overline{F_2(u,v)} du dv.$$

Applying this identity to

$$F_1(u,v) = f_1(u)\overline{g_1(v)} \text{ and } F_2(u,v) = f_2(u)\overline{g_2(v)},$$

then

$$(3.8)\quad (W(f_1,g_1),W(f_2,g_2))_{L^2} = \iint f_1(u)\overline{g_1(v)f_2(u)}g_2(v)dudv$$
$$= \left(\int f_1(u)\overline{f_2(u)}du\right)\left(\int g_1(v)\overline{g_2(v)}dv\right).$$

Q.E.D.

There is a nice connection between wave packets and Wigner transforms. This connection is crucial in the subsequent sections.

Lemma 3.5

$$(3.9)\quad \mathcal{P}_\lambda(f)(p,q)\overline{\mathcal{P}_\lambda(g)(p,q)} = \pi^{-n/2}\lambda^n \int W(f,g)(x,\lambda\xi)E_\lambda(x-p,\xi-q)dxd\xi$$

for $f,\ g \in L^2(\mathbf{R}^n)$*. Here,*

$$(3.10)\qquad E_\lambda(x,\hat{\xi}) = \exp\left\{-\lambda(|x|^2+|\hat{\xi}|^2)\right\}.$$

Proof: From the Moyal identity and Lemma (3.3), it follows that the LHS of (3.9) is equal to

$$\pi^{-n/2}\lambda^n \int W(f,g)(x,\lambda\xi)W\left(\rho(-p,q)\phi^\lambda,\rho(-p,q)\phi^\lambda\right)(x,\lambda\xi)dxd\xi$$

and it is seen that

$$(3.11)\qquad W\left(\rho(-p,q)\phi^\lambda,\rho(-p,q)\phi^\lambda\right)(x,\xi) = W\left(\phi^\lambda,\phi^\lambda\right)(x-p,\xi-q).$$

The last expression is equal to

$$(2\pi)^{n/2}2^{n/2}e^{-\lambda(x-p)^2-\frac{1}{\lambda}(\xi-q)^2}.$$

In fact, Lemma 3.1 implies

$$(3.12)\qquad W(e^{-x^2/2},e^{-x^2/2}) = \int e^{-i\xi\cdot p}e^{-\frac{1}{2}(x+\frac{p}{2})^2-\frac{1}{2}(x-\frac{p}{2})^2}dp$$
$$= \int e^{-i\xi\cdot p}e^{-x^2-\frac{1}{4}p^2}dp = 2^{n/2}(2\pi)^{n/2}e^{-x^2-\xi^2}.$$

Now, we are going to describe the interaction of the pseudodifferential operator with the Wigner transformation. By a direct calculation, we obtain

$$\begin{aligned}\partial_x W(u,v) &= W(u_x,v)+W(u,v_x)\\ &= 2W(u_x,v)-2i\xi W(u,v)\\ &= 2W(u,v_x)+2i\xi W(u,v),\end{aligned} \tag{3.13}$$

so that

$$\begin{aligned}W(\partial_x u,v) &= (\frac{1}{2}\partial_x+i\xi)W(u,v)\\ W(u,\partial_x v) &= (\frac{1}{2}\partial_x-i\xi)W(u,v)\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}W(xu,v) &= (x-i\frac{1}{2}\partial_\xi)W(u,v)\\ W(u,xv) &= (x+i\frac{1}{2}\partial_\xi)W(u,v).\end{aligned} \tag{3.15}$$

Therefore, we conclude that

$$\begin{aligned}W(i\partial_x^2 u,v)+W(u,i\partial_x^2 v) &= i\left\{(\frac{1}{2}\partial_x+i\xi)^2-(\frac{1}{2}\partial_x-i\xi)^2\right\}W(u,v)\\ &= -2\xi\partial_x W(u,v)\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}W(ix^2 u,v)+W(u,ix^2 v) &= i\left\{(x-i\frac{1}{2}\partial_\xi)^2-(x+i\frac{1}{2}\partial_\xi)^2\right\}W(u,v)\\ &= 2x\partial_\xi W(u,v).\end{aligned} \tag{3.17}$$

In order to develop the above calculation into more general case, we shall use the Weyl calculus. Let $a(x,\xi)$ and $b(x,\xi)$ be two symbols belonging to the standard symbol class $S^m_{1,0}(\mathbf{R}^n)$. Denote their corresponding pseudodifferential operators by $a^w(x,D)$ and $b^w(x,D)$:

$$a^w(x,D)u=\int e^{i(x-y)\xi}a(\frac{x+y}{2},\xi)u(y)dyd\xi/(2\pi)^n.$$

We denote the symbol of the product AB by $(a \circ b)(x,\xi)$ as usual. From a direct calculation, it follows that

$$(a^w(x,D)u,v)_{L^2} = (2\pi)^{-n}\int a(x,\xi)W(u,v)(x,\xi)dxd\xi.$$

We use the formula

$$(3.18)\quad (a\circ b)(x,\xi) = e^{i\{D_y\cdot D_\xi - D_x\cdot D_\eta\}/2}a(x,\xi)b(y,\eta)|_{y=x,\eta=\xi}$$
$$\sim ab + \sum_{j\geq 1}\frac{1}{j!}\{a,b\}_j(x,\xi),$$
$$\{a,b\}_j(x,\xi) = \left(\frac{-i}{2}(\partial y\cdot\partial_\xi - \partial_x\cdot\partial_\eta)^j\right)a(x,\xi)b(y,\eta)|_{y=x,\eta=\xi}.$$

