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WELL–POSEDNESS OF A FREE BOUNDARY PROBLEM IN THE LIMIT OF SLOW–DIFFUSION FAST–REACTION SYSTEMS

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ABSTRACT. We consider a free boundary problem obtained from the asymptotic limit of a FitzHugh–Nagumo system, or more precisely, a slow–diffusion, fast–reaction equation governing a phase indicator, coupled with an ordinary differential equation governing a control variable v. In the range $(-1,1)$, the v value controls the speed of the propagation of phase boundaries (interfaces) and in the mean time changes with dynamics depending on the phases. A new feature included in our formulation and thus made our model different from most of the contemporary ones is the nucleation phenomenon: a phase switch occurs whenever v elevates to 1 or drops to -1. For this free boundary problem, we provide a weak formulation which allows the propagation, annihilation, and nucleation of interfaces, and excludes interfaces from having (space–time) interior points. We study, in the one space dimension setting, the existence, uniqueness, and non–uniqueness of weak solutions. A few illustrating examples are also included.

1. INTRODUCTION

We consider the limit, as $\varepsilon \searrow 0$, of the reaction diffusion system

$$(1.1) \qquad \begin{cases} u_t^\varepsilon = \varepsilon\Delta u^\varepsilon + \varepsilon^{-1} f(u^\varepsilon, v^\varepsilon), \\ v_t^\varepsilon = D\Delta v^\varepsilon + g(u^\varepsilon, v^\varepsilon) \end{cases}$$

with typical f and g given by

$$(1.2) \qquad f(u,v) = F(u) - v, \quad F(u) = u(3/\sqrt[3]{2} - 2u^2), \quad g(u,v) = u - \gamma v - b,$$

where $D \geq 0$, $\gamma > 0$ and $b \in \mathbb{R}$ are constants. This system is often used to model the propagation of chemical waves in excitable or bistable or oscillatory media, where u and v represent the propagator and controller respectively [6]. When $D = O(\varepsilon)$, (1.1) was used by Tyson and Fife to study the Belousov-Zhabotinskii reagent [12]. When $D = 0$, (1.1) is the well–known FitzHugh-Nagumo model for nerve impulse propagation; see [5, 8, 10, 11, and references therein].

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The local minimum and maximum of the cubic function $F(u)$ in (1.2) is -1 and 1. If $v \in (-1, 1)$, the equation $f(u, v) = F(u) - v = 0$, for u, has three real roots, $h_-(v)$, $h_0(v)$ and $h_+(v)$, where $h_-(v) < h_0(v) < h_+(v)$.

As $\varepsilon \searrow 0$, Fife [6, Chapter 4], X.Y. Chen [4], and X. Chen [2] demonstrated that solution $(u^\varepsilon, v^\varepsilon)$ to (1.1) has a limit (u, v) with $u = h^\pm(v)$ in $Q^\pm$, where (v, Q^+, Q^-) solves the following free boundary problem (with $\varepsilon = 0$):

$$\text{(1.3)} \qquad \begin{cases} v_t &= D\Delta v + g(h_\pm(v), v) \qquad \text{in } Q^\pm, \\ \frac{\partial \Gamma}{\partial t} &= \{W(v) - \varepsilon\kappa\}\mathbf{N} \qquad \text{on } \Gamma = \cup_{t>0}\Gamma_t \times \{t\} = \partial Q^+ \cap \partial Q^- \end{cases}$$

where κ and $\mathbf{N}$ are, respectively, the mean curvature and the unit normal of $\Gamma(t)$, and $W(v)$ is the speed of the traveling wave $(W(v), U(\cdot; v))$ of

$$U_{zz} + W\,U_z + f(U, v) = 0 \text{ on } \mathbb{R}, \quad U(\pm\infty, v) = h_\pm(v),\ U(0, v) = h_0(v).$$

Classical solution of the free boundary problem (1.3) has been studied by Hilhorst, Nishiura, and Mimura [9] (1-D case), X. Y. Chen [2] ($\varepsilon > 0$), X. Chen [4] ($\varepsilon = 0$). In general interfaces may collide and annihilate each other and therefore (global in time) classical solutions may not exist. Giga, Goto and Ishii [7] introduced and established the existence of viscosity (weak) solutions to (1.3) where the interface Γ is defined as the zero level set of the viscosity solution ϕ to $\phi_t = W(v)\,|\nabla\phi| + \varepsilon|\nabla\phi|\mathrm{div}(\frac{\nabla\phi}{|\nabla\phi|})$ ($\varepsilon \geq 0$). This formulation takes care of topological changes such as the annihilation of interfaces. However, there is another phenomenon, the *nucleation*, needs to be considered.

A careful analysis of the original system (1.1) shows that, if $v(x, t) > 1$, then the phase state at x will immediately switch to the "$-$" phase (regardless of its neighbors' phase states). Similarly, if $v(x, t) < -1$, the phase state at x will switch to the "+" phase. We refer to this phenomenon as *nucleation* . This phenomenon was ignored in most of the past works. The main purpose of this paper is to take into account the nucleation phenomenon. For this purpose, we consider only the one space dimension case, and assume that $D = 0$, which corresponds to the FitzHugh–Nagumo system. More precisely, we consider

$$(\mathbf{P}) \qquad \begin{cases} v_t &= G^\pm(v) \quad \text{in } \Omega_\pm(t), \\ \frac{\partial \Gamma}{\partial t} &= W(v) \quad \text{on } \Gamma(t) = \partial\Omega_\pm(t), t > 0, \end{cases}$$

where $G^\pm(v) = g(h_\pm(v), v)$ for $\pm v \leq 1$.

In the next section we will provide a weak formulation for problem (**P**). Then in §3, we provide several illustrating examples. In the rest of the paper, we prove our main result roughly stated as follows:

> *If the initial speeds are not zero on all initial interfacial points, then problem* (**P**) *admits a unique, global in time, weak solution.*

If the initial speed at an interfacial point *is* zero, there are, in general, countably many solutions. The non-uniqueness of (**P**) is not due to our deficiency in the definition of weak solutions, but due to the nature of the problem; see §3.4 for more details.

2. A Weak Formulation of (P) and the Main Result

In the sequel, we denote by $B(x,r)$ an open ball centered at x with radius r, and by $\bar{B}(x,r)$ a closed ball. If $r \le 0$, then $B(x,r) = \emptyset$. Also $M := \sup_{v\in(-1,1)} |W(v)|$. The following weak formulation was originated from [3].

Definition 1. *Let $\mathcal{D}$ be a closed domain in $\mathbb{R} \times [0,\infty)$. We say that (v, Q^+, Q^-) is a (weak) solution to* (**P**) *in $\mathcal{D}$ if $v \in C^0(\mathcal{D})$, Q^+ and Q^- are disjoint and (relatively) open in $\mathcal{D}$, and the followings hold:*

(1) **(Dynamics)** *$v_t \in L^\infty(\mathcal{D})$ and $v_t = G^\pm(v)$ in $Q^\pm$;*
(2) **(Nucleation)** *$\{(x,t) \in \mathcal{D} \mid \pm v > 1\} \subset Q^\mp$;*
(3) **(Propagation)** *If $B(x_0, r_0) \times \{t_0\} \subset Q^\pm$ and $\pm v < 1$ in $\bar{B}(x_0, r_0 + M\delta) \times [t_0, t_0 + \delta] \subset \mathcal{D}$ for some $\delta > 0$, then $B(x_0, r_0 + c^\pm\delta) \times \{t_0 + \delta\} \subset Q^\pm$, where $c^\pm = \min\{\mp W(v(x,t)) \mid x \in \bar{B}(x_0, r_0 + M\delta), t \in [t_0, t_0 + \delta]\}$;*
(4) **(No Fattening)** *$\mathrm{m}(\Gamma) = 0$, where $\Gamma = \mathcal{D}\setminus(Q^+ \cup Q^-)$ and m denotes the Lebesgue measure in $\mathbb{R}^2$.*

Remark 2.1. The **nucleation** criterion implies $\pm v \le 1$ in $Q^\pm$. Suppose that $G^\pm(\pm 1) \ne 0$. Then since $Q^\pm$ is open, we obtain from the **dynamics** criterion that $\pm v < 1$ in $Q^\pm \setminus \partial\mathcal{D}$ and that any point $(x,t) \in \mathcal{D}\setminus\partial\mathcal{D}$ where $v = \pm 1$ cannot be an interior point of $\overline{Q^\pm}$. Thus, the **no fattening** criterion implies that $\{ (x,t) \in \mathcal{D} \setminus \partial\mathcal{D} \mid v(x,t) = \pm 1 \} \subset \overline{Q^\mp}$.

