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ENTROPY FOR EXACT C^*-DYNAMICAL SYSTEM

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1. 前置き

エルゴード理論における自己同型写像にたいする二つのエントロピーの概念は、非可換数学の枠組みの中で、作用素環の理論に持ち込まれた。

その一つは、力学的エントロピーで、最初 Connes-Stϕrmer([CS]) により、有限型 von Neumann 環 M 上の有限トレース τ と $\tau\alpha=\tau$ を満たす自己同型写像 α に対して Kolmogorov-Sinai 不変量の拡張として、導入された。このエントロピーを $H(\alpha)$ で記す。その後、このエントロピーの拡張した概念として、有限トレース τ を α-不変な状態 ϕ に置き換える事によって、いわゆる CNT-エントロピー $h_\phi(\alpha)$ が Connes-Narnhofer-Thirring([CNT]) によって C^*-環 A の自己同型写像 α に対して定義された。

他の一つは、位相的エントロピーの概念である。これは、それぞれ独立に、Hudetz, Tompsen, Voiculescu により、ある種の C^*-環 A 上の自己同型写像 α に対して状態には関係しない不変量として定義された。Nuclear な C^*-環 A にたいして定義された Voiculescu ([V2]) の位相的エントロピー $ht(\alpha)$ の概念は、その後 Exact な C^*-環にたいして Brown([B]) により拡張された。

Voiculescu の位相的エントロピー $ht(\alpha)$ を少し変形する事により、[Ch2] に於いては、Nuclear な C^*-環 A の状態 ϕ を保つ自己同型写像 α にたいして力学的エントロピー $ht_\phi(\alpha)$ を定義した。

ここでは、この概念を A が Exact な場合にまで拡張し、その基本的性質、及び、CNT-エントロピや位相的エントロピーとの関係についての結果について記す。更に、

環の構造と密接に関係する自己同形写像 α に対するこのエントロピー $ht_\phi(\alpha)$ と、環の構造の間にどのような関係が成り立っているのかという事柄に関する結果についても、報告する。

2. 定義 と基本的性質

2.1. Definition. Given a unital separable C^*-algebra A, a state ϕ of A, and a ϕ-preserving automorphism α of A, let π be a faithful *-representation of A on a Hilbert space H, and let $\xi \in H$ be a a cyclic unit vector for $\pi(A)$ such that $\phi(a) =< \pi(a)\xi, \xi >$. Let

$$CPA(A, B(H)) = \{(\varrho, \eta, C) : C\text{is a finite dimensional } C^* - \text{algebra and}$$
$$\varrho : A \to C, \eta : C \to B(H) \text{ are unital completely positive maps}\}.$$

Given $(\varrho, \eta, C) \in CPA(A, B(H))$, C has the state $\omega_\xi \circ \eta$ of C :

$$\omega_\xi \circ \eta(c) =< \eta(c)\xi, \xi > \quad \text{for all} \quad c \in C.$$

The von-Neumann entropy of the state $\omega_\xi \circ \eta$ is denoted by $S(\omega_\xi \circ \eta)$. For a finite subset $\omega \subset A$, and a $\delta > 0$, put

$$scp_\phi(\pi, \omega, \delta) = \inf\{S(\omega_\xi \circ \eta) : (\varrho, \eta, C) \in CPA(A, B(H))$$
$$\text{and } \|\eta \circ \varrho(a) - \pi(a)\| < \delta\|a\|, \text{ for all } a \in \omega\}.$$

The $scp_\phi(\pi, \omega, \delta)$ is defined to be ∞ if no such approximation exists.

The value $scp_\phi(\pi, \omega; \delta)$ does not depend on the choice of the representation $\pi : A \to B(H)$ by the following Lemma 2.1.1 so that we denote $scp_\phi(\pi, \omega; \delta)$ simply by $scp_\phi(\omega; \delta)$.

2.1.1. Lemma. *If $\pi_i : A \to B(H_i)$ is a *-representation for $i = 1,2$, and if $\xi_i \in H_i$ is a cyclic vector for $\pi_i(A)$ such that $\phi(a) =< \pi_i(a)\xi_i, \xi_i >$ for $i = 1,2$, then $scp_\phi(\pi_1, \omega; \delta) = scp_\phi(\pi_2, \omega; \delta)$.*

For a unital ϕ-preserving automorphism α of A, put

$$ht_\phi(\alpha, \omega\ ; \delta) = \overline{\lim_{\mathrm{N}\to\infty}} \frac{1}{\mathrm{N}} scp_\phi(\omega \cup \alpha(\omega) \cup \cdots \cup \alpha^{\mathrm{N}-1}(\omega); \delta)$$

and

$$ht_\phi(\alpha, \omega) = \sup_{\delta > 0} ht_\phi(\alpha, \omega\ ; \delta).$$

Then the entropy $ht_\phi(\alpha)$ of α is defined by

$$ht_\phi(\alpha) = \sup_{\omega \in \Omega} ht_\phi(\alpha, \omega),$$

where Ω is the set of all finite subsets of A.