It holds that

$$(3.19)\quad (a^w(x,D)b^w(x,D)u,v)$$
$$= \int a(x,\xi)W(b^w(x,D)u,v)(x,\xi)dxd\xi$$
$$= \int (a\circ b)(x,\xi)W(u,v)(x,\xi)dxd\xi$$
$$= \int\left\{a(x,\xi)b(x,\xi) - \frac{i}{2}\{a,b\} + \cdots\right\}W(u,v)(x,\xi)dxd\xi$$

modulo $S^{-\infty}(\mathbf{R}^n)$. Here,

$$\{a,b\} = \partial_\xi a(x,\xi)\partial_x b(x,\xi) - \partial_x a(x,\xi)\partial_\xi b(x,\xi).$$

In what follows , we will take

$$(3.20)\qquad a(x,\xi) = e_{\lambda,q,p}(x,\xi) = e^{-\lambda(x-q)^2-(\xi/\lambda-p)^2}.$$

We willl often drop the suffix q and p and use the notation $\mathcal{P}$ instead of $\mathcal{P}_\lambda$, in short. Let $b(x,\xi)$ belong to $S^0(g)$ for the metric $g = dx^2 + \frac{d\xi^2}{\lambda^2}$.

By use of an integration by parts, we obtain

$$(3.21)\quad (\chi(q,p)\mathcal{P}b(x,D)u,\mathcal{P}v) = \iint \chi(q,p)e_\lambda(x,\xi)b(x,\xi)W(u,v)(x,\xi)dxd\xi d\lambda dpdq$$
$$+\frac{i}{2}\iint \chi(q,p)e_\lambda(x,\xi)\{\partial_x b(x,\xi)\partial_\xi - \partial_\xi b(x,\xi)\partial_x\}W(u,v)(x,\xi) + O(\lambda^{-2}).$$

Here, we note that the terms corresponding to the second derivatives $\partial_x \partial_\xi b$ are cancelled out.

On the other hand, we obtain

$$(3.22)\quad (\mathcal{P}b(x,D)u, \mathcal{P}v)$$
$$= (b(q,\lambda p)\mathcal{P}u, \mathcal{P}v) + \frac{i}{2}\left(\{\partial_q b(q,\lambda p)\frac{\partial_p}{\lambda} - \frac{\partial_p}{\lambda}b(q,\lambda p)\partial_q\}\mathcal{P}u, \mathcal{P}v\right) + \cdots .$$

Here, the remainder terms contain the error terms like

$$\lambda(x-q)e_{\lambda,q,p}(x,\lambda\xi) = \partial_x e_{\lambda,q,p}(x,\lambda\xi) = -\partial_q e_{\lambda,q,p}(x,\lambda\xi).$$

For any real symbol b, we consider the evolution equation

$$\frac{1}{i}\partial_t u + b^w(x,D)u = 0.$$

Differentiating $W(u(t),u(t))$ in t, we have

$$(3.23)\quad \partial_t W(u(t),u(t)) = W(\partial u(t),u(t)) + W(u(t),\partial_t u(t))$$
$$= W(-ib^w u(t),u(t)) + W(u(t),-ib^w u(t))$$
$$= W(-ib^w u(t),u(t)) + \overline{W(-ib^w u(t),u(t))}$$
$$= -iW(b^w u(t),u(t)) + \overline{W(b^w u(t),u(t))}.$$

It follows that the Wigner distribution itself satisfies the transport equation modulo $(\lambda^{-\infty})$

$$(3.24)\qquad \{\partial_t + H_b + \cdots\}W(u,v) = 0, \quad H_b = \partial_\xi b\partial_x - \partial_x b\partial_\xi .$$

In fact, this is immediately verified for polynomial $b(x,\xi)$ without error terms because we can arbitrarily choose a test function $e = a \in \mathcal{S}(\mathbf{R}^n)$ in (3.21).

For the general case, it is helpful to use the Taylor expansion at $(q,\lambda p)$ up to any high order:

$$(3.25)\qquad b(x,\xi) = b_N(x,\xi;\ q,\lambda p) + R_N(x,\xi,q,\lambda p),$$

where

$$b_N(x,\xi;\ q,\lambda p) = \sum_{|\alpha|+|\beta|<N} \frac{(x-q)^\alpha(\xi-\lambda p)^\beta}{\alpha!\beta!}\partial_q^\alpha(\partial_p/\lambda)^\beta b(q,\lambda p) + R_N(x,\xi,q,\lambda p)$$

Since we can apply the formula (3.24) to $b_N(x,\xi;\ q,\lambda p)$, the hardest thing is to estimate the remainder terms to be $O(\lambda^{-N/2})$.