On the other hand, if one of $G^\pm(\pm 1)$, say $G^+(1)$ vanishes, then interior points in $\{(x,t) \mid v(x,t) = 1\}$ can have choices of being in Q^+ or Q^-, thereby creating non-uniqueness. To avoid this situation, in the sequel we shall always assume that $G^\pm(\pm 1) \ne 0$. Also, we shall work only on "compatible" initial conditions; namely, $\pm v(\cdot, 0) < 1$ in $\partial\mathcal{D} \cap Q^\pm$. The generation of interface indicates that initial conditions to (**P**) should always be compatible.

In the sequel, we need only the **dynamics**, **propagation**, and the following criteria (to replace the **nucleation** and **no fattening** criteria): $\{(x,t) \in \mathcal{D} \setminus \partial\mathcal{D} \mid \pm v(x,t) \ge 1\} \subset \overline{Q^\mp}$.

Remark 2.2. To understand better the **propagation** criterion, we first note that if $(x_0, t_0) \in Q^\pm$, then $\pm v(x_0, t_0) < 1$ and consequently, $\pm v < 1$ in some neighborhood of (x_0, t_0). Hence, letting δ approach zero we see that $Q^\pm$ shrinks/expands with a velocity at most/least $W(v)$. The (necessary) introduction of $M, \delta, c^\pm$, etc. enables us to let (x_0, t_0) approach the boundary of $Q^\pm$ and thus to conclude that the boundary of $Q^\pm$ will shrink/expand with a speed no more/less than $W(v)$. In particular, if Q^+ and Q^- share a common boundary, then it moves with a speed $W(v)$, in the direction from the "$-$" phase region to "$+$" phase region. Thus, in the case of classical solutions, this condition is compatible with the equation $\Gamma_t = W(v)$. We remark that, due to the **nucleation**

criterion and the assumption that $G^{\pm}(\pm 1) \neq 0$, the value of $W(v)$ for $|v| > 1$ and the value $G^{\pm}(v)$ for $\pm v \geq 1$ are not needed. Nevertheless, for $c^{\pm}$ to have a clear meaning, in the sequel, we assume that $W(v)$ has been extended for all $v \in \mathbb{R}$.

Throughout this paper, we always assume the followings:

(A1) $W \in C^1((-1,1))$, $W(0) = 0$, $W'(v) > 0$ *for all* $v \in (-1,1)$, *and* $M := \sup\{ |W(v)| \, | v \in (-1,1)\} < \infty$;

(A2) $G^+ \in C^0((-\infty, 1])$, $G^- \in C^0([-1,\infty))$, $G^{\pm}(\pm 1) \neq 0$, and $\pm G^{\pm}(v) > 0$ if $\pm W(v) \leq 0$.

The condition that $\pm G^{\pm}(v) > 0$ if $\pm W(v) \leq 0$ (i.e., if $\pm v \leq 0$) is crucial in our subsequent analysis. It implies that any interface will propagate without changing direction, until it annihilates with another approaching interface or meets a nucleation point.

In the sequel, we say that a (not necessarily bounded) function $T(\cdot)$ on $\mathbb{R}$ is *Lipschitz* if there exists a constant $L > 0$ such that $|T(x_1) - T(x_2)| \leq L\, |x_1 - x_2|$ for any $x_1, x_2 \in \mathbb{R}$; we write

$$|T'(x)| := \limsup_{y \to x} |\frac{T(y) - T(x)}{y - x}|.$$

Also, $\{(x,t)| \, x \in \mathbb{R}, \ t \geq T(x)\}$ is abbreviated as $\{t \geq T\}$. Our main result is as follows.

Theorem 1. (Existence and Uniqueness of Initial Value Problem) *Let $\Omega_{\pm} \subset \mathbb{R}$ and $v_0(x) : \mathbb{R} \to \mathbb{R}$ be given. Assume that Ω_+ and Ω_- are disjoint and open, that $\partial\Omega_+ = \partial\Omega_- =: \Gamma_0$ has finitely many points, and that $\Omega_+ \cup \Omega_- \cup \Gamma_0 = \mathbb{R}$. Also assume that $v_0(x)$ is bounded and Lipschitz continuous in $\mathbb{R}$, $\pm v_0 < 1$ in $\Omega^{\pm}$, and $W(v_0) \neq 0$ on Γ_0. Then problem* **(P)** *has a unique weak solution (v, Q^+, Q^-) in $\mathbb{R} \times [0, \infty)$ satisfying $v(x, 0) = v_0(x)$ on $\mathbb{R}$ and $\{x \,|\, (x,0) \in Q^{\pm}\} = \Omega_{\pm}$.*

In order to prove Theorem 1, we consider a more general problem, the Cauchy problem, where the initial value of v and the location of the phase regions are specified on a curve in the space-time domain.

Definition 2. *Let $T : \mathbb{R} \to [0, \infty)$ and $\psi : \mathbb{R} \to \mathbb{R}$ be functions and Ω_+, Ω_- be sets in $\mathbb{R}$. We say that (v, Q^+, Q^-) has Cauchy data $(T, \psi, \Omega_+, \Omega_-)$ if $v(x, T(x)) = \psi(x) \quad \forall x \in \mathbb{R}$ and $\{x \mid (x, T(x)) \in Q^{\pm}\} = \Omega_{\pm}$.*

To ensure the existence of a unique solution for the Cauchy problem, we provide, for the Cauchy data, a sufficient condition, which we call property **S**, defined as follows:

Definition 3. *A quadruple $(T, \psi, \Omega_+, \Omega_-)$ is said to have property* **S** *(solvable) and write $(T, \psi, \Omega_+, \Omega_-) \in$* **S** *if the followings hold:*

(S1) *Ω_+, $\Omega_- \subset \mathbb{R}$ are open and disjoint, $\partial\Omega_+ = \partial\Omega_- =: \Gamma_0$ consists of a finite number of points, and $\Omega_+ \cup \Omega_- \cup \Gamma_0 = \mathbb{R}$;*

(S2) *$\psi : \mathbb{R} \to \mathbb{R}$ is bounded, Lipschitz continuous, and $\pm\psi < 1$ in $\Omega_{\pm}$;*

(S3) *The function $T : \mathbb{R} \to [0, \infty)$ is Lipschitz continuous and satisfies $\pm W(\psi)\, |T'| < 1$ on $\bar{\Omega}_{\pm}$;*

(**S4**) $W(\psi) \neq 0$ *on* Γ_0.