2.1.2. Remark. A unital C^*-algebra A is exact if and only if for some C^*-algebra B there exists an embedding $\iota : A \to B$ which is nuclear, that is, for arbitrary $\epsilon > 0$ and for every finite set $\omega \subset A$ there exist a finite dimensional C^*-algebra C and unital completely positive maps $\rho : A \to C$ and $\eta : C \to B$ such that $||\iota(a) - \eta \circ \rho(a)|| < \epsilon$ for all $a \in \omega$. ([K2 : Theorem 4.1], [W]).

We remark that if A is exact, then the GNS-representatation $\pi_\phi : A \to B(H_\phi)$ of ϕ is nuclear map (so that the approximation approach for $scp_\phi(\omega; \delta)$ is reasonable).

Let C be a finite dimensional C^*-algebra. The rank of C is the dimension of a maximal abelian C^*-subalgebra of C and it is denoted by $rank(C)$. We denote by $M_{rank(C)}$ the matrix algebra which has the same rank and the same diagonal algebra with C.

2.2. Lemma. *Assume that C is a finite dimensional C^*-algebra and that $\eta : C \to B(H)$ is a unital completely positive map. Then there exists a unital completely positive map $\bar{\eta} : M_{rank(C)} \to B(H)$ such that*

$$\eta(a) = \bar{\eta}(a), \quad (a \in C) \quad \textit{and} \quad S(\psi \circ \eta) = S(\psi \circ \bar{\eta}),$$

for all state ψ of $B(H)$.

Remark. As a a consequence of Lemma 2.2, we may treat only the triplet $(\rho, \eta, C) \in CPA(A, B(H))$ such that C is some matrix algebra $M_n(\mathbb{C})$ in the definition of $scp_\phi(\omega; \delta)$.

2.3. A similar entropy (which we denote for a little while by $ht'_\phi(\alpha)$) as $ht_\phi(\alpha)$ was defined for an automorphism α on a nuclear C^*-algebra A preserving a state ϕ of A in [Ch2]. The definition was given by replacing the definition of $scp_\phi(\omega; \delta)$ to the $scp'_\phi(\omega; \delta)$ defined as follows : Let $CPA(A)$ the triplet (ϱ, η, C), where C is a finite dimensional C^*-algebra, and $\varrho : A \to C$ and $\eta : C \to A$ are unital completely positive maps. For a finite suset ω of A and a $\delta > 0$, let

$$scp'_\phi(\omega; \delta) = \inf\{S(\phi \circ \eta) : (\varrho, \eta, C) \in CPA(A), \|\eta \circ \varrho(a) - a\| < \delta, a \in \omega\}$$

then $ht'_\phi(\alpha, \omega\ ; \delta), ht'_\phi(\alpha, \omega)$ and $ht'_\phi(\alpha)$ are defined by the same formula as $ht_\phi(\alpha)$.

2.3. Proposition. *If A is nuclear and ϕ is a state of A whose GNS-representation is faithful, then*

$$ht'_\phi(\alpha) = ht_\phi(\alpha)$$

for every automorphism α of A with $\phi \circ \alpha = \phi$.

2.4. Proposition. *Let (A, α, ϕ) be a C^*-dynamical system, where A is exact and ϕ has faithful GNS-representation.*

(1) The monotonicity : If $B \subset A$ is a C^*-subalgebra with the same unit with A and $\alpha(B) = B$, then $ht_\phi(\alpha|_B) \leq ht_\phi(\alpha)$.

(2) $ht_\phi(\alpha^k) = |k| ht_\phi(\alpha)$ for all $k \in \mathbb{Z}$.

(3) The covariance property : $ht_\phi(\alpha) = ht_{\phi\circ\sigma}(\sigma^{-1} \circ \alpha \circ \sigma)$ for all $\sigma \in Aut(A)$.

