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Fibrations associated with $*$-homomorphisms of C*-algebras

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1 Introduction

We consider geometric counter parts of C*-algebras. By [1] and our studies, we have the following tabular of correspondences as a generalization of classical (commutative) known results:

algebra	commutative	non commutative
C*-algebra	compact Hausdorff space	UKB (=uniform Kähler bundle)
Automorphism group	homeomorphism group	isomorphism group of UKB
Hilbert C* -module	vector bundle	the atomic bundle on UKB
homomorphism	continuous map	?
subalgebra	fibration	?
representation	spectral measure	?

where the part of the group of automorphisms and modules are our studies [2, 3].

We want to fill ? in this tabular. C*-subalgebra and representation are treated as special studies of homomorphisms between two C*-algebras.

The aim of present paper is a presentation of studies of fibration structure associated with $*$-homomorphisms for two unital C*-algebras $\mathcal{A}$ and $\mathcal{B}$. We introduce three levels of studies as follows:

(i) one homomorphism $\phi \in \mathrm{Hom}(\mathcal{A},\mathcal{B})$ itself,

(ii) the space $\mathrm{Hom}(\mathcal{A},\mathcal{B})$ of all $*$-homomorphisms,

(iii) a bi-functor Hom.

We introduce examples of three levels (i), (ii), (iii). We show relations between them, quantum mechanics and non commutative geometry.

2 General setting

In [1],

Theorem 2.1 *Each unital C*-algebra $\mathcal{A}$ can be realized faithfully as a function subalgebra on the set $\mathcal{P}$ of all pure states on it as Gelfand representation*

$$f : \mathcal{A} \to \mathcal{F}(\mathcal{P}); \quad A \mapsto f_A,$$

$$f_A(\rho) \equiv \rho(A) \quad (\rho \in \mathcal{P})$$

with a $*$-product on the function space $\mathcal{F}(\mathcal{P})$ on $\mathcal{P}$.

This theorem brings us a possibility of a geometrical study of C*-algebra. Though the correspondence between a C*-algebra and $\mathcal{P}$ is not categorical, that is, it is not suitable for studies of morphisms and subalgebras. Hence we try to consider the following two possibilities:

(i) give up to study about their categorical property and study the phenomena which appear from morphisms and subalgebras.

(ii) find the other generalization of the correspondence between C*-algebras and its dual objects.

We review facts in commutative case once more. Let X and Y be two compact Hausdorff spaces. Then there is a one-to-one correspondence between $C(X,Y)$ and $\mathrm{Hom}(C(Y), C(X))$. For example, $f : X \to Y$ is surjective if and only if $f^* : C(Y) \to C(X)$ is injective. In this sense, subalgebra of commutative C*-algebra is corresponded to a fibration. Then there is a question what happens in the non commutative case.

	mapping space		*homomorphism space*
commutative case	$C(X,Y)$	$\Leftrightarrow$	$\mathrm{Hom}(C(Y), C(X))$
non commutative case	?	$\Leftrightarrow$	$\mathrm{Hom}(\mathcal{A}, \mathcal{B})$.

In general, for $\phi \in \mathrm{Hom}(\mathcal{A}, \mathcal{B})$, $\phi^*(\mathcal{P}_{\mathcal{B}}) \not\subset \mathcal{P}_{\mathcal{A}}$. This is one of problems of breakdown of the categorical equivalence in the commutative case. We treat this type of dual of morphisms in the next section.

3 A fibration associated with single morphism

As the case (i) in section 1, we explain easy examples. We obtain the following fibration from an inclusion $\iota : \mathbf{C}^2 \hookrightarrow M_2(\mathbf{C})$ as an observation of a naive spin system.

Proposition 3.1 *(Torus fibration for S^3) There is a continuous surjection $\mu : S^3 \to [0,1]$ of S^3 on a closed interval $[0,1]$ and this fibration induces a decomposition as follows:*

$$S^3 \cong \coprod_{z\in[0,1]} T_z$$

where $T_0 \cong T_1 \cong S^1$ and $T_z \cong T^2$ for $z \in (0,1)$.

