数理解析研究所講究録
1178 巻 2000 年 150-161

Smooth Solutions of Porous Medium Equations

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1 Introduction

In this paper, we are concerned with Cauchy problem of the porous medium equations:

$$(\mathrm{P})\left\{\begin{array}{ll} u_t = \operatorname{div}\left(u^{\ell}\nabla u\right), & (x,t)\in \mathbb{R}^N\times[0,\infty), \\ u(x,0)=u_0(x), & x\in\mathbb{R}^N, \end{array}\right. \tag{1}$$

$$\ell : \text{even natural number},$$

$$u_0(x)\in\bigcap_{m=0}^{\infty}H^m(\mathbb{R}^N).$$

This type of equations have been studied by so many people and many interesting results have been obtained so far. Among them, concerning the regularity of solutiions, the Hölder continuity of u has been well known. However, as for the estimate for the derivative of u itself, little is known. It is also well known that (P) does not admit any time global classical solutions for the case where $\ell \geq 1$ and $u_0 \geq 0$, (see Kalashnikov [8]). On the other hand, in our recent works [13], [14], and [15], we constructed a time local solution which is smooth with respect to time and space variables. Our aim here is to investigate the nonexistence of time global classical soutions and the existence of time local smooth solutions of (P). Our main results are given in the next section, and the sketch of their proofs are given in §3.

2 Main Results

As for the nonexistence of time global classical solutions, we work for the one dimensional problem:

$$
(\mathrm{P})_1 \begin{cases} u_t = \left(|u|^\ell u\right)_{xx}, & (x,t) \in \mathbb{R}^1 \times [0,\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^1, \end{cases} \tag{2}
$$

Under the assumptions that $\ell \geq 1, 0 \leq u_0 \in C^\infty(\mathbb{R}^1) \bigcap L^\infty(\mathbb{R}^1)$, Kalashnikov [8] showed that the solution u of $(\mathrm{P})_1$ does not satisfy $u_x \in C(\mathbb{R}^1 \times [0,\infty))$. If one looks at the Barenblatt solution, one can easily see that the condition that $\ell \geq 1$ is best psossible for the nonnegative initial data. From the view point of physics, the nonnegativity of the initial dta is very natural requirement. However, from the mathematical point of view, it is interesting to observe that if we are concerned with sign changing solutions, this condition $\ell \geq 1$ is not neccessary for the nonexistence of global classical solutions as follows.

Theorem 1 *Let* $v_0 = \int_{-\infty}^{x} u_0(\xi) d\xi$ *and assume*

(N.1) v_0 *is symmetric with respect to* $x = x_0$.

(N.2) $v_0 \in L^2(\mathbb{R}^1)$.

Then for each solution u of $(P)_1$, there exists a positive time T_0 such that $u(x, T_0) \notin C^1(\mathbb{R}^1)$.

As for the existence of time local smooth solutions, our basic assumptions imposed on the parameter ℓ and the initial data u_0 are the following (A.1) and (A.2).

(A.1) ℓ is an even natural number.

(A.2) $u_0(x) \in \bigcap_{m=0}^{\infty} H^m(\mathbb{R}^N)$.

Then our main result on the existence is stated as follows.

Theorem 2 *Let* (A.1) *and* (A.2) *be satisfied, then there exists a positive number T_0 depending on $\|u_0\|_{W^{2,\infty}(\mathbb{R}^N)}$ such that Cauchy problem* (P) *has a unique solution $u \in C^\infty([0,T_0] \times \mathbb{R}^N)$ satisfying*

$$
\sup_{0 \leq t \leq T_0} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \;\leq\; \|u_0\|_{L^\infty(\mathbb{R}^N)}. \tag{3}
$$