2.5. Proposition (Kolmogorov-Sinai Property). *If $(\omega_\iota)_{\iota\in I}$ is a net of finte subsets of A such that the linear span of $\bigcup_{\iota\in I}\bigcup_{n\in\mathbb{Z}} \alpha^n(\omega_\iota)$ is dense in A, then*

$$ht_\phi(\alpha) = \sup_\iota ht_\phi(\alpha, \omega_\iota).$$

2.6. Proposition. *Given C^*-dynamical systems (A_i, α_i, ϕ_i), where A_i is exact and ϕ_i has the faithful GNS-representation for $i = 1, 2$, we have*

$$\max\{ht_{\phi_1}(\alpha_1), ht_{\phi_2}(\alpha_2)\} \leq ht_{\phi_1\otimes\phi_2}(\alpha_1 \otimes \alpha_2) \leq ht_{\phi_1}(\alpha_1) + ht_{\phi_2}(\alpha_2).$$

2.7. Relations among Other Entropies.

The relation between the Connes-Narnhofer-Thirring dynamical entropy $h_\phi(\alpha)$ and the Brown-Voiculescu topological entropy $ht(\alpha)$ was obtained by Dykema [Dy]. We give here more presice relation.

2.7.1. Proposition. *Let (A, α, ϕ) be a C^*-dynamical systems such that A is exact and ϕ has the faithful GNS-representation. Then*

$$h_\phi(\alpha) \leq ht_\phi(\alpha) \leq ht(\alpha).$$

To prove that $h_\phi(\alpha) \leq ht_\phi(\alpha)$, we review the definition of the CNT-entropy $h_\phi(\alpha)$. Let $\gamma : M_k(\mathbb{C}) \to A$ be a unital completely positive map, where $M_k(\mathbb{C})$ is the $k \times k$ matrices.

An abelian model $\mathcal{A} = (B, P, \mu, B_1, B_2, \cdots, B_n)$ for $(A, \phi, (\alpha^i \circ \gamma)_{i=0}^{n-1})$ consists of an abelian finite dimensional C^*-algebra B, a unital completely positive map

$P : A \to B$, a state μ such that $\mu \circ P = \phi$ and *-subalgebras $B_1, B_2, \cdots B_n$ of B which contain the same identity of B. Put $P_j = E_j \circ P \circ \alpha^j \circ \gamma$ and

$$s_\mu(P_j) = S(\mu|_{B_j}) - \sum_{i=1}^{m_j} \mu(p_i^{(j)}) S(\phi \circ \gamma | \phi_i^{(j)}).$$

Here m_j is the dimension of B_j, $\{p_i^{(j)}; i = 1, \cdots m_j\}$ is the minimal projections of B_j generating B_j, and $\{\phi_i^{(j)}; i = 1, \cdots, m_j\}$ is states of $M_k(\mathbb{C})$ obtained by the method that

$$P_j(x) = E_j \circ P \circ \alpha^{j-1} \circ \gamma(x) = \sum_{i=1}^{m_j} \phi_i^{(j)}(x) p_i^{(j)}, \quad (x \in M_k(\mathbb{C}))$$

for the μ - conditional expectation $E_j : B \to B_j$. The entropy $H(\mathcal{A})$ of such an abelian model $\mathcal{A}$ is defined by

$$H(\mathcal{A}) = S(\mu|_{\vee_{j=1}^n B_j}) - \sum_{j=1}^{n} s_\mu(P_j).$$

Here we need the following 2.7.2 and 2.7.3.

2.7.2. Remark. The value $H(\mathcal{A})$ does not change when we replace B and $\{B_j\}_{j=1,\cdots,n}$ by Be and $\{B_j e\}_{j=1,\cdots,n}$ respectively for the support projection e of μ. Hence we may assume that the state μ in $\mathcal{A}$ is faithful.

Letting $H_\phi((\alpha^j \circ \gamma)_{j=0}^{n-1}) = \sup_{\mathcal{A}} H(\mathcal{A})$ and $h_{\phi,\alpha}(\gamma) = \lim_{n\to\infty} \frac{1}{n} H_\phi((\alpha^j \circ \gamma)_{j=0}^{n-1})$, the $h_\phi(\alpha)$ is defined by the supremum over all posible γ's of the values $h_{\phi,\alpha}(\gamma)$.

2.7.3. Lemma. *Let A be a C^*-subalgebra of $B(H)$ containing the identity operator, and let $\xi \in H$ be a cyclic vector for A. Let B be a finite dimensional abelian C^*-algebra, and let μ be a faithful state of B. Then every completely positive linear map $P : A \to B$ with $\mu \circ P = \omega_\xi$ has a completely positive extension $P' : B(H) \to B$ with $\mu \circ P' = \omega_\xi$.*

Remark. Remark that the three entropies $h_\phi(\alpha), ht_\phi(\alpha), ht(\alpha)$ are different by examples given in [Ch2 : Examples 2.6.4, 2.6.5].