Proof. Define a map

$$f : S^3 = S(\mathbf{C}^2) \to [0,1]$$

by

$$f(z_1, z_2) \equiv |z_1| \quad ((z_1, z_2) \in S^3).$$

Then f is surjective and continuous. The fibration by f gives the statement of proposition. ∎

Specially, a map f' which is induced by f

$$f' : \mathbf{C}P^1 \equiv S^3/U(1) \to [0,1]$$

is just the restriction of dual map $\iota : \mathbf{C}^2 \hookrightarrow M_2(\mathbf{C})$ to the set $\mathbf{C}P^1$ of pure states of $M_2(\mathbf{C})$. $[0,1]$ is the set of all(mixed)states of $\mathbf{C}^2$. The fiber of f' is regarded as lost freedom in the observation of quantum states of $M_2(\mathbf{C})$. The image of f' is the probability which the spin takes value, $0(=$ down $)$ or $1(=$ up $)$. In this sense, we note that $[0,1]$ is a 1-simplex.

In other point of view, Proposition 3.1 represents the locus of deformation of torus T^2 in $\mathbf{C}^2 \cong \mathbf{R}^4$. Or, S^3 can be considered as a locus of T^2 in $\mathbf{R}^4$ which moves with 1-parameter. At point of the start and the end in $[0,1]$, T^2 collapses S^1 by pinching out one of two cycles of T^2.

In the same way, we have a little bit general result.

Proposition 3.2 *Any $2n+1$-dimensional sphere has a singular fibration on a n-simplex with $k+1$-torus as a fiber at a point on the interior of k-subsimplex.*

In general, the normal (mixed)state space of $\mathcal{L}(\mathcal{H})$ has a fibration on a Hilbert simplex with flag manifold as the fiber. This is proved by the uniqueness of diagonalization of positive normalized trace class operator as its ordered eigen values for a complete orthonormal basis. In this way, the state space of an algebra of observables has a fibration with respect to the spectrum of the commutative algebra affiliated with Hamiltonian. This is just an observation in a quantum system. We can watch lost freedom in an observation as a flag manifold.

4 Fibrations of homomorphism spaces

4.1 Morphisms between matrix algebras

We show easy examples as the case (ii) in section 1. We consider $H_{2,3}$ defined by the set of all $*$-homomorphisms:

$$H_{2,3} \equiv \mathrm{Hom}(M_2(\mathbf{C}), M_3(\mathbf{C})).$$

The content of $H_{2,3}$ is classified two types, $\phi = 0$ or not. Define $\Lambda_{2,3} \equiv H_{2,3}/\sim$ where $\phi \sim \phi'$ if there is a $*$-automorphism g of $M_2(\mathbf{C})$ such that $\phi' = \phi \circ g$. Remark that the automorphism group of $M_2(\mathbf{C})$ is isomorphic to the projective unitary group $PU(2) \equiv U(2)/U(1)$. Denote the natural projection

$$\pi : H_{2,3} \to \Lambda_{2,3}.$$

Then we have a fibration $(H_{2,3}, \pi, \Lambda_{2,3})$. We have the following proposition

Proposition 4.1 *There is the following equivalence of fibration:*

$$(H_{2,3}, \pi, \Lambda_{2,3}) \cong (PV_2(\mathbf{C}^3) \cup \{0\}, \tilde{\nu}, \mathbf{C}P^2 \cup \{0\})$$

where $(PV_2(\mathbf{C}^3), \nu, \mathbf{C}P^2)$ is a principal $PU(2)$-bundle, $PV_2(\mathbf{C}^3)$ is projective Stiefel manifold consisting of $U(1)$-orbits of 2-orthonormal basis of $\mathbf{C}^3$ and $\tilde{\nu}$ is the extension of ν by $\tilde{\nu}(0) = 0$.