Theorem 2. *Let* $(T, \psi, \Omega_+, \Omega_-) \in \mathbf{S}$. *Then* (**P**) *has a unique solution on* $\{t \geq T\}$ *with Cauchy data* $(T, \psi, \Omega_+, \Omega_-)$.

Note that Theorem 1 is just a special case of Theorem 2 with $T \equiv 0$. .

Remark 2.3. 1. The condition (**S1**) (except the finiteness of Γ_0) is necessary to ensure the uniqueness of a solution. Here for simplicity, we assume that Γ_0 consists of finitely many points. We expect that this is general enough in real applications, and in the special case when Γ_0 does consist of infinitely many points, a unique solution can be obtained by taking the limit of the unique solution with Γ_0 finite.

2. As mentioned earlier, condition (**S2**) is only a compatibility condition for the existence of a solution.

3. Condition (**S3**) is simply a non–characteristic condition on the curve where Cauchy data is given for the pde $\Gamma_t = W(v)$ (regarding Γ as the zero level set of ϕ which solves $\phi_t = |\nabla\phi| W(v)$).

4. Finally condition (**S4**) is one of the keys in our uniqueness proof. Indeed, as can be seen from a non-uniqueness example given in §3.4, if (**S4**) does not hold, there exist, in general, infinitely many solutions.

The rest of the paper is organized as follows. In §3, we give several examples to illustrate the generic behavior of solutions to (**P**). Also, we give a non-uniqueness example demonstrating the necessity of (**S4**) for the uniqueness. §§4–5 are dedicated to the proof of Theorem 2.

3. Examples of Solutions

There are three distinguished cases according to the combination of the signs of $G^{\pm}(\pm 1)$ [6, Chapter 4].

(1) $G^+(1) < 0$ and $G^-(-1) > 0$. This is referred to as a *Bistable* case, since there exists an equilibrium in each of the "$\pm$" phase. Also $G^{\pm}(1) < 0 < G^{\pm}(-1)$ and the equation $v_t = G^{\pm}(v)$ imply that v cannot reach ± 1, so that nucleation will not occur.

(2) $G^+ > 0$ in $(-\infty, 1]$ and $G^- < 0$ in $[-1, \infty)$. This case is called *Oscillatory* since the phase at any point switches between "+" and "−" phases infinitely many times.

(3) Neither (1) nor (2). We call this case *Excitable* since nucleation can occur, and at any fixed point x, the phase changes only finitely many times and v eventually rests at one of the zeros of $G^{\pm}$.

For convenience, we use $\Phi^{\pm}(\alpha, t)$ to denote the solution of the following ode

$$(3.1) \qquad \begin{cases} \Phi^{\pm}_t & = & G^{\pm}(\Phi^{\pm}), \\ \Phi^{\pm}|_{t=0} & = & \alpha \end{cases} \quad \Longleftrightarrow \quad t = \int_{\alpha}^{\Phi^{\pm}(\alpha,t)} \frac{ds}{G^{\pm}(s)}.$$

3.1. **The Oscillatory Case.** For simplicity, we assume $W(v) = v$, $G^+ \equiv 1$, and $G^- \equiv -1$. Then $\Phi^{\pm}(\alpha, t) = \alpha \pm t$. We consider the initial value $v(x, 0) = \frac{1}{2}\cos(\omega x)$, $\Omega_+ = \mathbb{R}$ and $\Omega_- = \emptyset$, where ω is a parameter.

When $\omega = 1$, the solution is also periodic in time, and is given by

$$v(x,t) = (-1)^j(1 - T_{j+1}(x) + t), \qquad \forall x \in \mathbb{R},\ t \in [T_j(x), T_{j+1}(x)],\ j = 0, 1,, \cdots,$$

$$Q^+ = \{(x,t)\,|x \in \mathbb{R},\ T_{2k}(x) < t < T_{2k+1}(x), k \geq 0\} \cup \mathbb{R} \times \{0\},$$

$$Q^- = \{(x,t)\,|\ x \in \mathbb{R},\ T_{2k+1}(x) < t < T_{2k+2}(x), k \geq 0\},$$

where $T_0 \equiv 0$ and $T_j(x) = 2j - 1 - \frac{1}{2}\cos x$ for all integer $j \geq 1$.

Notice that initially the system is uniformly in "+" phase state. At each $x \in \mathbb{R}$, the phase switches between the "+" phase and the "−" phase at time $t = T_j(x)$, $j = 1, 2, \cdots$; all of these phase changes are due to nucleation. In this particular example, the effect of propagation of interface is totally suppressed by nucleation. Indeed, the speed of propagation of interface is $|W(v)|\ |_\Gamma = 1$, whereas the "speed" due to nucleation is $|\frac{dx}{dt}| = |\frac{dx}{dT_j(x)}| = |\frac{2}{\sin(x)}|$.

If $\omega > 2$, then both nucleation and propagation play roles in the evolution of the interface. Consider a half period interval $[0, \pi/\omega]$. Let $x^* = \frac{1}{\omega}\arcsin(2/\omega)$. Then at each $x \in [0, x^*]$, the phase switches due to nucleation from "+" to "−" at time $T = 1 - v_0(x)$ at which $v = 1$. At each $x \in (x^*, \pi/\omega]$, the phase can change either by nucleation which occurs at time $1 - v_0(x)$, or by the propagation of interface from neighboring points, depending on which occurs earlier. Indeed, solving equation, for $t = \hat{T}(z)$,

$$\begin{cases} \frac{dz}{d\hat{T}(z)} = \hat{T} + v_0(z) = \hat{T} + \frac{1}{2}\cos(\omega z), & z > x^* \\ \hat{T}(x^*) = 1 - v_0(x^*), \end{cases}$$

we see that $\hat{T}(x) < 1 - v_0(x)$ for $x \in (x^*, x^{**})$ where $x^{**} > x^*$ is the point $\hat{T}(x^{**}) = 1 - v(x^{**})$. Hence, the first layer of interface (in $x \in [0, \pi/\omega]$) is given by $t = 1 - v_0(x)$ for $x \in [0, x^*]$, $t = \hat{T}(x)$ for $x \in [x^*, \min\{\pi/\omega, x^{**}\}]$ and $t = 1 - v_0(x)$ for $x \in (\min\{\pi/\omega, x^{**}\}, \pi/\omega]$ (if it is not empty).

For other layers of the interface, the idea is similar, but the computation is much more involved.

3.2. **The Bistable Case.** We assume that $W(v) = v$, $G^+(v) = \frac{1}{2} - v$, and $G^-(v) = -\frac{1}{2} - v$. Solving (3.1) gives $\Phi^{\pm}(\alpha, t) = \pm\frac{1}{2}(1 - e^{-t}) + \alpha e^{-t}$.

We consider initial value given by $\Omega_+ = (1,2) \cup (3,4) \cup (5, \infty)$, $\Omega_- = \mathbf{R}\backslash\bar{\Omega}_+$ and $v(x, 0) = -1/2$ for $x \leq 4$, and $= \frac{1}{2}$ for $x > 5$, $= -\frac{1}{2} + (x - 4)$ for $x \in (4, 5]$. We denote the interface curve starting from $x = i$, $i = 1, 2, \ldots, 5$ as s_i. We further assume that all the interface curves retain their initial directions. That is, s_1, s_3 are decreasing and the remaining ones are increasing. In addition, s_2 and s_3 intersect at some time $t > 0$. Then we can obtain the regions Q^+, Q^- and the interface of the solution as follows.