3. Crossed Products

In this section, we estimate the entropy for some automorphisms on the C^*-crossed productd of an exact C^*-algebra A by a discrete countable amenable group G, and we need some results of this section in the next section. Remark that if A is exact and G is amenable then the crossed product is exact by Kirchberg [K1].

The statements in this section are analogous of [BC : Lemma 3.1] and [Ch3 : Cor. 3.4, 3.5]. In the latter, we obtained an estimate of the topological entropy for an automorphism on the crossed product by using the entropic invariant $h(\theta)$ in [Ch3] for an automorphism θ of an amenable discrete group G. Here we discuss on our dynamical entropy by using the entropy $ha(\theta)$ by Brown and Germain [BG] :

Let $K \subset G$ be a finite subset, and for an $\delta > 0$ we denote by $\mathcal{F}(K, \delta)$ the set of functions f on G such that $f(g) \geq 0, (g \in G)$, $|supp(f)| < \infty$, $||f||_1 = 1$ and $\sum_g |f(h^{-1}g) - f(g)| < \delta, (h \in K)$, where $supp(f)$ is the support of f and $|K|$ means the cardinality of K. Let $ra(K, \delta) = \inf\{|supp(f)| : f \in \mathcal{F}(K, \delta)\}$. A function f on G is minimal for (K, δ) if $f \in \mathcal{F}(K, \delta)$ and if $|supp(f)| = ra(K, \delta)$. Let

$$ha(\theta, K, \delta) = \limsup_{n\to\infty} \frac{1}{n} \log(ra(\bigcup_{i=0}^{n-1} \theta^i(K), \delta)),$$

$ha(\theta, K) = \sup_{\delta>0} ha(\theta, K, \delta)$. Then the entropy $ha(\theta)$ is defined as the supremum of $ha(\theta, K)$ over all finite subset K of G. To fix our notations, we review the deinition of the reduced crossed product. Let A be a C^*-algebra acting on a Hilbert space H, and let α be an action of a discrete countable group G on A, that is, $\alpha : G \to \mathrm{Aut}(A)$ is a homomorphism. Then the reduced crossed product $A \rtimes_\alpha G$ is the C^*-algebra on $l^2(G, H)$ generated by $\pi(A) \cup \lambda_G$. Here $\pi : A \to B(l^2(G, H))$ is a faithful *-

reprensentation defined by $(\pi(a)\xi)(g) = \alpha_g^{-1}(a)\xi(g), (\xi \in l^2(G,H))$, and $\lambda : G \to B(l^2(G,H))$ is a unitary repsentation given by $(\lambda_g\xi)(h) = \xi(g^{-1}h), (\xi \in l^2(G,H))$. There exists the faithful conditional expectation E of $A \rtimes_\alpha G$ onto $\pi(A)$ such that $E(\pi(a)\lambda_g) = 0$ for all $g \in G$ except the unit e and $a \in A$. Given a state ϕ of A, we denote by the same notation ϕ the state $\phi \circ \pi^{-1}$ on $\pi(A)$. Then we have the state $\phi \circ E$ of $A \rtimes_\alpha G$. For a subset ω of A and a subset K of G, we let $\omega_K = \{\pi(a)\lambda_g : a \in \omega, g \in K\}$.

3.1. Lemma. *Let A be an exact unital C^*-algebra, and let ϕ be a state of A whose GNS-representation π_ϕ is faithful. Let G be a discerete countable amenable group, and let α be an action of G on A with $\phi \circ \alpha = \phi$. Given a finite set $K \subset G$ and $\delta > 0$, let $F = supp(f)$ for some minimal function f for $(K, \delta^2/2)$. If ω is a finite subset in the unit ball of A, then*

$$scp_{\phi\circ E}(\omega_K, \delta) \leq scp_\phi(\bigcup_{g\in F} \alpha_g^{-1}(\omega), \frac{\delta}{2}) + \log(|F|).$$

3.2. From a view point of entropy, an interesting example α is an automorphism of the shift type, that is, α is the automorphism of $\coprod_{i\in\mathbb{Z}} G_i$ (restricted direct product) induced by the map $i \in \mathbb{Z} \to i+1$. Here $G_i, (i \in \mathbb{Z})$ is a copy of a finite group G_0.

We consider a condition which is satisfied by such an automorphism. Let G be a discrete group and let $\theta \in \mathrm{Aut}(G)$.