Proof. For $0 \neq \phi \in H_{2,3}$, the image $\phi(I_2)$ of the unit I_2 of $M_2(\mathbf{C})$ is a 2-dimensional projection on $\mathbf{C}^3$. On the other hand, for a two dimensional subspace $V \subset \mathbf{C}^3$, we obtain a isomorphism $\mathrm{End}_{\mathbf{C}}(V) \cong M_2(\mathbf{C})$ up to automorphism of $M_2(\mathbf{C})$. Hence we have an equivalence

$$\begin{aligned}\Lambda_{2,3}^{\times} \equiv\ & \{\phi \in H_{2,3} : \phi \neq 0\}/\sim \\ \cong\ & \{\phi(I_2) : \phi \in H_{2,3} \setminus \{0\}\} \\ \cong\ & G_2(\mathbf{C}^3) \\ \cong\ & \mathbf{C}P^2.\end{aligned}$$

For $\phi, \phi' \in H_{2,3}^{\times}$ such that $\phi \sim \phi'$, then $\phi^{-1} \circ \phi' \in \mathrm{Aut}M_2(\mathbf{C}) \cong PU(2)$. The action of $PU(2)$ on the fiber on $0 \neq z \in \Lambda_{2,3}$ is free. Hence

$$\pi^{-1}(z) \cong PU(2) \quad (0 \neq z \in \Lambda_{2,3}).$$

The correspondence between $H_{2,3}^{\times}$ and $PV_2(\mathbf{C}^3)$ is given to calculate the isotropy subgroup of $PU(3)$-(left)action on $H_{2,3} \setminus \{0\}$. ∎

In the same way, we have

Proposition 4.2 *There is an equivalence of fibrations:*
$(H_{2,4}, \pi, \Lambda_{2,4})$

$$\cong (\{0\} \cup PV_2(\mathbf{C}^4) \cup FM_2(\mathbf{C}^4),\ \tilde{\nu},\ \{0\} \cup G_2(\mathbf{C}^4) \cup G_2(\mathbf{C}^4)/\mathbf{Z}_2)$$

where $FM_2(\mathbf{C}^4)$ is a homogeneous space defined by

$$FM_2(\mathbf{C}^4) \equiv PU(4)/P(U(2) \otimes I_2).$$

The fiber of this fibration is equal to $PU(2)$ except $\{0\}$.

$FM_2(\mathbf{C}^4)$ is the space of all fermions on $\mathbf{C}^4$ because an element of $FM_2(\mathbf{C}^4)$ is a homomorphism determined by a partial isometry v on $\mathbf{C}^4$ which satisfies the canonical anti-commutation relation:

$$\{v, v^*\} = I_4, \quad \{v, v\} = 0, \quad \{v^*, v^*\} = 0$$

where $\{\cdot,\cdot\}$ is the anti-commutator on $M_4(\mathbf{C})$. $FM_2(\mathbf{C}^4)$ has no name in usual text book of geometry and it is not known other realization like a Stiefel manifold yet.

For $n, m \in \mathbf{N}$, $H_{n,m}$ is described as a union of space of $n-1$-chain of B-construction for the category of projections on $\mathbf{C}^m$. For example, an element of $FM_2(\mathbf{C}^4)$ is a 1-chain, that is an edge between two orthogonal projections on $\mathbf{C}^4$. $0 \neq \phi \in H_{3,6}$ corresponds to 2-chain(,or 2-simplex) of the set of projections on $\mathbf{C}^6$.

A matrix algebra has one-point spectrum. Hence it is regarded as a point with some kind of internal freedom. $H_{2,3}$ is a non commutative mapping space between non commutative points with internal symmetries $PU(2)$ and $PU(3)$, respectively.

4.2 Structure theorem of representation space of Cuntz algebra

Let $\mathcal{O}_n$ be a Cuntz algebra with Cuntz generator $\{s_i\}_{i=1}^n$, $n \geq 2$, and $\mathcal{H}$ a separable infinite dimensional Hilbert space. Denote $\mathrm{Rep}(\mathcal{O}_n, \mathcal{H})$ the set of all unital $*$-representations of $\mathcal{O}_n$ on $\mathcal{H}$.

We show that $\mathrm{Rep}(\mathcal{O}_n, \mathcal{H})$ is realized as a fiber product of representation space of matrix algebra and space of some partial isometries. Remark $\mathrm{Rep}(\mathcal{O}_n, \mathcal{H}) = \mathrm{Hom}(\mathcal{O}_n, \mathcal{L}(\mathcal{H}))$.