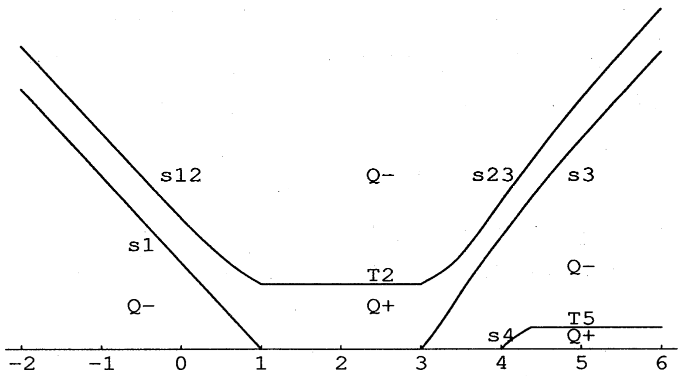

FIGURE 1. Interfaces for the excitable case example

Below and on $x = s_1(t)$, $v(x,t) = \Phi^-(v_0(x),t) = -\frac{1}{2}$. Hence solving $s_1' = W(v(s_1,t)) = -\frac{1}{2}$ gives $s_1(t) = -\frac{t}{2}+1$ for all $t \geq 0$. Similarly, $v(x,t) = \Phi^+(v_0(x),t) = \frac{1}{2}$ for $x \geq s_5(t) = 5+\frac{1}{2}t$, $t \geq 0$.

Below and on s_2 and s_3, $v(x,t) = \Phi^-(v_0(x),t) = -\frac{1}{2}$, so that $s_2(t) = 2+\frac{1}{2}t$ and $s_3(t) = 3-\frac{1}{2}t$ for $0 \leq t \leq 1$. At $t=1$, $s_2 = s_3 = \frac{5}{2}$ and the two interfaces annihilate.

Below $x = s_4(t)$ and above $x = s_5(t)$, $v(x,t) = \Phi^-(v_0(x),t)$ for $x \in (4,5)$ and $v(x,t) = \Phi^-(v(x,T_5(x)), t-T_5(x))$ for $x > 5$ where $t = T_5(x) = 2(x-5)$ is the inverse of $x = s_5(t) = 5+\frac{1}{2}t$. Hence, the inverse $t = T_4(x)$ of $x = s_4(t)$ solves $\frac{dx}{dT_4(x)} = -\Phi^-(v_0,T_4)$ for $x \in [4,5]$ and $\frac{dx}{dT_4(x)} = -\Phi^-(\Phi^+(v_0,T_5),T_4-T_5) = \frac{1}{2}-e^{-T_4+2(x-5)}$ for $x > 5$. This equation has a unique monotonic solution $T_4(x)$ for all $x \geq 4$ and it satisfies $T_4(x) > T_5(x)$ for all $x > 5$.

Finally, the region above the curves $x = s_1, s_2, s_3$, and s_4 belongs to Q^+ and v can be obtained by solving $v_t = G^+(v)$ together with known "initial" values on $x = s_1, s_2, s_3, s_4$.

It is easy to verify that such obtained (v, Q^+, Q^-) is a solution to the given initial value problem, and is the only solution by Theorem 1.

3.3. **The Excitable Case.** We take $W(v) = v$, $G^+ \equiv 1$, and $G^-(v) = -\frac{1}{2} - v$. Then $\Phi^+(\alpha,t) = \alpha+t$ and $\phi^-(\alpha,t) = -\frac{1}{2}+(\alpha+\frac{1}{2})e^{-t}$. We consider an initial data given by $\Omega_- = (-\infty,1) \cup (3,4)$, $\Omega_+ = \mathbf{R}\backslash\bar{\Omega}_-$, and $v(x,0) = -\frac{1}{2}$ for $x \leq 3$, $= \frac{1}{2}$ for $x > 4$, and $= -\frac{1}{2}+(x-3)$ for $x \in (3,4)$.

Figure 1 shows the regions Q^+, Q^- and the interface of the solution to this initial value problem.

Below and on $x = s_1$, $v(x,t) = -\frac{1}{2}$. Consequently, $s_1(t) = 1-\frac{1}{2}t$.

The interface $x = s_4(t) = 4+\frac{1}{2}t+\frac{1}{2}t^2$ for $t \in [0,\frac{1}{2}]$ is due to propagation, and the interface $t = T_5(x) \equiv \frac{1}{2}$ for $x > 4\frac{3}{8}$, on which $v = 1$, is due to nucleation.

Below $x = s_3$ (and above $x = s_4, t = T_5$), v can be calculated by $v_t = G^-(v)$ and $s_3' = -W(v(s_3,t))$. One can show that $s_3' > 0$ for all $t \geq 0$ and $s_3'(t) \to \frac{1}{2}$ as $t \to \infty$.

For $x \in [1,3]$, $v = \Phi^+(v_0,t) = -\frac{1}{2}+t$ for all $t < \frac{3}{2}$ and nucleation occurs at $t = T_2(x) \equiv \frac{3}{2}$.

For $s_1 < x < 1$ the interface at $\{x = 1, t = \frac{3}{2}\}$ will propagate, while nucleation may take a role. Calculation under the assumption of nucleation and propagation respectively tell us that only propagation takes a role. Hence below $x = s_{12}$ and above $x = s_1$, $v = \Phi^+(v(x, T_1(x)), t - T_1(x)) = -\frac{1}{2} + t - T_1(x)$ where $T_1(x) = 2(1-x)$ is the inverse of $x = s_1(t)$. Solving $s_{12}' = -W(v(s_{12}, t))$ with initial value $s_{12}(\frac{3}{2}) = 1$ then gives $s_{12}(t) = \frac{3}{2} - \frac{1}{2}t + \frac{1}{4}e^{3-2t}$ for all $t \geq \frac{3}{2}$. Now we can check that on $x = s_{12}$, $v = \frac{1}{2}(1 + e^{3-2t}) < 1$ for all $t > \frac{3}{2}$, and hence the interface $x = s_{12}$ is indeed due to propagation. Similarly we can calculate s_{23}.

We remark that in a general situation, the calculation of s_{12}, T_2, and s_{23} is much more involved, and should be proceeded as follows:

(i) Pretend that $v_t = G^+(v)$ for the rest of the domain and find a curve $t = T^*(x)$ on which $v = 1$. Nucleation occurs only at points on the curve $t = T^*(x)$.

(ii) At every point $(y, T^*(y))$, calculate an interface $t = h(y, T^*(y); \cdot)$ based solely on propagation.

(iii) Take the infimum of $h(y, T^*(y); \cdot)$ for all y. This infimum is then the required interface.

3.4. **A Non–uniqueness Example.** We consider a bistable case where $W(v) = v$ and $G^\pm(v) = \pm\frac{1}{2} - v$. Then $\Phi^\pm(\alpha, t) = \pm\frac{1}{2}(1 - e^{-t}) + \alpha e^{-t}$. We consider the initial value $\Omega_+ = (0, \infty)$, $\Omega_- = (-\infty, 0)$, and $v(x, 0) = v_0(x) \equiv 0$. Note that $W(v(x, 0)) = 0$ on $\Gamma_0 = \{0\}$ so that Theorem 1 cannot be applied.

This initial value problem has infinitely many solutions. We next construct explicitly two of them.

The first solution we are going to give has only one interface, which is given by $x = s_1(t) := -\frac{1}{2}(t + e^{-t} - 1)$, $Q^\pm = \{\pm(x - s_1) > 0\}$, $v = \Phi^-(v_0(x), t) = \frac{1}{2}(e^{-t} - 1)$ for $x < s_1$, $= \Phi^+(v_0, t) = \frac{1}{2}(1 - e^{-t})$ for $x \geq 0$, and $= \Phi^+(v(x, T_1(x)), t - T_1(x)) = \frac{1}{2}(e^{-t} + 1 - 2e^{T_1(x) - t})$ for $s_1 < x < 0$, where $t = T_1(x)$ is the inverse function of $x = s_1(t)$. It is easy to verify that $s_1'(t) = W(v(s_1, t))$ and that (v, Q^+, Q^-) is a solution.