4 Schrödinger equations with quadratic Hamiltonian

We consider the following essentially self-adjoint operator on $C_0^\infty(\mathbf{R}^n)$:

$$H = h^w(x, D_x), \quad h(x,\xi) = \sum_{|\alpha+\beta|\le 2} a_{\alpha,\beta} x^\alpha \xi^\beta,$$

where $a_{\alpha,\beta}$ are any real constants. We consider the Cauchy problem:

$$\frac{1}{i}\partial_t u + h^w(x, D)u = 0, \qquad u(0,x) = u_0(x). \tag{4.1}$$

Let us denote its self-adjoint extension on $L^2(\mathbf{R}^n)$ by the same letter H. According to Stone's theorem or Hille-Yosida's theorem, the solution $e^{-itH}u_0$ to the equation (4.1) is continuous with valued in $L^2(\mathbf{R}^n)$. Moreover, we see that $e^{-itH}u_0 \in C^1(\mathbf{R}; D(H))$ if u_0 belongs to the domain $D(H)$ of H. However, we require more regularity of the solution. In fact, we need

Lemma 4.1 *If the initial function u_0 belongs to $\mathcal{S}(\mathbf{R}^n)$, the solution $v(t)$ to the Cauchy problem (4.1) belongs to $C^1(\mathbf{R};\ \mathcal{S}(\mathbf{R}^n))$.*

Proof: When $h = |\xi|^2/2$, it holds that

$$u(t,x) = (2\pi)^{-n} \int e^{ix\cdot\xi} e^{-it|\xi|^2/2} \hat{u}(\xi) d\xi.$$

This formula leads us to the conclusion of this Lemma in this simple case. Q.E.D.

Define the Hamilton flow

$$\phi^t(x,\xi) = (X(t,x,\xi), \Xi(t,x,\xi))$$

by the solution to the equation

$$\begin{cases} \dfrac{dX}{dt} = \partial_\xi h(X,\Xi) \\ \dfrac{d\Xi}{dt} = -\partial_x h(X,\Xi), \ X(0) = x, \ \Xi(0) = \xi. \end{cases} \tag{4.2}$$

We are going to see this phenomenon more precisely.

To make the situation to be clear, we define the life span of smoothing effects as follows: Let $H = h^w(x, D)$. For $u(t) = e^{-itH}u_0$

$$t_c = \inf\{t > 0, \quad u(t) \notin C^\infty(\mathbf{R}^n) \text{ with } \exists u_0(x) \in \mathcal{E}' \cap L^2(\mathbf{R}^n)\}.$$

Especially, we define $t_c = +\infty$ if the above set is empty. In what follows, $\dot{T}^*(\mathbf{R}^2)$ denotes $T^*(\mathbf{R}^2)\backslash\{0\}$. We consider the Schrödinger operator H_0 with magnetic vector potential:

$$H_0 = \frac{1}{2}\left\{-(\nabla - iAx)^2 + \langle Ex, x\rangle\right\},$$

where A is a constant real skew symmetric and E is a constant real symmetric matrices. Moreover, we assume that they commute each other. This means that essentially, we may assume that $n = 2$ and

$$A = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix} \quad \text{and} \quad E = \varepsilon Id,$$

where μ and ε stand for positive constants.

Theorem 4.2 *Let $H = H_0$. It holds that*

$$t_c = \begin{cases} \pi/\sqrt{\mu^2 + \varepsilon} & \text{if } \mu^2 + \varepsilon > 0 \\ +\infty & \text{if } \mu^2 + \varepsilon \leq 0. \end{cases}$$

Furthermore, if $\mu^2 + \varepsilon > 0$ and $u_0 \in \mathcal{E}' \cap L^2(\mathbf{R}^n)$, then it holds that

$$WFu(t) = \emptyset, \quad \forall t \notin t_c\mathbf{Z}. \tag{4.3}$$

and

$$WFu(t_c\ell) = \{(x, \xi) \in \dot{T}^*(\mathbf{R}^2); \quad ((-1)^\ell e^{At_c\ell}x, (-1)^\ell e^{At_c\ell}\xi) \in WFu_0\} \tag{4.4}$$

for any $\ell \in \mathbf{Z}$.

Corollary 4.3 *If $A = 0$ and $\varepsilon > 0$, then*

$$WFu(t_c) = \{(x, \xi) \in \dot{T}^*(\mathbf{R}^2); \quad (-x, -\xi) \in WFu_0\}.$$

If $E = 0$, then

$$WFu(t_c) = WFu_0.$$

Remark 4.1 *A, Jensen firstly proved the corresponding result for $A = E = 0$ in 1986. His method is completely different from ours. However, he did not give any microlocal results.*

We can generalize the above result as follows. We consider a self adjoint operator H on $L^2(\mathbf{R}^n)$ as a second order perturbation of H_0 such that the symbol of $\tilde{H} = H - H_0$

$$\sigma(\tilde{H}) = \sum_{|\alpha|\leq 2} a_\alpha(x)\xi^\alpha. \tag{4.5}$$

satisfies the following condition:

$$a_\alpha(x) \in S(\langle x\rangle^{\kappa-|\alpha|};\ \frac{dx^2}{\langle x\rangle^2}).$$

The conclusion (4.3) and (4.4) are valid for $u(t) = e^{-iHt}u_0$ if $\kappa < 2$ and $\kappa < 1$, respectively.

Moreover, we can characterize the wave front set of the solution at the midpoints of two adjacent critical times.

Theorem 4.4 *Let $\mu^2 + \varepsilon > 0$. Then,*

$$WFu(t_c(\ell+\frac{1}{2})) = \{(x,\xi) \in \dot{T}^*(\mathbf{R}^2);\quad ((-1)^\ell De^{At_c/2}x, (-1)^\ell D^{-1}e^{At_c/2}\xi) \in WF\hat{u}_0\}.$$

Here, $\hat{u}_0$ denotes the Fourier transform of u_0:

$$\hat{u}_0(\xi) = \int e^{-ix\xi}u_0(x)dx.$$

Now, we state creation of new singularities. Let us consider the special initial data

$$u_0(x) = e^{i(\Gamma x, x)} \in \mathcal{S}'(\mathbf{R}^2),$$

where Γ stands for the constant real symmetrix matrix. We denote its (real) two eigenvalues by γ_1 and γ_2, which may coincide.