Let $G_{\infty,\infty}(\mathcal{H})$ be the set of all projections on $\mathcal{H}$ with both rank and co-rank ∞. We call $G_{\infty,\infty}(\mathcal{H})$ *the Grassmanian on $\mathcal{H}$ type of* (∞,∞). Let $H_n(\mathcal{H})$ be the set of all unital $*$-homomorphisms from $M_n(\mathbf{C})$ to $\mathcal{L}(\mathcal{H})$. Let $PI(I, G_{\infty,\infty}(\mathcal{H}))$ be the set of all partial isometries consisting v satisfying

$$v^*v = I, \quad vv^* \in G_{\infty,\infty}(\mathcal{H}).$$

Then we have three fibrations on $G_{\infty,\infty}(\mathcal{H})$ as follows:

$$\begin{array}{rcl} \pi: & \mathrm{Rep}(\mathcal{O}_n, \mathcal{H}) & \to G_{\infty,\infty}(\mathcal{H}), \\ \nu: & H_n(\mathcal{H}) & \to G_{\infty,\infty}(\mathcal{H}), \\ \mu: & PI(I, G_{\infty,\infty}(\mathcal{H})) & \to G_{\infty,\infty}(\mathcal{H}) \end{array}$$

defined by

$$\begin{array}{lll} \pi(\phi) \equiv & \phi(s_1 s_1^*) & (\phi \in \mathrm{Rep}(\mathcal{O}_n, \mathcal{H})), \\ \nu(\psi) \equiv & \psi(E_{11}) & (\psi \in H_n(\mathcal{H})), \\ \mu(v) \equiv & vv^* & (v \in PI(I, G_{\infty,\infty}(\mathcal{H}))) \end{array}$$

where $\{E_{ij}\}_{i,j=1}^n$ is the canonical matrix unit of $M_n(\mathbf{C})$. Define the fiber product of $(\,H_n(\mathcal{H}), \nu, G_{\infty,\infty}(\mathcal{H})\,)$ and $(PI(I, G_{\infty,\infty}(\mathcal{H})), \mu, G_{\infty,\infty}(\mathcal{H})\,)$ by

$$H_n(\mathcal{H}) \times_{G_{\infty,\infty}(\mathcal{H})} PI(I, G_{\infty,\infty}(\mathcal{H})) \equiv \{\,(\psi, v) \in H_n(\mathcal{H}) \times PI(I, G_{\infty,\infty}(\mathcal{H})) \,:\, \nu(\psi) = \mu(v)\,\}.$$

We denote the natural projection

$$p : H_n(\mathcal{H}) \times_{G_{\infty,\infty}(\mathcal{H})} PI(I, G_{\infty,\infty}(\mathcal{H})) \to G_{\infty,\infty}(\mathcal{H}).$$

$(\, H_n(\mathcal{H}) \times_{G_{\infty,\infty}(\mathcal{H})} PI(I, G_{\infty,\infty}(\mathcal{H})),\, p,\, G_{\infty,\infty}(\mathcal{H})\,)$ is a fibration, too.

Theorem 4.1 *There is the following equivalence of fibrations:*

$$(\mathrm{Rep}(\mathcal{O}_n, \mathcal{H}), \pi, G_{\infty,\infty}(\mathcal{H})\,)$$
$$\cong (\, H_n(\mathcal{H}) \times_{G_{\infty,\infty}(\mathcal{H})} PI(I, G_{\infty,\infty}(\mathcal{H})),\, p,\, G_{\infty,\infty}(\mathcal{H})\,).$$

Proof. Define a map

$$\theta : \mathrm{Rep}(\mathcal{O}_n, \mathcal{H}) \to H_n(\mathcal{H}) \times_{G_{\infty,\infty}(\mathcal{H})} PI(I, G_{\infty,\infty}(\mathcal{H})),$$
$$\theta(\phi) \equiv (\, \{\phi(s_i s_j^*)\}_{i,j=1}^n,\, \phi(s_1)\,) \quad (\phi \in \mathrm{Rep}(\mathcal{O}_n, \mathcal{H}))$$

where we identify a matrix unit and an element in $H_n(\mathcal{H})$.