3 Proofs

3.1 Sketch of proof of Theorem 1

Let u be a global solution of $(\mathrm{P})_1$ and put $v(x,t)=\int_{-\infty}^{x}u(\xi,t)d\xi$, then we get $v_x(x,t)=u(x,t)$. Hence it holds that $(v_x)_t=(|v_x|^\ell v_x)_{xx}$. Integrating this over $(-\infty,x)$, we find that v satisfies

$$(\mathrm{E})\begin{cases} v_t=\left(|v_x|^\ell v_x\right)_x, & (x,t)\in\mathbb{R}^1\times[0,\infty),\\ v(x,0)=v_0(x)=\int_{-\infty}^{x}u_0(\xi)d\xi, & x\in\mathbb{R}^1, \end{cases} \tag{4}$$

Multiplying (4) by v and integrating over $(0,t)$, we get

$$\frac{1}{2}\|v(t)\|_{L^2}^2+\int_0^t\|v_x(s)\|_{L^p}^p ds\le\frac{1}{2}\|v_0\|_{L^2}^2, \tag{5}$$

where we put $p=\ell+2$. Furthermore, multiplication of (4) gives

$$\|v_t(t)\|_{L^2}^2+\frac{1}{p}\frac{d}{dt}\|v_x(t)\|_{L^p}^p=0,$$

whence follows that $\|v_x(t)\|_{L^p}^p$ is monotone decreasing. Then, in view of (5), we deduce

$$\|v_x(t)\|_{L^p}^p\le\frac{1}{2t}\|v_0\|_{L^2}^2. \tag{6}$$

On the other hand, integrating the identity $\left(\frac{1}{q}|v(x)|^q\right)_x=|v|^{q-2}v\cdot v_x$ over $(-\infty,x)$, we get

$$\frac{1}{q}|v(x)|^q=\int_{-\infty}^{x}|v|^{q-2}vv_x d\xi\le\|v\|_{L^2}^{q-1}\|v_x\|_{L^p},\quad\text{with}\quad q=\frac{3p-2}{p}.$$

Consequently, by virtue of (5) and (6), we obtain

$$\|v(t)\|_{L^\infty}\le\left(q\|v(t)\|_{L^2}^{q-1}\|v_x(t)\|_{L^p}\right)^{1/q}\to 0\quad\text{as}\quad t\to\infty. \tag{7}$$

Here, by using the symmetricity of the initial data and the uniquness of solution of (E), we can conclude that $v(x,t)$ is also symmetric with respect to $x=x_0$. Hence it is easy to see that $v_x(x_0,t)=0$ for all $t\ge 0$. Suppose now that $v(\cdot,t)\in C^2(\mathbb{R}^1)$ for all $t\in[0,\infty)$, then we have

$$\begin{aligned}\frac{d}{dt}v(x_0,t)=\frac{\partial}{\partial t}v(x,t)|_{x=x_0} &= \left(|v_x|^{p-2}v_x(x,t)\right)_x|_{x=x_0}\\ &= (p-1)|v_x(x_0,t)|^{p-2}v_{xx}(x_0,t)=0.\end{aligned}$$

Hence it follows that $v(x_0,t)\equiv v_0(x_0)$ for all $t\in[0,\infty)$, which contradicts the fact (7). Thus it is proved that there exxists a positive time T_0 such that $v(x,T_0)\notin C^2(\mathbb{R}^1)$, i.e., $u(x,T_0)\notin C^1(\mathbb{R}^1)$.

□

3.2 Sketch of proof of Theorem 2

We here give a brief sketch for prove Thorem 2.

Step1 (Existence of C^∞-solutions of approximate equations)

We introduce the following approximate equations for (P).

$$(\mathrm{P})^\varepsilon \begin{cases} u_t - \operatorname{div}\left((\varepsilon + u^\ell)\nabla u\right) = \ell u^{\ell-1}|\nabla u|^2, & (x,t) \in \mathbb{R}^N \times [0,T], \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

where ε is a positive parameter.

In order to assure the existence of C^∞-solutions for $(\mathrm{P})^\varepsilon$, we have only to show the following proposition.