Theorem 4.5 *If $\mu^2 + \varepsilon = 0$, then $u(t) = e^{-itH_0}u_0$ is smooth if and only if $2t \notin \{-\gamma_1^{-1},\ \gamma_2^{-1}\}$. Here, we define $\gamma^{-1} = +\infty$ if $\gamma = 0$.*

Theorem 4.6 *Assume that* $A = 0$ *and* $\varepsilon > 0$. *Then* $u(t) = e^{-itH_0}u_0$ *is smooth if and only if* $-2\varepsilon^{-1}\tan(\varepsilon t)$ *does not coincide with the inverse of any eigenvalues of* Γ.

These phenomena do not occur under the presence of some magnetic filed.

Theorem 4.7 *Assume that* $\mu^2 + \varepsilon > 0$. *Let* $D = \sqrt{\mu^2 + \varepsilon}$ *then* $u(t) = e^{-itH_0}u_0$ *is smooth if and only if* $-2D^{-1}\tan(Dt)$ *does not coincide with the inverse of any real eigenvalues of* $\Gamma - A$.

Corollary 4.8 *Assume that* $\Gamma - A$ *does not have any real eigenvalues. Then,* $u(t) = e^{-itH_0}u_0$ *is smooth at any* $t \in \mathbf{R}$.

We can show the similar results to Theorem 4.5–4.7 when $\mu^2+\varepsilon < 0$ by replacing tan by tanh.

5 Microlocal conservation law

The essential part for proving Theorem 4.2 is to show an inequality (or identity) along the Hamilton flow for the symmetrized wave packet of the solution, introduced in the previous section. It follows that the wave front sets of solutions at time t is completely determined by the initial datum and the behavior of the Hamilton flow.

Theorem 5.1 *Let* $\Phi(x)$ *be a real-valued function in* $\mathcal{S}(\mathbf{R}^n)$ *and* $u(t) = e^{-itH}u_0$. *Then, we have*

$$\begin{aligned}(5.1)\quad &\int W(u(t))(x,\lambda\hat{\xi})W(\Phi)\left(\lambda^{1/2}(x-q),\lambda^{1/2}(\hat{\xi}-p)\right)dxd\hat{\xi}\\ &=\int W(u(0))(\phi^{-t}(x,\lambda\hat{\xi}))W(\Phi)\left(\lambda^{1/2}(x-q),\lambda^{1/2}(\hat{\xi}-p)\right)dxd\hat{\xi}\quad \textit{for any } \lambda > 1.\end{aligned}$$

When Φ is equal to $\exp(-|x|^2/2)$, it follows that

$$W(\Phi)\left(\lambda^{1/2}(x-q),\lambda^{1/2}(\hat{\xi}-p)\right) = E_\lambda(x-q,\hat{\xi}-p).$$

To prove the identity (5.1), we need the following lemma.

Lemma 5.2 *Let $v \in C^1(\mathbf{R};\ \mathcal{S}(\mathbf{R}^n))$ be a solution to*

$$(\partial_t + ih^w(x, D))v = 0.$$

Then, it satisfies

$$\{\partial_t + H_h\}W(v(t))(x, \xi) = 0, \tag{5.2}$$

where H_h denotes the Hamilton vector of $h(x, \xi)$:

$$H_h = \partial_\xi h \partial_x - \partial_x h \partial_\xi.$$

Proof: We only consider the simplest case that $h(x, \xi) = |\xi|^2/2$ and Φ is the Gaussian function. Define

$$f(t, x, \xi) = W(v(t))(x, \xi). \tag{5.3}$$

By a simple calculation, we observe that

$$\begin{aligned}\partial_t f(t, x, \xi) &= \int e^{-iy\xi}\left\{v_t(x + \frac{1}{2}y)\overline{v}(x - \frac{1}{2}y) + v(x + \frac{1}{2}y)\overline{v}_t(x - \frac{1}{2}y)\right\} dy \\ &= \int e^{-iy\xi}\frac{i}{2}\Delta v(x + \frac{1}{2}y)\overline{v(x - \frac{1}{2}y)}dy + \int e^{-iy\xi}v(x + \frac{1}{2}y)\overline{\frac{i}{2}\Delta v(x - \frac{1}{2}y)}dy \\ &= \int e^{-iy\xi}\left\{2i\Delta_y v(x + \frac{1}{2}y)\overline{v}(x - \frac{1}{2}y) + v(x + \frac{1}{2}y)\overline{2i\Delta_y v}(x - \frac{1}{2}y)\right\} dy.\end{aligned}$$

An integration by parts with respect to y gives

$$\begin{aligned}&\partial_t f(t, x, \xi) \\ &= \int e^{-iy\xi}\left\{-2\xi \cdot \nabla_y v(x + \frac{1}{2}y)\overline{v}(x - \frac{1}{2}y) + v(x + \frac{1}{2}y)\overline{2\xi\nabla_y v}(x - \frac{1}{2}y)\right\} dy \\ &= \int e^{-iy\xi}\left\{-\xi \cdot \nabla_x v(x + \frac{1}{2}y)\overline{v}(x - \frac{1}{2}y) - v(x + \frac{1}{2}y)\overline{\xi\nabla_x v}(x - \frac{1}{2}y)\right\} dy.\end{aligned} \tag{5.4}$$

Notice that

$$ye^{-iy\xi} = i\nabla_\xi e^{-iy\xi}.$$

This leads us to the desired conclusion of Lemma 5.2.