3.3. Condition (*) for (G, θ) : Given a finite set $K \subset G$ and $\delta > 0$, there exist a finite subgroup $L \subset G$ such that for all all $n \in \mathbb{N}$ we can choose a a minimal function f_n for $(\cup_{i=0}^{n-1}\theta^i(K), \delta)$ whose support $supp(f_n)$ is contained in the product set $L\theta(L)\cdots\theta^{n-1}(L)$.

The pair (G, θ) in 3.2 satisfies (*) by taking the smallest subgroup $L \supset K$ for given finite set $K \subset G$.

3.4. Let A be an exact unital C^*-algebra, and let ϕ be a state of A whose GNS-representation π_ϕ is faithful. Let G be a discerete countable amenable group, and let α an action with $\phi \circ \alpha_g = \phi$ for all $g \in G$. In the next Theorem, we study the entropy for a $\phi \circ E$-preserving automorphism γ of $A \rtimes_\alpha G$ which satisfies that

$$\gamma(\pi(A)) = \pi(A) \quad \text{and} \quad \gamma(\lambda_G) = \lambda_G.$$

This condition is equivalent to that $\gamma \circ \pi$ is a ϕ-preserving automorphism of A.

Remark that we can construct such an automorphism γ from a ϕ-preserving automorphism of A as in [Ch3], [DS]. In the section 4, we treat such a γ which asises through the reduced free product construction.

Theorem.

(1) If $\gamma(\lambda_g) = \lambda_g$ for all $g \in G$, then

$$ht_{\phi \circ E}(\gamma) = ht_\phi(\gamma|_{\pi(A)}).$$

(2) Assume that γ commutes with $Ad\lambda_g$ for all $g \in G$.

(2.1) If (λ_G, γ) satisfies (*), then

$$ht_{\phi \circ E}(\gamma) \leq ht_\phi(\gamma|_{\pi(A)}) + ha(\gamma|_{\lambda_G}).$$

(2.2) If G is abelian and if a finite subset of G is contained in a finite subgroup, then the inequality in (2.1) holds.

4. Entropy of Free Products

In this section, we investigate entropies for automorphisms which arise naturally by the free product construction. (See [BC, Ch1, Ch3, D2, DS, S1, S2] for other kind of computations of entropies for automorphisms on the reduced free product C^*-algebras.

4.1. For a set I, let $A_i, i \in I$ be a unital C^*-algebra with a state ϕ_i whose GNS representation is faithful. The reduced free product $(A, \phi) = \underset{i \in I}{*}(A_i, \phi_i)$ defined by Voiculescu [V1] (see also [VDN]) is the pair of a unital C^*-algebra A with unital embeddings $A_i \hookrightarrow A$ for all $i \in I$ and a state ϕ such that

(i) $\phi|_{A_i} = \phi_i$, for all $i \in I$,

(ii) the family $(A_i)_{i \in I}$ is free in (A, ϕ),

(iii) A is generated by the family $(A_i)_{i \in I}$,

(iv) the GNS reprentation of ϕ is faithful on A.

Here, the statement (ii) means that $\phi(a_1 a_2 \cdots a_n) = 0$, whenever $a_j \in A_{\iota_j}, \phi(a_j) = 0$ and $\iota_j \neq \iota_{j+1}$ for $j \in \{1, 2, \cdots, n-1\}$. The state ϕ is denoted by $\underset{i \in I}{*}\phi_i$.

In the case where all ϕ_i are tracial state, $\underset{i \in I}{*}\phi_i$ is a tracial state of A ([Av]).

A reduced word a in $(A_i)_{i \in I}$ is an element in A given by an expression of the form $a = a_1 a_2 \cdots a_n$, where $n \geq 1, a_i \in A_{\iota_i}, \phi_{\iota_i}(a_i) = 0$ and $\iota_1 \neq \iota_2, \cdots, \iota_{n-1} \neq \iota_n$. The number n is called the length of the reduced word and the set $\{\iota_1, \iota_2, \cdots, \iota_n\}$ is called the alphabet for the word.

The linear span of all reduced words in $(A_i)_{i \in I}$ is dense in A. Let α_i be a $*$-automorphism of A_i, and let ϕ_i be an α_i-invariant state of A_i. Then there exists a ϕ-preserving automorphism α of the algebra A such that $\alpha(a_1 a_2 \cdots a_n) = \alpha_{\iota_1}(a_1)\alpha_{\iota_2}(a_2) \cdots \alpha_{\iota_n}(a_n)$ whenever $a_j \in A_{\iota_j}, \phi(a_j) = 0$ and $\iota_j \neq \iota_{j+1}$ for $j \in \{1, 2, \cdots, n-1\}$. The automorphism α is denoted by $\underset{i \in I}{*}\alpha_i$.