Proposition 1 *For any $T > 0, n \in \mathbb{N}, \varepsilon \in (0,1]$ and $u_0 \in H^{2n+1}(\mathbb{R}^N)$, $(\mathrm{P})^\varepsilon$ has a unique solution u^ε belonging to $W^{2,k}(0,T; H^{2(n-k+1)}(\mathbb{R}^N))$ for all $k = 0, 1, \ldots, n$.*

To prove Proposition 1, we reduce $(\mathrm{P})^\varepsilon$ to the following evolution equations in $H_n = H^{2n}(\mathbb{R}^N)$:

$$(\mathrm{P})^\varepsilon \begin{cases} u_t + \varepsilon A u + u^\ell A u = \ell u^{\ell-1}|\nabla u|^2, & t \in [0,T], \\ u(0) = u_0. \end{cases}$$

Here $A = A_n$ is an operator in H_n defined by $A_n = -\Delta$ and $D(A_n) = H^{2n+2}(\mathbb{R}^N)$, and the inner product of H_n is given by $(u,v)_{H_n} = (u,v)_{L^2(\mathbb{R}^N)} + (A_0^n u, A_0^n v)_{L^2(\mathbb{R}^N)}$. In what follows we always assume $u_0 \in D(A_n^{1/2}) = H^{2n+1}(\mathbb{R}^N)$ and denote A_n and A_0 simply by A.

In solving $(\mathrm{P})^\varepsilon$, we regard the terms $u^\ell A u$ and $\ell u^{\ell-1}|\nabla u|^2$ as perturbations for $\varepsilon A u$. We first solve the following equation with the perturbation $u^\ell A u$.

$$(\mathrm{P})_0^\varepsilon \begin{cases} u_t + \varepsilon A u + u^\ell A u = f, & t \in [0,T], \\ u(0) = u_0, \end{cases}$$

where f is a given function in $L^2(0,T; H_n)$.
To solve $(\mathrm{P})_0^\varepsilon$, we introduce another auxiliary equation:

$${}^\lambda(\mathrm{P})_0^\varepsilon \begin{cases} u_t + \varepsilon A u + \lambda u^\ell A u = h + f, & t \in [0,T], \ \lambda \in [0,1], \\ u(0) = u_0. \end{cases}$$

If ${}^{\lambda}(\mathrm{P})_0^{\varepsilon}$ has a unique solution u^h, we define the operator ${}^{\lambda}\mathcal{F}_{\eta_0}$ by the following correspondence:

$$ {}^{\lambda}\mathcal{F}_{\eta_0} \ : \ h \mapsto u^h \mapsto -\eta_0 (u^h)^{\ell} A u^h, \qquad \eta_0 \in \mathbb{R}. $$

By making use of the fact that $u^{\ell}Au$ can be decomposed into the sum of the monotone part and the small perturbation part, we can show the following lemma.

Lemma 2 *There exist a (sufficiently small) positive number η_0 and a positive number R independent of λ such that ${}^{\lambda}\mathcal{F}_{\eta_0}$ becomes a contraction mapping from K_R^T into itself, where $K_R^T = \left\{ v \in L^2(0,T;H_k); \|v\|_{L^2(0,T;H_k)} \le R \right\}$.*

It is clear that ${}^{\lambda}(\mathrm{P})_0^{\varepsilon}$ with $\lambda = 0$ has a unique solution, so ${}^{0}\mathcal{F}_{\eta_0}$ is well defined. Hence ${}^{0}\mathcal{F}_{\eta_0}$ has a fixed point by the contraction mapping principle, which implies that ${}^{\lambda}(\mathrm{P})_0^{\varepsilon}$ with $\lambda = \eta_0$ admits a unique solution, so ${}^{\eta_0}\mathcal{F}_{\eta_0}$ is well defined. Hence ${}^{\lambda}\mathcal{F}_{\eta_0}$ with $\lambda = 2\eta_0$ admits a unique solution. Thus repeating this procedure finite times, we can construct a unique solution of $(\mathrm{P})_0^{\varepsilon}$.