We are going to prove Theorem 5.1. Since $h^w(x, D)$ is an essentially self-adjoint operator on $L^2(\mathbf{R}^n)$, we can approximate the solution u by the sequence of solution

$v_n \in C^1(\mathbf{R};\ \mathcal{S}(\mathbf{R}^n))$, $n = 1, 2, \ldots$. Therefore, it suffices to prove the identity (5.1) for the solution $v \in C^1(\mathbf{R};\ \mathcal{S}(\mathbf{R}^n))$. Lemma 5.2 implies

$$\partial_t W(v(t))(\phi^t(x,\xi)) = 0.$$

Hence,

$$W(u(t))(\phi^t(x,\lambda\hat{\xi})) = W(u(0))(x,\lambda\hat{\xi}) \text{ and } W(u(t))(x,\lambda\hat{\xi}) = W(u(0))(\phi^{-t}(x,\lambda\hat{\xi})).$$

Integrating the last identity against the Gaussian function $E_\lambda(x-q,\hat{\xi}-p)$ in t from 0 to t, we obtain the desired identity. This completes the proof of Theorem 5.1.

Remark 5.1 *For perturbed operators, we can no more expect any identity analogous to the one in Theorem 5.1. Fortunately, we can replace it by an inequality which assures the same conclusion of Theorem 4.2 when $A = 0$ and $V(x) \in S(\langle x\rangle^\kappa, dx^2/\langle x\rangle^2)$ with $\kappa < 2$ ([17]). The sublinear case (when $\kappa < 1$) can be much easily analyzed than the other case because the difference between the Hamilton flows corresponding to the unperturbed operator and the perturbed one is bounded.*

6 Outline of Proof of Theorems 4.2–4.7

Finally, we are going to see how the wave packet identity (5.1) leads us to our results.

The first step is the following result. For $\varepsilon > 0$, let us denote ε neighborhood of any set U of $\mathbf{R}^n$ by U_ε:

$$U_\varepsilon = \{x \in \mathbf{R}^n;\ |x-y| < \varepsilon,\ \exists y \in U\}.$$

Proposition 6.1 *Let K be any compact set of $\mathbf{R}^{2n}$ and ε an arbitrary positive number. Suppose that*

$$\lim_{\lambda\to\infty} \sup_{(q,p)\in K} |X(t,q,\lambda p)| = \infty$$

and $\phi^{-t}(x,\lambda\xi)$ *is holomorphic in* $\{|\mathrm{Im}\xi|<\sigma/\lambda\}$ *for some* $\sigma>0$. *Then, There exists a positive number* δ *such that for any* $u\in L^2(\mathbf{R}^n)$, $\exists C_u>0$,

$$\sup_{(q,p)\in K}\int W(u)(\phi^{-t}(x,\lambda\xi))E_\lambda(x-q,\xi-p)dxd\xi \tag{6.1}$$
$$\leq C_u\lambda^n\sup_{(x,\xi)\in K_\varepsilon}\sup_{|y|<\varepsilon\lambda}|u(y+\phi^{-t}(x,\lambda\xi))|^2+\mathcal{O}(e^{-\delta\lambda}),\qquad \textit{as } \lambda\to\infty\quad .$$

Proof: For simplicity, we shall treat only the simplest case $A=E=0$. We consider the region $S_r=\{z\in\mathbf{R}^n;\ |z|\geq r\lambda\}$ with $r>0$. Then, there exist a finite open covering $\{S_{r,j}\}_{j=1}^N$ of S_r such that each $S_{r,j}$ is a conic subset such that either $\inf_{z\in S_{r,j}}y\cdot e_{k_j}>0$ or $\sup_{z\in S_{r,j}}y\cdot e_{k_j}<0$ for some $1\leq k_j\leq n$. Here, e_j is the j-th canonical base of $\mathbf{R}^n$: Its j-th component is equal to one and the others are zero. Then, the integral

$$\int e_{\lambda,(q,p)}(\phi^t(y,\lambda\eta))W(u,v)(y,\lambda\eta)d\eta \tag{6.2}$$
$$=\iint e_{\lambda,(q,p)}(\phi^t(y,\lambda\eta))e^{-iy\lambda\eta}u(y+z/2)\overline{u}(y-z/2)dzd\eta$$

is equal to the finite sum of the similar integrals I_j over the conic set $S_{r,j}$. We are going to substitute each integral I_j by a new one taken over a path

$$\Gamma_j=\{\zeta=\eta+\varepsilon_j ie_{k_j}\lambda^{-1}\tau;\ 0\leq\tau<\tau_0\}$$

in the space $\mathbf{C}^n$. Here, $\varepsilon_j=1$ or -1 and it is determined such that

$$\sup_{y\in S_{r,j}\cap S^{n-1}}\varepsilon_j y\cdot e_{k_j}<0,$$

which we denote by δ_j. On Γ_j, the integrand of I_j becomes

$$e^{e_jy\tau}\exp\{-\lambda(\mathrm{Re}X(-t,y,\lambda\zeta)-q)^2+\lambda|\mathrm{Im}X(-t,y,\lambda\zeta)|^2\} \tag{6.3}$$
$$\times\exp\{-\lambda(\lambda^{-1}\mathrm{Re}\Xi(-t,y,\lambda\zeta)-p)^2+\lambda|\lambda^{-1}\mathrm{Im}\Xi(-t,y,\lambda\zeta)|^2\}$$
$$\times u(x+z/2)\overline{u}(x-z/2)e^{i\varepsilon_jy\lambda\eta}\exp\{2i(\mathrm{Re}X(-t,y,\lambda\zeta)-q)(\mathrm{Im}X(-t,y,\lambda\zeta))\}$$
$$\times\exp\{2i(\lambda^{-1}\mathrm{Re}\Xi(-t,y,\lambda\zeta)-p)(\lambda^{-1}\mathrm{Im}\Xi(-t,y,\lambda\zeta))\}.$$