Theorem. *Let I be a set, and for every $\iota \in I$ let A_ι be a unital finite dimensional C^*-algebra with a state ϕ_ι whose GNS-representation is faithful. Let*

$$(A, \phi) = \underset{\iota \in I}{*} (A_\iota, \phi_\iota).$$

(1) If $\omega \subset A$ is a finite subset of reduced words in $(A_\iota)_{\iota \in I}$, then

$$scp_\phi(\omega, \delta) = 0, \quad \text{for all} \quad \delta > 0.$$

(2) For every ϕ-preserving automorphism α of A, we have that

$$h_\phi(\alpha) = ht_\phi(\alpha) = 0.$$

Let G be a countable discrete group, and let λ the left regular representation of G. We denote by τ_G the trace of the C^*-algebra $C_r^*(G)$ generated by λ_G defined by $\tau_G(\lambda_g) = 0$ for all $g \in G$ except the unit. An automorphism $\theta \in \mathrm{Aut}(G)$ induces the automorphism $\hat{\theta} \in \mathrm{Aut}(C_r^*(G))$ by $\hat{\theta}(\lambda_g) = \lambda_{\theta(g)}$ for all $g \in G$.

4.2. Proposition. *Let B be a finite dimensional C^*-algebra with a state ψ whose GNS-representation is faithful. Let G be an amenable discrete group. Then*

$$ht_{\tau_G}(\hat{\theta}) \le ht_{\tau_G * \psi}(\hat{\theta} * \beta) \le ha(\theta),$$

for all $\theta \in \mathrm{Aut}(G)$ and $\beta \in \mathrm{Aut}(B)$ with $\psi \circ \beta = \psi$.

Proof. First, we prove that $ht_{\tau_G * \psi}(\hat{\theta} * \beta) \le ha(\theta)$.

Let

$$(\mathcal{A}, \varphi) = (C_r^*(G), \tau_G) * (B, \psi).$$

Let $A_g = \lambda_g B \lambda_g^*$ for all $g \in G$, and let A be the C^*- subalgebra of $\mathcal{A}$ generated by $\{A_g : g \in G\}$. We denote by ϕ the resriction $\varphi|_A$ of the state φ to A, and by ϕ_g the state $\phi|_{A_g}$. Then A is isomorphic to the C^*-algebra which is obtained from the reduced free product construction, that is, $(A, \phi) \cong \underset{g \in G}{*}(A_g, \phi_g)$, and $\mathcal{A}$ is isomorphic to the crossed product $A \rtimes_\alpha G$ ([CD : Claim 4]). Here we define the action α of G by $\alpha_g(x) = \lambda_g x \lambda_g^*$ for all $x \in A$. Then $\phi \circ \alpha(x) = \phi(x)$ for all $x \in A$ because ϕ is the restriction of $\tau_G * \psi$. In this situation, we have that $\varphi = \phi \circ E$, where $E : \mathcal{A} \to A$ is the φ-conditional expectation such that $E(\lambda_g) = 0$ for all $g \in G$ except the unit. For the sake of simplicity, we denote $\hat{\theta} * \beta$ by γ. Then $\gamma(A) = A$ and $\gamma(\lambda_g) = \lambda_{\theta(g)}$ for all $g \in G$. It is clear that $\varphi \circ \gamma = \varphi$.

First, we apply Lemma 3.1 to compute $ht_{\phi \circ E}(\gamma)$. Let $\omega \subset A$ be a finite set and let $K \subset G$ be a finite set. Let W be the set of all reduced words in $(A_g)_{g \in G}$. We may assume by Proposition 2.5 that $\omega \subset W$. Also we may assume that K contains the unit 1 of G and all elements in ω has the norm less than 1. For an $n \in \mathbb{N}$, let $\omega(\gamma, n) = \cup_{i=0}^{n-1} \gamma^i(\omega)$ and $K(\theta, n) = \cup_{i=0}^{n-1} \theta^i(K)$. Then

$$\omega_K \cup \gamma(\omega_K) \cup \cdots \cup \gamma^{n-1}(\omega_K) \ \subset \ \omega(\gamma, n)_{K(\theta, n)}.$$

Given $\delta > 0$ and $n \in \mathbb{N}$, let $F = F(\theta, n)$ be the support of some minimal function for $(K(\theta, n), \delta^2/2)$. Then by Lemma 3.1, we have that

$$scp_{\phi \circ E}(\omega(\gamma, n)_{K(\theta, n)}, \delta) \leq scp_\phi(\bigcup_{g \in F} \alpha_g^{-1}(\omega(\gamma, n), \frac{\delta}{2}) + \log |F|.$$

On the other hand, A_g is finite dimensional for all $g \in G$, and if $\omega \subset W$ then $\bigcup_{g \in F} \alpha_g^{-1}(\omega(\gamma, n)) \subset W$. Hence we have by Theorem 4.1 (1)

$$scp_\phi(\bigcup_{g \in F} \alpha_g^{-1}(\omega(\gamma, n), \frac{\delta}{2}) = 0.$$

This implies that $ht_\varphi(\hat{\theta} * \beta) = ht_{\phi \circ E}(\gamma) \leq h(\theta)$.