This solution can be obtained as the limit of unique solutions to a sequence of initial value problems of (**P**). Indeed, for any small positive ϵ, let $(v_\epsilon, Q_\epsilon^+, Q_\epsilon^-)$ be solution to (**P**) with initial data $\Omega^- = (-\infty, 0)$, $\Omega_+ = (0, \infty)$ and $v_\epsilon(x, 0) = -\epsilon$. By Theorem 1, $(v_\epsilon, Q_\epsilon^+, Q_\epsilon^+)$ exists and is unique, and $Q^\pm$ is given by $Q_\epsilon^\pm = \{\ \pm(x - s^\epsilon(t)) > 0\ \}$ where $s^\epsilon(t) = s_1(t) + \epsilon(e^{-t} - 1)$. Hence, as $\epsilon \searrow 0$, $(v^\epsilon, Q_\epsilon^+, Q_\epsilon^-) \to (v, Q^+, Q^-)$.

The second solution we shall present here has three interfaces, which are given by $t = T_1(x), T_2(x)$, and $T_{1*}(x)$, where $T_{1*}(x) = T_1(-x)$ for $x \geq 0$, and $T_2(x)$ solves

$$(3.2)\quad \frac{dx}{dT_2(x)} = -\tfrac{1}{2}(e^{-T_2} + 1 - 2e^{T_1 - T_2}) \text{ and } T_2(x) > T_1(x) \quad \forall x < 0, \qquad \lim_{x \nearrow 0} T_2(x) = 0.$$

By considering T_1 as the independent variable and writing $\frac{dT_2}{dT_1} = \frac{dT_2}{dx}\frac{dx}{dT_1} = \frac{1 - e^{-T_1}}{1 + e^{-T_2} - 2e^{T_1 - T_2}}$, we can show that (3.2) has a unique solution T_2 for all $x < 0$; we omit the details.

This solution, again, can be obtained as a limit of unique solutions of a sequence of initial value problems. Consider, for every small positive ϵ, the initial value $(v_0^\epsilon, \Omega_+^\epsilon, \Omega_-^\epsilon)$

given by $\Omega_-^\epsilon = (-\infty, -\epsilon) \cup (0, \epsilon)$, $\Omega_+^\epsilon = \mathbb{R} \setminus \bar{\Omega}_-^\epsilon$, and $v_0^\epsilon(x) = -\epsilon$ for $x < -\epsilon$, $= \epsilon + 2x$ for $x \in (-\epsilon, 0]$, and $= \epsilon$ for $x > 0$. Since $W(v_0^\epsilon) \neq 0$ on $\Gamma_0^\epsilon := \{-\epsilon, 0, \epsilon\}$, by Theorem 1, this initial value problem has a unique solution $(v_\epsilon, Q_\epsilon^+, Q_\epsilon^-)$. Simple calculation shows that this solution has three interfaces, given by $t = T_1^\epsilon(x), T_2^\epsilon(x)$ and $T_1^{\epsilon*}$, where $t = T_1^\epsilon(x)$ is the inverse of $x = s^\epsilon(t) := -\frac{1}{2}(t + e^{-t} - 1) + \epsilon(e^{-t} - 1)$, $T_1^{\epsilon*}(x) = T_1^\epsilon(-x)$ and $T_2^\epsilon(x)$ solves a differential equation analogous to (3.2) for $x \leq -\epsilon$ whereas for $x \in (-\epsilon, 0]$, T_2 is monotonically decreasing and $T_2(-\epsilon) = O(\sqrt{\epsilon})$. Sending $\epsilon \searrow 0$, we can show that $(v_\epsilon, Q_\epsilon^+, Q_\epsilon^-)$ approaches the second solution we gave.

In a similar manner, we can obtain solutions with arbitrary odd number of interfaces. All these solutions are classical for $t > 0$.

Remark 3.1. We believe that every weak solution in our definition is "physical" in the sense that it can be obtained as a limit of a sequence of solutions of (1.1) as $\varepsilon \to 0$. For example, consider the second solution (v, Q^+, Q^-) we constructed, and also the solution $(v_\epsilon, Q_\epsilon^+, Q_\epsilon^-)$ we mentioned. Since for every fixed $\epsilon > 0$, $(v_\epsilon, Q_\epsilon^+, Q_\epsilon^-)$ is unique, one can show, by the analysis in [1, 2, 3], that there exists a sequence $\{(u_\epsilon^\varepsilon(x,0), v_\epsilon^\varepsilon(x,0))\}_{\varepsilon>0}$ of initial values to (1.1) such that, as $\varepsilon \to 0$, the solutions $(u_\epsilon^\varepsilon, v_\epsilon^\varepsilon)$ to (1.1) with these initial values have the limit $(v_\epsilon, Q_\epsilon^+, Q_\epsilon^-)$ (namely, $v_\epsilon^\varepsilon \to v_\epsilon$ in $\mathbb{R} \times [0, \infty)$ and $u_\epsilon^\varepsilon \to h^\pm(v_\epsilon)$ in $Q_\epsilon^\pm$). Upon selecting a subsequence from the double indexes (ε, ϵ), we then can conclude that the second solution can be obtained as a limit of the solutions of (1.1) as $\varepsilon \to 0$.

4. Dynamics of Interfaces

In this section, we study the evolution of the interface according to the motion equation $\Gamma_t = W(v)$ and the nucleation mechanics. We investigate the shrinkage of the "+" phase region and the expansion of the "−" phase region. (The opposite phase change is analogous.)

4.1. **Shrinkage of the "+" phase region.** We denote by $\Phi^\pm(\alpha, t)$ the solution to (3.1). For convenience, we extend $G^\pm(v)$ by zero for $\pm v \geq 2$ and by a linear interpolation for $\pm v \in (1, 2)$. Also, we extend $W(v)$ by the constant $W(\pm 1)$ for all $\pm v > 1$. Since the values of $G^+(v)$ for $v > 1$, $G^-(v)$ for $v < -1$, and $W(v)$ for $|v| \geq 1$ are not used for any solution to (**P**), these extensions will not affect our final result.

Consider (**P**) with Cauchy data $(T, \psi, \Omega_+, \Omega_-)$ in the domain $\{t \geq T(x)\} := \{(x,t) \mid x \in \mathbb{R}, t \geq T(x)\}$. Let $(a, b) \subset \Omega_+$ be an interval such that $a, b \notin \Omega_+$, $W(v(a, T(a))) > 0$, and $W(v(b, T(b))) > 0$, so that "initially" (i.e., $t = T$) the "+" phase region is shrinking.

Propagation and annihilation of interfaces. Let's assume, for the moment, that there is no nucleation. Then interfaces started at $(a, T(a))$ and $(b, T(b))$ can be written as $x = s^{\mathrm{R}}(t)$ and $x = s^{\mathrm{L}}(t)$ respectively, where

$$\frac{ds^{\mathrm{R}}}{dt} = W(v(s^{\mathrm{R}}, t)), \qquad -\frac{ds^{\mathrm{L}}}{dt} = W(v(s^{\mathrm{L}}, t)). \tag{4.1}$$

The curve $t = T(\cdot)$, on which the Cauchy data is given, is "characteristic" to equations in (4.1) at points where $|\frac{dx}{dT}| = W(\psi)$. For this reason, we impose the "non–characteristic" condition $\pm W(\psi)|T'| < 1$ on $\bar{\Omega}_\pm$.