If $\tau_0>0$ is taken to be small enough, we see that

$$\sup_{y\in S_{r,j},\ |y|\geq\delta_jr\lambda}e^{y\tau_0}e^{\lambda|\mathrm{Im}X(-t,y,\lambda\zeta)|^2}e^{\lambda^{-1}|\mathrm{Im}\Xi(-t,y,\lambda\zeta)|^2}=O(e^{-\delta_jr\lambda}). \tag{6.4}$$

Indeed, the absolute value of the left hand side of (6.4) is less than

$$\exp\{-\delta_j r\tau_0\lambda\}\exp\{\tau_0^2\lambda\}\exp\{\tau_0^2\lambda^{-1}\}\leq C\exp\{-(\delta_j r\tau_0-\tau_0^2)\lambda\}, \tag{6.5}$$

where C is a positive constant independent of τ_0 and λ.

Now, it remains to check the validity of this deformation. It is easy to see that the left hand side of the expression (6.1) decreases of order exponentially as $\lambda \to \infty$ whenever the distance between (x, ξ) and (q, p) are greater than a positive constant c_0. This indicates that it suffices to consider the case where (x, ξ) is in a compact set K' of $\mathbf{R}^n \times (\mathbf{R}^n \backslash \{0\})$ since (q, p) runs over the compact set K. Therefore, we may assume that $(y, \lambda\eta)$ in the expression of (6.1) belongs to the image of K' under the map $\phi^t(x, \lambda\xi)$: Namely,

$$y \in \{x + t\lambda\xi;\ (x, \xi) \in K'\} \text{ and } \eta \in \{\lambda\xi;\ (x, \xi) \in K'\}. \tag{6.6}$$

From the Riemann-Lebesgue Theorem, it follows that for any compact K of $\mathbf{R}^n \times (\mathbf{R}^n \backslash \{0\})$,

$$\lim_{\eta\to\pm\infty}\sup_{\tilde{\phi}^t(y,\lambda\eta)\in K}\int_{\eta}^{\eta+i\sigma} d\zeta \int_{|z|\geq\varepsilon\lambda} e^{-iy\lambda\zeta} e_{\lambda,(q,p)}(\phi^t(y, \lambda\zeta))u(y + z/2)\overline{u}(y - z/2)dzd\zeta = 0$$

for each $\lambda > 1$.

Since the same reasoning as for deriving (6.5) implies that the integrand of the above integral, $e^{-iy\lambda\zeta}e_{\lambda,(q,p)}(\phi^t(y, \lambda\zeta))$, is a bounded function in (y, ζ), by the dominant convergence theorem, we can deform the path of the integral (6.13) into the following way.

$$\begin{aligned}\int_{\mathbf{R}^n} dy \int_{\mathbf{R}^n} d\eta \int_{|z|\geq\varepsilon\lambda^{-1+\kappa}} K(y, z, \eta, \lambda)dz \\ = \sum_{j=1}^{n}\int_{\mathbf{R}^n} dy \int_{\mathbf{R}^n+i\sigma\lambda^{1-\delta}e_j} d\zeta \int_{z\in S_{r,j}} K(y, z, \eta, \lambda)dz,\end{aligned} \tag{6.7}$$

where $K(y, z, \eta, \lambda)$ denotes the integrand of (6.13). Q.E.D.

When the Hamilton flow has no holomorphic extension in ξ, we can show the analogous result to (6.1) by use of integration by parts. Indeed,

Proposition 6.2 *Let K be a compact set of $\dot{T}^*(\mathbf{R}^n)$. Suppose that*

$$\lim_{\lambda\to\infty} \sup_{(q,p)\in K} |X(t,q,\lambda p)| = \infty.$$

Then, there exists a compact set $\tilde{K} \supset K$ such that

$$\text{(6.8)} \quad \sup_{(q,p)\in K} \int W(u(0))(\phi^{-t}(x,\xi))E_\lambda(x-q,\xi-\lambda p)dxd\xi$$
$$\le C_N\lambda^n \sup_{(q,p)\in\tilde{K}} \sup_{2|y|<\phi^{-t}(q,\lambda p)} |u(y+\phi^{-t}(q,\lambda p))|^2 + O(\lambda^{-N})$$

for any $N \in \mathbf{N}$ and $\lambda > 1$.

Proof: The formula given in the next section tells us that

$$X(t) = e^{-tA}\{T(t)x + S(t)\xi\}, \quad \Xi(t) = e^{-tA}\{T'(t)x + S'(t)\xi\}$$

and that

$$\begin{pmatrix} T(t) & S(t) \\ T'(t) & S'(t) \end{pmatrix}^{-1} = \begin{pmatrix} S'(t) & -S(t) \\ -T'(t) & T(t) \end{pmatrix}$$

if $\mu^2 + \varepsilon \ne 0$. The left hand side of (6.1) is estimated by

$$\int \chi_{\hat{q},\hat{p},r}(q,p)W(u_0)\left(X(-t,x,\lambda\xi), \Xi(-t,x,\lambda\xi)/\lambda\right)E_\lambda(x-q,\xi-p)dxd\xi dpdq$$

for any nonnegative function $\chi_{\hat{q},\hat{p},r}(q,p) \in C_0^\infty(\mathbf{R}^{2n})$ such that

$$\chi_{\hat{q},\hat{p},r}(q,p) = 1 \quad \text{on } K.$$

By a change of variables, the last expression is equal to

$$\text{(6.9)} \quad I = \int \chi_{\hat{q},\hat{p},r}(q,p)W(u_0)(x,\lambda\xi)\exp[-\lambda\{(X(t)x-q)^2 + (\lambda^{-1}\Xi(t)-p)^2\}]$$
$$\times Q_\lambda(t,x,\xi,p,q)W(u_0)(x,\lambda\xi)dxd\xi dpdq.$$