To solve the original approximate equation $(\mathrm{P})^{\varepsilon}$, we introduce another mapping $\mathcal{S}$ defined by the following correspondence:

$$ \mathcal{S} \ : \ h \mapsto u^h \mapsto \ell (u^h)^{\ell-1} |\nabla u^h|^2, $$

where u^h is a unique solution of $(\mathrm{P})_0^{\varepsilon}$ with $\lambda = 1$ and $f = 0$.

By using much the same arguments as for ${}^{\lambda}\mathcal{F}_{\eta_0}$, we can show that there exist a positive number R and a sufficiently small $T_0 > 0$ such that $\mathcal{S}$ becomes a contraction from $K_R^{T_0}$ into itself. Since $\mathcal{S}$ does not involve any small parameter such as η_0 for ${}^{\lambda}\mathcal{F}_{\eta_0}$, we need the smallness of T_0. However, by the standard energy estimates for $(\mathrm{P})^{\varepsilon}$, we can establish a priori bound for $\|A^{1/2}u(t)\|_{H_k}$, which assures that the local solution on $[0,T_0]$ can be continued up to $[0,T]$. Thus the first step is completed.

Step2 (A priori estimates)

The basic tool here is the " L^{∞}-energy method" introduced in [13], [14] and [15].

(i) Estimate for $\|u(\cdot,t)\|_{L^{\infty}}$

Multiplication of $(\mathrm{P})^{\varepsilon}$ by $|u|^{r-2}u$ gives

$$ \|u\|_{L^r}^{r-1} \cdot \frac{d}{dt}\|u\|_{L^r} + \varepsilon(r-1)\int |u|^{r-2}|\nabla u|^2\,dx + (r-1)\int |u|^{\ell+r-2}|\nabla u|^2\,dx \ = \ 0, $$

whence follows

$$\frac{d}{dt}\|u\|_{L^r} \leq 0.$$

Then we get

$$\sup_{0\leq t\leq T} \|u(\cdot,t)\|_{L^r(\mathbf{R}^N)} \leq \|u_0\|_{L^r(\mathbf{R}^N)} \quad \text{for all } r \in [2,\infty]. \tag{8}$$

(ii) Estimate for $\|\nabla u(\cdot,t)\|_{L^\infty}$ and $\|D^2u(\cdot,t)\|_{L^\infty}$ $(2\leq r\leq\infty)$

Here and in what follows, we use the notations:

$$D_i = \frac{\partial}{\partial x_i}, \quad u_i = D_i u, \quad u_{i,j} = D_i D_j u,$$
$$D^\ell u D^\ell w = \sum_{|\alpha|=\ell} D^\alpha u D^\alpha w, \qquad |D^\ell u| = (D^\ell u D^\ell u)^{1/2},$$

and also use the summation convention.

The direct energy method does not work for this case. However, we can apply the argument of Oleinik and Kruzhkov [11] based on the change of variables and the maximum principle to get a priori bound of $\|\nabla u\|_{L^\infty}$. For example, Theorem 11.16 of Lieberman [10] assures that these exists a constant $C_{1,\infty}$ depending only on $\|u_0\|_{L^\infty}, \|\nabla u_0\|_{L^\infty}, \|u_{0ij}\|_{L^\infty}, \ell$ and ε such that for $i,j\in\{1,\cdots N\}$

$$\sup_{0\leq t\leq T}\|\nabla u(t)\|_{L^\infty} + \sup_{0\leq t\leq T}\|u_{ij}(t)\|_{L^\infty} \leq C_{1,\infty}$$

On the other hand, multiplication of $(\mathrm{P})^\varepsilon$ by $-\Delta u$ and the integration by parts yield