Suppose we know a priori that s^{R} and s^{L} are monotonic. Then the region below $x = s^{\mathrm{R}}$ and $x = s^{\mathrm{L}}$ is in Q^+ (since nucleation is ignored). Hence, solving $v_t = G^+(v)$ in this region gives $v(x,t) = \Phi^+(\psi(x), t - T(x))$. Consequently, (4.1) can be solved uniquely in terms of (T, ψ, a, b). As a part of a guess–and–check process, we shall show below in Lemma 4.1 that such uniquely obtained functions s^{R} and s^{L} are indeed strictly monotonic. For this we need the condition that $\pm G^\pm(v) > 0$ for $\mp v \geq 0$. In such a manner, we obtain a whole component of the interface being the union of the curve $x = s^{\mathrm{R}}(t)$ for $t \in [T(a), t^*]$ and the curve $x = s^{\mathrm{L}}(t)$ for $t \in [T(b), t^*]$, where t^* is the time such that $s^{\mathrm{R}}(t^*) = s^{\mathrm{L}}(t^*)$, i.e., the time of annihilation of the two interfaces starting from $(a, T(a))$ and $(b, T(b))$ respectively.

Note that the union of the two curves $x = s^{\mathrm{R}}(t)$ and $x = s^{\mathrm{L}}(t)$ for $t \leq t^*$ is a graph in x. Hence, it is convenient to use the inverse function of $x = s^{\mathrm{R,L}}$. We denote by $t = h(y, \mu; x)$ the inverse of $x = s^{\mathrm{R}}(y, \mu; t)$ for $x \geq y$ and $x = s^{\mathrm{L}}(y, \mu; t)$ for $x \leq y$, where $s^{\mathrm{R,L}}(y, \mu; t)$ are solutions to (4.1) with initial data $s^{\mathrm{R,L}}(y, \mu; \mu) = y$. Then $h(y, \mu; \cdot)$ solves

$$(4.2)\quad \operatorname{sgn}(x-y)\frac{dx}{dh(y,\mu;x)} = W(\Phi^+(\psi(x), h - T(x))) \quad \text{for } x \in \mathbb{R}\setminus\{y\}, h(y,\mu;y) = \mu,$$

where $\operatorname{sgn}(z) = 1$ if $z > 0$ and $\operatorname{sgn}(z) = -1$ if $z < 0$. The whole component of the interface mentioned earlier then can be written as $t = H(x)$ for $x \in (a, b)$, where $H(x) = \min\{h(a, T(a); x),\ h(b, T(b); x)\}$. The lens shape region $\{T(x) \leq t < H(x)\}$ is one component of Q^+.

Nucleation of phase regions. Next we take into account the nucleation. Let $y \in (a, b)$ be an arbitrary fixed point. If the phase at y is not affected by the expansion of neighboring "−" phase regions, then, due to the nucleation mechanics, it will change from the "+" phase to the "−" phase at time $T^*(y)$ at which $v = 1$. Once the phase at y is changed, the new "−" phase region $\{y\}$ will expand to change the phase of its neighboring points. Hence, at any point $x \in (a, b)$, the phase will be changed at a time no later than $h(y, T^*(y); x)$, or more precisely, no later than $H(T, \psi, a, b; x)$ defined by

$$(4.3)\quad H(T,\psi,a,b;x) := \begin{cases} \inf\{\, h(y, T^*(y); x) \,|\, y \in [a,b]\cap\mathbb{R} \,\} & \text{if } x \in (a,b), \\ T(x) & \text{if } x \notin (a,b), \end{cases}$$

$$(4.4)\quad T^*(y) := \begin{cases} \sup\{t \geq T(y) \,|\, \Phi^+(\psi(y), \tau - T(y)) < 1\, \forall \tau \in [T(y), t)\} & \text{if } y \in (a,b), \\ T(y) & \text{if } y \notin (a,b). \end{cases}$$

Here we have used the obvious notation $[a, b] \cap \mathbb{R}$ to include cases where $a = -\infty$ and/or $b = \infty$. We also use the extension $h(y, T^*(y); \cdot) \equiv \infty$ if $T^*(y) = \infty$. As it turns out, $t = H(T, \psi, a, b; x)$ is precisely the first time of phase change from "+" to "−" at point

$x \in (a, b)$. Here we state without proof the well-definedness and a few properties of $H(x) = H(T, \psi, a, b; x)$.

Lemma 4.1. *Let $\psi \in L^\infty(\mathbb{R} \to \mathbb{R})$ and $T : \mathbb{R} \to [0, \infty)$ be Lipschitz, and $(a, b) \subseteqq \mathbb{R}$ be an interval such that*

$$\psi < 1 \quad in\ (a,b), \qquad W(\psi)\,|T'| < 1 \ on\ [a,b] \cap \mathbb{R}. \tag{4.5}$$

(1) For any $y \in [a, b] \cap \mathbb{R}$ and $\mu \in [T(y), \infty)$ satisfying $W(\Phi^+(\psi(y), \mu - T(y))) > 0$, problem (4.2) admits a unique solution $h(y, \mu; x)$ for all $x \in [a, b] \cap \mathbb{R}$, and the solution satisfies

$$T < h < \infty, \quad \frac{\operatorname{sgn}(x-y)}{h'} = W(\Phi^+(\psi, h - T)) > 0 \quad on \quad ([a,b] \setminus \{y\}) \cap \mathbb{R}. \tag{4.6}$$

(2) Assume in addition to (4.5) that

$$W(\psi(a)) > 0 \text{ if } \ a \in \mathbb{R}, \quad W(\psi(b)) > 0 \text{ if } \ b \in \mathbb{R}. \tag{4.7}$$

Define T^* as in (4.4) and H as in (4.3). Then either $\{(a, b) = \mathbb{R}, T^* \equiv \infty, H \equiv \infty\}$ or $H < \infty$ on $\mathbb{R}$ and the followings hold:

- **(a)** *For each $x \in [a, b] \cap \mathbb{R}$, there exists $y^x \in [a, b] \cap \mathbb{R}$ such that $H(\cdot) = h(y^x, T^*(y^x); \cdot)$ on the closed interval with end points x and y^x;*
- **(b)** *$H > T$ on (a, b), $W(\Phi^+(\psi, H - T)) > 0$ on $[a, b] \cap \mathbb{R}$, and $\Phi^+(\psi, t - T) < 1$ on $\{(x, t) \mid x \in (a, b), T(x) \le t < H(x)\}$;*
- **(c)** *For any $x_1 \in (a, b)$, there exists $\delta_0 = \delta_0(x_1) > 0$ such that for all $\delta \in (0, \delta_0)$,*

$$H(x_2) \ge H(x_1) - \delta \quad \forall\, x_2 \in B(x_1, c(\delta)\delta),$$

where $c(\delta) = \min_{\bar{B}(x_1, M\delta) \times [H(x_1) - \delta, H(x_1)]} \{W(\Phi^+(\psi, t - T))\} > 0$;
- **(d)** *H is Lipschitz continuous on $\mathbb{R}$.*

4.2. **Expansion of the "−" phase region.** For any point $(x_0, t_0) \in Q^-$, there are two driving forces that may change the phase at x_0. The first is an external force coming from the neighboring points on the "+" phase, but it will not be large enough to change the phase at x_0 if v at x_0 is positive. The other is an internal force due to nucleation, yet it will not change the phase at x_0 if $v > -1$. Thus, as long as $v > 0$ at x_0, the "−" phase at x_0 will not change. Consequently, $v(x_0, t) = \Phi^-(v(x_0, t_0), t - t_0)$ is valid at least up to the time v becomes zero. Based on this idea, we can prove the following lemma concerning the expansion of the "−" phase region.