Since $ht_{\tau_G}(\hat{\theta}) \leq ht_\varphi(\hat{\theta} * \beta)$ by Proposition 2.4, we have the conclusion. □

4.3. If (A, α, ϕ) is a C^*-dynamical system, and (H, π, ξ) is the GNS-triplet of ϕ, and $\bar{\alpha}$ the extension of α to the von Neumann algebra $M = \pi(A)''$, then $h_\phi(\alpha) = h_{\omega_\xi}(\bar{\alpha})$, ([CNT]). Furthermore, if ϕ is a tracial state of A, then $h_{\omega_\xi}(\bar{\alpha})$ is the Connes-Stϕrmer entropy $H(\bar{\alpha})$ of a finite von Neumann algebra M ([CS]).

For an automorphism θ of an discrete group G, we denote by $\bar{\theta}$ the automorphism of the group von Neumann algebra $L(G)$ induced by θ.

Let (M_i, φ_i) be a von Neumann algebra with a faithful state for $i \in I$. The free product $(M, \varphi) = \underset{i \in I}{*}(M_i, \varphi_i)$ has the same structure as in 4.1 ([V1, VDN]). And if α_i is an automorpism of M_i with $\varphi_i \circ \alpha_i = \varphi_i$ for $i \in I$, then we have the automorphism $\underset{i \in I}{*}\alpha_i$ of M with the same property as in 4.1.

Corollary. *Let B, ψ, β be the same as in Proposition 4.2. Assume that G is discrete and abelian, and $\theta \in \mathrm{Aut}(G)$. Then*

$$h_{\tau_G}(\hat{\theta}) = ht_{\tau_G}(\hat{\theta}) = ht_{\tau_G * \psi}(\hat{\theta} * \beta) = h_{Top}(\hat{\theta}) = ha(\theta) = ht(\hat{\theta}).$$

In the case where ψ is a tracial state, we have that

$$H(\bar{\theta} * \beta) = h_{\tau_G * \psi}(\hat{\theta} * \beta) = ht_{\tau_G * \psi}(\hat{\theta} * \beta) = h_{Top}(\hat{\theta}).$$

Proof. Peters [P] introduced an entropy $h(\alpha)$ for an automorphism α of an abelian discrete group G and he proved that $h(\alpha)$ equals the Kolmogorov-Sinai entropy for $\hat{\alpha}$ which is nothing but the classical topological entropy $h_{Top}(\hat{\alpha})$. On the other hand, by [BG : Theorem 4.1] $ha(\alpha) = h_{Top}(\hat{\alpha})$. Hence we have

$$h_{\tau_G}(\hat{\theta}) \leq ht_{\tau_G}(\hat{\theta}) \leq ht_{\tau_G * \psi}(\hat{\theta} * \beta) \leq ha(\theta) = h_{Top}(\hat{\theta}) = ht(\hat{\theta}) = h_{\tau_G}(\hat{\theta})$$

by combining Proposition 4.2 and known results in [V2].

If ψ is a tracial state, then $\tau_G * \psi$ is a tracial state of $\mathcal{A}$ in the proof of Proposition 4.2. By the definition in [CS], $H(\cdot)$ is monotone, i.e. if $N \subset M$ is a von Neumann subalgebra such that $\alpha(N) = N$ for given automorphism α of M then $H(\alpha|_N) \leq H(\alpha)$. Hence by the avobe fact for CNT-entropy and Proposition 2.7.1, we have

$$h_{\tau_G}(\hat{\theta}) = H(\bar{\theta}) \leq H(\bar{\theta} * \beta) = h_{\tau_G * \psi}(\hat{\theta} * \beta) \leq ht_{\tau_G * \psi}(\hat{\theta} * \beta) = h_{\tau_G}(\hat{\theta}).$$

These inequality implies the desired equality. $\square$

4.4. Theorem. *Let B be an exact C^*-algebra, and let ψ be a state of B whose GNS-representation is faithfuln. Let G be an amenable discrete group. If β is an automorphism of B preserving ψ, then*

$$ht_{\underset{g\in G}{*}\psi_g}(\underset{g\in G}{*}\beta_g) = ht_{\tau_G * \psi}(id_G * \beta).$$

Here, β_g and ψ_g are copies of β and ψ respectively for all $g \in G$, and id_G is the identity automorphism of $C_r^*(G)$.