Here,

$$\text{(6.10)} \quad Q_\lambda(t,x,\xi,p,q) = \chi_{\hat{q},\hat{p},r}\left(e^{-tA}q, e^{-tA}p\right)$$
$$\times \exp[-\lambda\{(T(t)x + S(t)\lambda\xi - q)^2 + (\lambda^{-1}T'(t)x + S'(t)\xi - p)^2\}].$$

We have used the fact that e^{-tA} is an orthogonal matrix. The function Q_λ is exponentially decaying in λ outside of

$$\{T(t)x + S(t)\lambda\xi - q = 0, \quad \lambda^{-1}T'(t)x + S'(t)\xi - p = 0\}.$$

Thus, it suffices to look into the integrand of I on the following set:

$$\Sigma = \left\{(x,\xi,q,p);\ (T(t)x + S(t)\lambda\xi - q)^2 + (\lambda^{-1}T'(t)x + S'(t)\xi - p)^2 \geq \delta > 0\right\}.$$

We split the Wigner transform into two pieces:

$$W(u_0)(x,\lambda\xi) = \int e^{-i\lambda y\cdot\xi} u_0(x+y/2)\overline{u}_0(x-y/2)dy \tag{6.11}$$

$$= \int_{|x|\geq|y|}\{\cdots\}dy + \int_{|x|\leq|y|}\{\cdots\}dy$$

$$= W_1 + W_2. \tag{6.12}$$

We consider the two integrals $\int Q_\lambda W_j$, $j = 1,2$, which are denoted by I_1 and I_2, respectively. Hence, $I = I_1 + I_2$.

We are going to estimate I_1. Since $|x| \geq |y|$ implies $|x \pm y/2| \geq |x|/2$ and $\langle x\rangle^N u(x)$ is bounded for any $N > 0$, it is easily verified that

$$|W_1(x,\xi)| \leq C|x|^n \sup_{|y|\leq|x|} |u_0(x+y)|^2.$$

From now on, we assume that $S(t)$ is nonsingular. For any $\delta > 0$ and a compact set of $\mathbf{R}^{2n}\backslash\{0\}$, K_δ stands for the δ neighborhood of K. It holds that

$$|x| \geq C\lambda|\hat{p}|.$$

From this, it follows that

$$|I_1| \leq C\lambda^n \sup_{(q,p)\in K_\varepsilon} \sup_{2|y|<\phi^{-t}(q,\lambda p)} |u(y+\phi^{-t}(q,\lambda p))|^2. \tag{6.13}$$

Now, we are going to estimate concerning I_2. When $|x| \leq |y|$, we use the identity

$$\langle\lambda y\rangle^{-2N}(1-\Delta_\xi)^N e^{-i\lambda y\cdot\xi} = e^{-i\lambda y\cdot\xi}$$

and an integration by parts in ξ to obtain

$$(6.14)\quad \int W_2 Q_\lambda(t,x,\xi,q,p)dxd\xi dpdq$$
$$= \int W(u_0)(x,\xi)\int \langle\lambda x\rangle^{-2N}(1-\triangle_\xi)^N Q_\lambda(t,x,\xi,q,p)dpdqdxd\xi.$$

According to Leibniz's rule and the identity

$$(6.15)\quad \partial_\xi \exp[-\lambda\{(T(t)x+S(t)\lambda\xi-q)^2+(\lambda^{-1}T'(t)x+S'(t)\xi-p)^2\}]$$
$$=\{-\lambda S(t)\partial_q - S'(t)\partial_p \exp[-\lambda\{(T(t)x+S(t)\lambda\xi-q)^2+(\lambda^{-1}T'(t)x+S'(t)\xi-p)^2,\}]$$

an integration by parts in (q,p) gives

$$|\int \langle\lambda x\rangle^{-2N}(1-\triangle_\xi)^N Q_\lambda(t,x,\xi,q,p)dpdq| \le C\int \langle x\rangle^{-2N}|Q_\lambda(t,x,\xi,q,p)|dpdq.$$

Using the same reasoning as in the previous case, it holds that $|x|\ge C\lambda|\hat{p}|$ on Σ. Therefore, we can conclude for any $N>0$,

$$I_2 = \mathcal{O}(\lambda^{-N}) \text{ as } \lambda\to\infty.$$

provided that $S(t)$ is nonsingular. Q.E.D.

The following is a direct consequence of Proposition 6.1 or 6.2.