Then we have

$$\begin{aligned} \nu(\{\phi(s_i s_j^*)\}_{i,j=1}^n) &= \phi(s_1 s_1^*) \\ &= \phi(s_1)\phi(s_1)^* \\ &= \mu(\phi(s_1)). \end{aligned}$$

Hence the image of θ is in $H_n(\mathcal{H}) \times_{G_{\infty,\infty}(\mathcal{H})} PI(I, G_{\infty,\infty}(\mathcal{H}))$. On the other hand,

$$\theta^{-1}(\psi, v) = \{v, \psi(E_{21})v, \psi(E_{31})v, \ldots, \psi(E_{n1})v\}$$

for $((\psi, v) \in H_n(\mathcal{H}) \times_{G_{\infty,\infty}(\mathcal{H})} PI(I, G_{\infty,\infty}(\mathcal{H}))$ where we identify a Cuntz generator and an element in $\mathrm{Rep}(\mathcal{O}_n, \mathcal{H})$. Clearly, $(\theta, id_{G_{\infty,\infty}(\mathcal{H})})$ is a (set theoretical)fibration isomorphism in Theorem 4.1. ∎

4.3 General case

In general, for two C*-algebra $\mathcal{A}$ and $\mathcal{B}$, $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$ becomes the union of homogeneous space of the inner automorphism group $\mathrm{Inn}\mathcal{B}$ of $\mathcal{B}$ with the isotropy subgroup H_ϕ defined by

$$\begin{aligned} H_\phi \equiv&\ \mathrm{Inn}\mathcal{R}_\phi \\ \subset&\ \mathrm{Inn}\mathcal{B}, \end{aligned}$$

$$\begin{aligned} \mathcal{R}_\phi \equiv&\ \phi(\mathcal{A})' \cap \mathcal{B} \\ =&\ \{b \in \mathcal{B} : [a, b] = 0,\, a \in \phi(\mathcal{A})\}. \end{aligned}$$

for $\phi \in \mathrm{Hom}(\mathcal{A}, \mathcal{B})$. We have calculated concrete examples for $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$.

We note that a similar theorem is known in a (commutative)mapping space between differential manifold(section 13 in [4]).

The Weyl form of the canonical quantization is a pair (ϕ, ψ) of faithful representation of $C(S^1)$ on a separable infinite dimensional Hilbert space $\mathcal{H}$ such that they are transformed by adjoint map of (abstract)Fourier (inverse)transformation each other. That is, the canonical commutation relation can be regarded as a special position of two points in $\mathrm{Hom}(C(S^1), \mathcal{L}(\mathcal{H}))$. Since Fourier transformation is a generator of $\mathbf{Z}_4$, it seems a rotation of angle 90° in a 2-dimensional plane. Hence a pair of points associated with CCR is seemed in a position of 90° in a some 2-dimensional plane in $\mathrm{Hom}(C(S^1), \mathcal{L}(\mathcal{H})$ around center. When a pair (ϕ, ψ) takes a commutative position, the algebra generated by $\phi(C(S^1))$ and $\psi(C(S^1))$ is isomorphic to $C(T^2)$. Deformation of $C(T^2)$ to $\mathcal{A}_\theta$ can be treated in the space $\mathrm{Hom}(C(S^1), \mathcal{L}(\mathcal{H}))$(or the quotient space of $\mathrm{Hom}(C(S^1), \mathcal{L}(\mathcal{H})))$.

In this way, many examples of C*-algebra with some generators can be stated as a position of of representations of commutative algebras.

5 A categorical extension of Gelfand transformation

In the case (iii) in section 1, we show the following categorical reformulation of Gelfand representation of commutative unital C*-algebras:

Fact 5.1 *(Gelfand representation) Denote categories $\mathcal{LCH}$, $\mathcal{CH}$ and $\mathcal{CC}_1^*$ categories of locally compact Hausdorff spaces, compact Hausdorff spaces and unital commutative C*-algebras, respectively.*

Then we have the following natural equivalence

$$1_{\mathcal{CC}_1^*} \cong \mathcal{LCH}_{\mathbf{C}} \circ (\mathcal{CC}_1^*)_{\mathbf{C}},$$

$$1_{\mathcal{CH}} \cong (\mathcal{CC}_1^*)_{\mathbf{C}} \circ \mathcal{LCH}_{\mathbf{C}}|_{\mathcal{CH}},$$

where $\mathcal{X}_Y$ is the contravariant principal representation of a category $\mathcal{X} = \mathcal{LCH}, \mathcal{CC}_1^*$ by an object Y in $\mathcal{X}$ and $1_{\mathcal{X}}$ is the identity functor on $\mathcal{X}$.