Proof. Our proof is a similar line to the proof of [Ch3 : Theorem 4.3]. Let $\mathcal{A}, A$ and $A_g (g \in G)$ be the algebras obtained by the same method in the proof of Proposition 4.2 from G and B. Then $\mathcal{A}$ is decomposed into the crossed product $A \rtimes_\alpha G$. This time, the automorphism $\gamma = id_G * \beta$ of $\mathcal{A} = A \rtimes_\alpha G$ satisfies $\gamma(A) = A$ and $\gamma(\lambda_g) = \lambda_g$ for all $g \in G$. The the state $\tau_G * \psi$ is nothing but the extension of the state $\underset{g\in G}{*}\psi_g$ by the conditional expectation E from $\mathcal{A}$ to A. Here ψ_g is the state of A_g given by $\psi_g(\lambda_g b \lambda_g^*) = \psi(b), (g \in G, b \in B)$, and so ψ_g coinsides with ϕ_g. Since all conditions Theorem 3.4 (1) are satisfied, we have that $ht_{\tau_G * \psi}(\gamma) = ht_{\tau_G * \psi}(\gamma|_A)$. On the other hand, $\gamma|_A$ behaves as $\underset{g\in G}{*}\beta_g$, where β_g is the automorphism of A_g defined by $\beta_g(\lambda_g b \lambda_g^*) = \lambda_g \beta(b) \lambda_g^*$ for all $g \in G$. These imply the conclusion. $\square$

4.5. Corollary. *If G is an abelian discrete group and $\theta \in \mathrm{Aut}(G)$, then for each positive integer k, we have*

$$h_{\tau_G}(\hat{\theta}) = h_{\tau_G * \cdots * \tau_G}(\underbrace{\hat{\theta} * \cdots * \hat{\theta}}_{k}) = ht_{\tau_G * \cdots * \tau_G}(\underbrace{\hat{\theta} * \cdots * \hat{\theta}}_{k}) = H(\underbrace{\bar{\theta} * \cdots * \bar{\theta}}_{k}).$$

In particular, if σ is the Bernoulli shift of an infinite product X of the n-point space for an integer n and μ is the state on $C(X)$ given by the product of the uniform measure, then for each $k \in \mathbb{N}$

$$h_{\mu * \cdots * \mu}(\underbrace{\sigma * \cdots * \sigma}_{k}) = ht_{\mu * \cdots * \mu}(\underbrace{\sigma * \cdots * \sigma}_{k}) = \log n = H(\underbrace{\bar{\sigma} * \cdots * \bar{\sigma}}_{k}).$$

Proof. We apply Corollary 4.3 to $B = C^*(\mathbb{Z}_k)$ of the cyclic group $\mathbb{Z}_k$ and ψ which is the trace $\tau_{\mathbb{Z}_k}$. we denote it by τ_k. Then

$$h_{\tau_G}(\hat{\theta}) = ht_{\tau_G * \tau_k}(\hat{\theta} * id_{\mathbb{Z}_k}).$$

On the other hand, we apply Theorem 4.2 to $B = C_r^*(G)$ and $\beta = \hat{\theta}$. Let us take $\mathbb{Z}_k$ as the group G in Theorem 4.2 . Then

$$\begin{aligned} ht_{\tau_G * \tau_k}(\hat{\theta} * id_{\mathbb{Z}_k}) &= ht_{\tau_g * \cdots * \tau_G}(\underbrace{\hat{\theta} * \cdots * \hat{\theta}}_{k}) \\ &\geq h_{\tau_G * \cdots * \tau_G}(\underbrace{\hat{\theta} * \cdots * \hat{\theta}}_{k}) = H(\underbrace{\bar{\theta} * \cdots * \bar{\theta}}_{k}) \\ &\geq H(\bar{\theta}) = h_{\tau_G}(\hat{\theta}). \end{aligned}$$

If we consider $\coprod_{i \in \mathbb{Z}} G_i$ as the gourp G, where $G_i = \mathbb{Z}_n$ for all $i \in \mathbb{Z}$, then we have the result on the Bernoulli shift. $\square$

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We remark that Theorem 4.4 and Corollary 4.5 are extended to more general automorphisms, and extended versions of Proposition 4.2, Corollary 4.3, Theorem 4.4 and Corollary 4.5 are proved by different methods.