Lemma 4.2. *Let (v, Q^+, Q^-) be a solution to* (**P**) *and $(x_0, t_0) \in \overline{Q^-}$ be a point such that $W(v(x_0, t_0)) > 0$. Let $[A, B]$ be a finite interval such that $x_0 \in (A, B)$ and the equation, for $h(\cdot)$,*

$$\operatorname{sgn}(x - x_0)\frac{d\,x}{d\,h(x)} = W(v(x, h(x))) > 0 \quad \forall\, x \in [A, B] \setminus \{x_0\}, \qquad h(x_0) = t_0 \tag{4.8}$$

has a solution on $[A, B]$. *Then for all* $x \in [A, B]$ *and* $t \in (h(x), h(x) + \int_{v(x,h(x))}^{0} \frac{ds}{G^-(s)})$,

$$(x,t) \in Q^- \quad \textit{and} \quad v(x,t) = \Phi^-(v(x,h(x)), t - h(x)).$$

We omit the proof here.

4.3. **A local existence and uniqueness result.** The following theorem shows that the curve $t = H(T, \psi, a, b; x)$ defined in (4.3) is actually a component of the interface, and the solution can be uniquely solved below and near $t = H$.

Theorem 3. *Let* $(T, \psi, \Omega_+, \Omega_-) \in \mathbf{S}$ *and* $(a, b) \subset \Omega_+$ *be an interval such that* $a \notin \Omega_+$, $b \notin \Omega_+$, *and (4.7) holds. Let* $H(x) = H(T, \psi, a, b; \cdot)$ *be defined as in Lemma 4.1. Set*

$$\mathcal{D} = \{(x,t) \mid x \in (a,b), T(x) \le t < H(x)\},$$

$$\hat{T} = H, \quad \hat{\psi} = \Phi^+(\psi, H - T), \quad \hat{\Omega}_- = (\Omega_- \cup [a,b]) \cap \mathbb{R}, \quad \hat{\Omega}_+ = \Omega_+ \setminus (a,b),$$

$$E_\eta = \left\{(x,t) \mid x \in (a-\eta, b+\eta), H(x) < t < H(x) + \int_{\hat{\psi}(x)}^{0} \frac{ds}{G^-(s)}\right\}.$$

Then the followings hold:

(I) $(\hat{T}, \hat{\psi}, \hat{\Omega}_+, \hat{\Omega}_-) \in \mathbf{S}$, $\hat{T} \gneqq T$, *and* $\hat{\Gamma}_0 = \Gamma_0 \setminus \{a, b\}$ *where* $\hat{\Gamma}_0 := \partial\hat{\Omega}_\pm$ *and* $\Gamma_0 := \partial\Omega_\pm$;

(II) *If* (v, Q^+, Q^-) *is a solution to* (**P**) *on* $\{t \ge T(x)\}$ *with Cauchy data* $(T, \psi, \Omega_+, \Omega_-)$, *then*

(a) $\mathcal{D} \subset Q^+$ *and* $v(x,t) = \Phi^+(\psi(x), t - T(x))$ *on* $\bar{\mathcal{D}}$,

(b) $E_\eta \subset Q^-$ *and* $v(x,t) = \Phi^-(\hat{\psi}(x), t - H(x))$ *on* $\bar{E}_\eta$ *for some* $\eta > 0$, *and*

(c) *the following defined* $(\hat{v}, \hat{Q}^+, \hat{Q}^-)$ *solves* (**P**) *on* $\{t \ge \hat{T}(x)\}$ *with Cauchy data* $(\hat{T}, \hat{\psi}, \hat{\Omega}_+, \hat{\Omega}_-)$:

$$\hat{v} = v, \quad \hat{Q}^- = Q^- \cup \{(x, H(x)) \mid x \in [a,b]\}, \quad \hat{Q}^+ = Q^+ \setminus \mathcal{D};$$

(III) *If* $(\hat{v}, \hat{Q}^+, \hat{Q}^-)$ *is a solution to* (**P**) *on* $\{t \ge \hat{T}(x)\}$ *with Cauchy data* $(\hat{T}, \hat{\psi}, \hat{\Omega}_+, \hat{\Omega}_-)$, *then the following defined* (v, Q^+, Q^-) *is a solution to* (**P**) *on* $\{t \ge T(x)\}$ *with Cauchy data* $(T, \psi, \Omega_+, \Omega_-)$:

$$v(x,t) = \begin{cases} \hat{v}(x,t) & \textit{if } t \ge \hat{T}(x), \\ \Phi^+(\psi(x), t - T(x)) & \textit{if } T(x) \le t < \hat{T}(x), \end{cases}$$

$$Q^- = \hat{Q}^- \setminus \{(x, \hat{T}(x)) \mid x \in [a,b]\}, \qquad Q^+ = \hat{Q}^+ \cup \mathcal{D};$$

(IV) (**P**) *has a unique solution on* $\{t \ge T(x)\}$ *with Cauchy data* $(T, \psi, \Omega_+, \Omega_-)$ *if and only if* (**P**) *has a unique solution on* $\{t \ge \hat{T}(x)\}$ *with Cauchy data* $(\hat{T}, \hat{\psi}, \hat{\Omega}_+, \hat{\Omega}_-)$.

The proof is very tedious and is omitted.

5. Proof of Theorem 2

The idea of the proof of Theorem 2 is to use repeatedly Theorem 3 (and it's companion for the case $(a,b) \subset \Omega_-$) to reduce the problem into a simple case where $\Gamma_0 = \partial\Omega_\pm = \emptyset$. Then use again Theorem 3 for the case $(a,b) = \mathbb{R}$ to construct, layer by layer in the space-time domain, a unique solution.

Proof of Theorem 2. Let $(T, \psi, \Omega_+, \Omega_-) \in \mathbf{S}$ be given. We prove the existence of a unique solution to **(P)** on $\{t \geq T(x)\}$ with Cauchy data $(T, \psi, \Omega_+, \Omega_-)$ in two steps.

Step 1. We assume that $\Gamma_0 \neq \emptyset$; otherwise, we go directly to Step 2.

First we find a maximal connected component (a,b), of either Ω_+ or Ω_-, for which we can apply Theorem 3 (or its companion for " $-$ ") to transfer the Cauchy problem to a simpler one.

We assign every point in the set $\Sigma := \{-\infty\} \cup \Gamma_0 \cup \{\infty\}$ a letter either "R" or "L", depending on the initial direction (Right or Left) of the motion of interface at that point. As a default, we assign "R" to $\{-\infty\}$ and "L" to $\{\infty\}$. Since $W(\psi) \neq 0$ on Γ_0, the assignment is well–defined. Now appending all the letters assigned to Σ in the same order as the corresponding points in Σ appeared on the real line, we obtain a word consisting of two letters, "R" and "L". By the default, this word begins with "R" and ends with "L". Hence, there is a first place where the letter "R" is followed by "L". Let's denote the corresponding points by a and b respectively. Then either (i) $(a,b) \subset \Omega_+$, $W(\psi(a)) > 0$ (if a is finite) and $W(\psi(b)) > 0$ (if b is finite), or (ii) $(a,b) \subset \Omega_-$, $W(\psi(a)) < 0$ (if a is finite) and $W(\psi(b)) < 0$ (if b is finite). Without loss of generality, we assume that (i) happens.