Corollary 6.3 *Let K be a compact set of $\dot{T}^*(\mathbf{R}^n)$. Suppose that*

$$\lim_{\lambda\to\infty}\sup_{(q,p)\in K}|X(t,q,\lambda p)|=\infty.$$

If $u(x)\in L^2(\mathbf{R}^n)$ satisfies

$$\forall N\in\mathbf{N},\ \exists C>0\ \ /\ \ |u(x)|\le C(1+|x|)^{-N}\quad \textit{for all } x\in\mathbf{R}^n,$$

then, for every compact set of $\mathbf{R}^n\times\mathbf{R}^n\backslash\{0\}$,

$$\int W(u)(\phi^{-t}(x,\lambda\xi))E_\lambda(x-q,\xi-p)dxd\xi = O(\lambda^{-N}),\quad \forall N>1.$$

As for the proof of reconstruction of singularities (4.4), we have to use the same idea as in the proof of Theorem 2.2 for the general case. We also omit this for saving pages. However, we note that if $A=0$, the proof is almost trivial.

For proving Theorems 4.5–4.7, we shall use the following useful relation.

Lemma 6.4 *Let Γ be a real symmetric matrix. Then, it holds that*

$$W(e^{i(\Gamma x, x)}u_0) = W(u_0)(x, \xi - 2\Gamma x).$$

Here, $W(u)$ denotes $W(u,u)$.

Let

$$B(t) = \cos(Dt) + D^{-1}2(\Gamma - A)\sin(Dt).$$

We observe that

$$X(t, x, 2\Gamma x) = e^{-tA}B(t)x, \quad \Xi(t, x, 2\Gamma x) = e^{-tA}B'(t)x.$$

Therefore, Theorems 4.5–4.7 follows from the next result.

Lemma 6.5 *Let $u(t) = e^{-iH_0 t}e^{i(\Gamma x, x)}$ and t be fixed. Then, $u(t)$ belongs to $C^\infty(\mathbf{R}^2)$ if and only if the matrix $B(t)$ has no zero eigenvalue. Furthermore, if $B(t)$ has zero eigenvalue, then*

$$WFu(t) = \{(0, p);\ p \in \mathbf{R}^2 \backslash \{0\}, \quad p = B'(t)q,\ q \in \ker B(t) \backslash \{0\}\}.$$

Proof: If the dimension of the subspace

$$H = \{x \in \mathbf{R}^2;\ B(t)x = 0\}$$

is positive, then one can find a couple of nonzero vector q and p such that

$$X(t, \lambda q, 2\Gamma\lambda q) = 0$$

and

$$\Xi(t, \lambda q, 2\Gamma\lambda q)/\lambda = p.$$

These relation imply that

$$(6.16) \quad |\mathcal{P}_\lambda u(t)|^2(0,p) = \int E_\lambda(X(t, x, 2\Gamma x), \Xi(t, x, 2\Gamma x) - p)dx$$
$$= \int E_\lambda(B(t)(x - \lambda q), \lambda^{-1}B'(t)(x - \lambda q))dx.$$

Since

$$|B(t)x|^2 + |B'(t)x|^2 \geq \delta|x|^2, \quad \delta > 0,$$

we can show that the integral (6.16) does not decay rapidly.

If the contrary case holds, we can easily show that (6.16) is rapidly dereasing. Q.E.D.

7 Hamilton flows

As described in the previous section, the behavior of the Hamilton flow is important. In this section, we shall give the explicit formula for the solution.

When $\mu^2+\varepsilon>0$, let $D=\sqrt{E+A^2}$, which is a positive scalar matrix. It follows that

Lemma 7.1 *Assume that $\mu^2+\varepsilon>0$. Then, the unique solution to (4.2) is given by*

$$X(t)=e^{-At}\left\{\left(\cos(Dt)-2AD^{-1}\sin(Dt)\right)x+D^{-1}\sin(Dt)\xi\right\} \tag{7.1}$$

and

$$\Xi(t)=e^{-At}\left\{(-D\sin(Dt)-2A\cos(Dt))\,x+\cos(Dt)\xi\right\}. \tag{7.2}$$

Here,

$$\cos B=\frac{e^{Bi}+e^{-Bi}}{2},\quad \sin B=\frac{e^{Bi}-e^{-Bi}}{2i}.$$

Lemma 7.2 *Let $\mu^2+\varepsilon=0$. Then, the unique solution to (4.2) is given by*

$$X(t)=e^{-At}(x+t\xi),\quad \Xi(t)=e^{-At}\xi. \tag{7.3}$$

Lemma 7.3 *Assume that $\mu^2+\varepsilon<0$. Let $\sqrt{-(\mu^2+\varepsilon)}=J$. Then, the unique solution to (4.2) is given by*

$$X(t)=e^{-At}\left\{\left(\cosh(Jt)-2AJ^{-1}\sinh(Jt)\right)x+J^{-1}\sinh(Jt)\xi\right\} \tag{7.4}$$

and

$$\Xi(t)=e^{-At}\left\{(J\sinh(Jt)-2A\cosh(Jt))\,x+\cosh(Jt)\xi\right\}. \tag{7.5}$$

Let $\tilde{\phi}^t(x,\xi)$ be the Hamilton flow corresponding to the perturbed Hamiltonian (4.5). Then, it holds that

Lemma 7.4 *Let K be any compact set of $\dot{T}^*(\mathbf{R}^{2n})$. If $\kappa<1$, then*

$$\lim_{\lambda\to\infty}\sup_{(x,\xi/\lambda)\in K}|\tilde{\phi}^t(x,\xi)-\phi^t(x,\xi)|=0.$$

If $1\le\kappa<2$, then

$$\sup_{(x,\xi/\lambda)\in K}|\tilde{\phi}^t(x,\xi)-\phi^t(x,\xi)|=\mathcal{O}(\lambda^{\kappa-1}),\qquad \text{as } \lambda\to\infty.$$

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