Remark that $\mathbf{C}$ is the set of all complex numbers. $\mathbf{C}$ is 1-dimensional commutative unital C*-algebra and a locally compact Hausdorff space which is homeomorphic to 2-dimensional Euclid space $\mathbf{R}^2$. In the above fact, a covariant functor $\mathrm{Gel} \equiv \mathcal{LCH}_{\mathbf{C}} \circ (\mathcal{CC}_1^*)_{\mathbf{C}}$ is a transformation of a category of $\mathcal{CC}_1^*$. We call Gel the *Gelfand transformation.* This reformulation gives us a question what is non commutative version of Gel.

The reason of the success of this reformulation is caused by regarding the set of pure states as the character of a commutative C*-algebra. Because of this reason, we consider that a generalization of Gelfand transformation to non commutative case may be suitable by principal representation of category $\mathcal{C}_1^*$ of unital (non commutative)C*-algebras. We claim that a generalization of the category $\mathcal{CH}$ of compact Hausdorff spaces with respect to generalized Gelfand transformation is a category of some kind of 2-step fibrations: Let $F_1 \equiv (\mathcal{C}_1^*)_*^*$ be the principal contravariant-covariant functor of a category $\mathcal{C}_1^*$. For $\mathcal{A}, \mathcal{B} \in \mathcal{C}_1^*$, define

$$\begin{aligned} F_2(\mathcal{A},\mathcal{B}) \equiv & \ F_1(\mathcal{A},\mathcal{B}))/\mathrm{Inn}\mathcal{B}, \\ F_3(\mathcal{A},\mathcal{B}) \equiv & \ \{\mathrm{Ker}\phi : \phi \in F_1(\mathcal{A},\mathcal{B})\} \\ = & \ \{\mathrm{Ker}\phi : [\phi] \in F_2(\mathcal{A},\mathcal{B})\}. \end{aligned}$$

Let r, s be natural projections between them

$$F_1(\mathcal{A},\mathcal{B}) \stackrel{r}{\to} F_2(\mathcal{A},\mathcal{B}) \stackrel{s}{\to} F_3(\mathcal{A},\mathcal{B}).$$

Proposition 5.1 (i) $F^{(1)} \equiv (F_1, r, F_2)$ *is a contravariant-covariant bi-functor from* $\mathcal{C}_1^*$ *to* $\mathcal{FIB}^{(1)}$.

(ii) $F^{(2)} \equiv (F_1, r, F_2, s, F_3)$ *is a contravariant functor from* $\mathcal{C}_1^*$ *to* $\mathcal{FIB}^{(2)}$.

where $\mathcal{FIB}^{(n)}$ *is the category of n-step fibrations which is a $n+1$-chain* $(\{X_i\}_{i=0}^n, \{p_j\}_{j=1}^n)$ *of surjections between spaces:*

$$X_0 \stackrel{p_1}{\to} X_1 \stackrel{p_1}{\to} \cdots \stackrel{p_n}{\to} X_n.$$

We note that $F_2 = F_1$ in commutative case. The notion of F_3 appears in the theory of null ideal sequence. F_2 and F_3 are some kinds of generalization of spectrum and primitive spectrum, respectively. Now we have not yet success to define good topology for them. We must calculate examples of this categorical fibration more.

References

[1] R.Cirelli, A.Manià and L.Pizzocchero, *A functional representation of noncommutative C^*-algebras* Rev.Math.Phys. Vol. 6, No.5 (1994) 675-697.

[2] K.Kawamura *Structure theorem of the group of automorphisms of C^*-algebras*, math.OA/9809109.

[3] K.Kawamura *Serre-Swan theorem for non-commutative C^*-algebras*, math.OA/0002160.

[4] P.W.Michor, *Manifolds of Differentiable Mappings*, Shiva Publishing Limited.