Now with the given $(T, \psi, \Omega_+, \Omega_-) \in \mathbf{S}$ and such (uniquely) chosen interval (a,b), we can apply Theorem 3 to obtain a new Cauchy data $(\hat{T}, \hat{\psi}, \hat{\Omega}_+, \hat{\Omega}_-) \in \mathbf{S}$ such that **(P)** with Cauchy data $(T, \psi, \Omega_+, \Omega_-)$ has a unique solution if and only if **(P)** with Cauchy data $(\hat{T}, \hat{\psi}, \hat{\Omega}_+, \hat{\Omega}_-)$ has a unique solution. One notices that $\hat{\Gamma}_0 := \partial\hat{\Omega}_\pm = \Gamma_0 \setminus \{a,b\}$ has at least one point less than Γ_0 does.

Applying this process finitely many times, we then find $(\tilde{T}, \tilde{\psi}, \tilde{\Omega}_+, \tilde{\Omega}_-) \in \mathbf{S}$ such that either $\tilde{\Omega}_- = \mathbb{R}$ or $\tilde{\Omega}_+ = \mathbb{R}$, and that problem **(P)** on $\{t \geq T(x)\}$ with Cauchy data $(T, \psi, \Omega_+, \Omega_-)$ is equivalent to **(P)** on $\{t \geq \tilde{T}(x)\}$ with Cauchy data $(\tilde{T}, \tilde{\psi}, \tilde{\Omega}_+, \tilde{\Omega}_-)$.

Step 2. Assume either $\Omega_- = \mathbb{R}$ or $\Omega_+ = \mathbb{R}$. Without loss of generality, we assume that $\Omega_+ = \mathbb{R}$. We consider separately the following three cases: (i) $G^+(1) < 0$; (ii) $G^+(1) > 0$ and $G^-(-1) > 0$; (iii) $G^+(1) > 0$ and $G^-(-1) < 0$.

Case (i): $G^+(1) < 0$. This case is either *bistable* (when $G^-(-1) > 0$) or *excitable* (when $G^-(-1) < 0$).

Since $\psi < 1$ on $\Omega_+ = \mathbb{R}$, the definition of T^* in (4.4) gives $T^*(\cdot) \equiv \infty$, so that $H(T, \psi, -\infty, \infty; \cdot) \equiv \infty$. By Theorem 3 **(II)**(a) with $(a,b) = \mathbb{R}$, the unique solution is given by

$$(5.1) \quad Q^- = \emptyset, \quad Q^+ = \{(x,t) \mid x \in \mathbb{R}, t \geq T(x)\}, \; v(x,t) = \Phi^+(\psi(x), t - T(x)) \text{ in } Q^+.$$

Case (ii): $G^+(1) > 0$ and $G^-(-1) > 0$. This corresponds to an *excitable* case.

By Lemma 4.1 with $(a,b) = \mathbb{R}$, either $H(\cdot) = H(T,\psi,-\infty,\infty;\cdot) \equiv \infty$ or $H(x) < \infty$ for all $x \in \mathbb{R}$.

If $H \equiv \infty$, there is a unique solution and it is given by (5.1).

If $H(x) < \infty$ for all $x \in \mathbb{R}$, we first apply Theorem 3 to $(T,\psi,\mathbb{R},\emptyset)$ and then apply a companion of Theorem 3 for the "$-$" phase change for $(H,\Phi^+(\psi,H-T),\emptyset,\mathbb{R})$ to conclude that there is a unique solution, given by

$$Q^- = \{t > H(x)\}, \quad Q^+ = \{T(x) \le t < H(x)\},$$

$$v(x,t) = \begin{cases} \Phi^+(\psi(x), t - T(x)), & (x,t) \in \overline{Q^+}, \\ \Phi^-(v(x, H(x)), t - H(x)), & (x,t) \in Q^-. \end{cases} \tag{5.2}$$

Case (iii): $G^+(1) > 0$ and $G^-(-1) < 0$. We consider three different situations:

(iii)(a) $\max_{[-1,0]}\{G^-\} \ge 0$;

(iii)(b) $G^- < 0$ on $[-1,\infty)$ and $\min_{[0,1]}\{G^+\} \le 0$;

(iii)(c) $G^- < 0$ on $[-1,\infty)$ and $G^+ > 0$ on $(-\infty,1]$.

As we shall see, cases (iii)(a) and (iii)(b) are *excitable* and (iii)(c) is *oscillatory.*

Case (iii)(a). If $T_1 = H(T,\psi,-\infty,\infty;x)$ is finite, then by Lemma 4.1 (2) (b), $\psi_1 := \Phi^+(\psi,H-T) > 0$ on $\mathbb{R}$. It then follows $T_1^*(y) \equiv \infty$ where

$$T_1^*(y) := \sup\{t \ge T_1(y) \mid \Phi^-(\psi_1(y),\tau - T_1(y)) > -1 \ \forall \tau \in [T_1(y),t)\} \quad \forall y \in \mathbb{R}. \tag{5.3}$$

Hence, same as the case (ii), the solution is unique, given by (5.1) (when $H \equiv \infty$) or (5.2) (when $H < \infty$).

Case (iii)(b). If $T_1 := H(T,\psi,-\infty,\infty;\cdot) \equiv \infty$. Then the unique solution is given by (5.1).

Suppose $T_1(x) < \infty$ for all $x \in \mathbb{R}$. Then T_1^* defined by (5.3) is bounded, since $G^- < 0$ on $[-1,\infty)$. Applying the companion Theorem 3 for the "$-$" case and using a similar reasoning as above we then conclude that there is a finite $T_2(\cdot) > T_1(\cdot)$ such that the solution is given uniquely by $Q^+ = \{T(x) \le t < T_1\} \cup \{t > T_2\}$, $Q^- = \{T_1(x) < t < T_2(x)\}$, and $v = \Phi^+(\psi,t-T)$ in $\{t \le T_1\}$, $v = \Phi^-(\psi_1,t-T_1)$ in Q^-, and $v = \Phi^+(\psi_2,t-T_2)$ in $\{t \ge T_2\}$ where $\psi_2 = \Phi^-(\psi_1,T_2-T_1)$.

Case (iii)(c). Same as before, we first apply Theorem 3 to obtain $(T_1,\psi_1,\Omega^1_+,\Omega^1_-) := (H(T,\psi,-\infty,\infty;\cdot),\Phi^+(\psi,H-T),\emptyset,\mathbb{R}) \in \mathbf{S}$. Note that $T_1 = H \le T^* < \infty$ since $G^+ > 0$ on $(-\infty,1]$. Applying a companion of Theorem 3 for the Cauchy data $(T_1,\psi_1,\Omega^1_+,\Omega^1_-)$ we then obtain $(T_2,\psi_2,\Omega^2_+,\Omega^2_-)$ where $\Omega^2_+ = \mathbb{R}$ and $\Omega^2_- = \emptyset$, and $T_2 < \infty$ since $G^- < 0$ on $[-1,\infty)$. Repeating this process we obtain a sequence $\{(T_j,\psi_j,\Omega^j_+,\Omega^j_-)\}_{j=1}^\infty$ in $\mathbf{S}$, where $T_j < T_{j+1} < \infty$ for all j, $\Omega^j_+ = \emptyset$ if j is odd, $\Omega^j_+ = \mathbb{R}$ if j is even. Hence in $\cup_{j=1}^\infty\{T(x) \le t \le T^j(x)\}$ the solution is uniquely determined.

With a considerable amount of technical effort, one can show that $\lim_{j\to\infty} T_j(x) = \infty$ for any $x \in \mathbb{R}$, and therefore complete the proof of Theorem 2. $\square$

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