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \int(\varepsilon+u^\ell)(\Delta u)^2\,dx &\leq \ell\int u^{\ell-1}|\nabla u|^2|\Delta u|\,dx\\ &\leq \frac{1}{4}\int u^\ell|\Delta u|^2dx + \ell^2\int u^{\ell-2}|\nabla u|^4dx\\ &\leq \frac{1}{4}\int u^\ell|\Delta u|^2dx + \ell^2 M_{1,\infty}^\ell|\nabla u|_{L^2}^2,\end{aligned}$$

where $M_{1,\infty} = \sup_{0\leq t\leq T_0}\Big(\|\nabla u(t)\|_{L^\infty} + \|u(t)\|_{L^\infty}\Big)$.

Hence, by Gronwall's inequality, we obtain the a priori bound for $|\nabla u|_{L^2}$. Therefore, noting the inequality $\|\nabla u\|_{L^r} \leq \|\nabla u\|_{L^2}^{\frac{2}{r}}\|\nabla u\|_{L^\infty}^{\frac{r-2}{r}}$, we deduce the boundness of $\|\nabla u\|_{L^r}$ for all r in $[2,\infty]$.

Next multiplication of $\nabla(\mathrm{P})^\varepsilon$ by $-\nabla\Delta u$ and the integration by parts yield

$$\begin{aligned}
&\frac{d}{dt}\|\Delta u\|_{L^2}^2 + \int(\varepsilon+u^\ell)|\nabla\Delta u|^2\,dx \\
&\leq \ell\int u^{\ell-1}|\nabla u||\Delta u||\nabla\Delta u|\,dx + \ell(\ell-1)\int u^{\ell-2}|\nabla u|^3|\nabla\Delta u|\,dx \\
&\qquad + \int 2\ell u^{\ell-1}u_i\nabla u_i\nabla\Delta u\,dx \\
&\leq I_1 + I_2 + I_3
\end{aligned}$$

$$\begin{aligned}
I_1 &\leq \frac{1}{4}\int u^\ell|\nabla\Delta u|^2\,dx + \ell^2\int u^{\ell-2}|\nabla u|^2|\Delta u|^2 \\
&\leq \frac{1}{4}\int u^\ell|\nabla\Delta u|^2\,dx + \ell^2 M_{1,\infty}^\ell\|\Delta u\|_{L^2}^2 \\
I_2 &\leq \frac{\varepsilon}{4}\int|\nabla\Delta u|^2\,dx + \ell^2(\ell-1)^2\varepsilon^{-1}\int u^{2(\ell-2)}|\nabla u|^6\,dx \\
&\leq \frac{\varepsilon}{4}\int|\nabla\Delta u|^2\,dx + \ell^4 M_{1,\infty}^{2\ell}\varepsilon^{-1}\|\nabla u\|_{L^2}^2 \\
I_3 &\leq \frac{1}{4}\int u^\ell|\nabla\Delta u|^2\,dx + 4\ell^2\int u^{\ell-2}|u_i|^2|\nabla u_i|^2\,dx \\
&\leq \frac{1}{4}\int u^\ell|\nabla\Delta u|^2\,dx + 4\ell^2 M_{1,\infty}^\ell\|D^2u\|_{L^2}^2
\end{aligned}$$

Therefore

$$\begin{aligned}
&I_1 + I_2 + I_3 \\
&\leq \frac{1}{2}\int(\varepsilon+u^\ell)|\nabla\Delta u|^2\,dx + \ell^2 M_{1,\infty}^\ell\left(\|\Delta u\|_{L^2}^2 + 4\|D^2u\|_{L^2}^2\right) + \ell^4 M_{1,\infty}^{2\ell}\varepsilon^{-1}\|\nabla u\|_{L^2}^2.
\end{aligned}$$

Hence, by virtue of the elliptic estimate:

$$\|D^2v\|_{L^r} \leq C_r\left(\|\Delta v\|_{L^r} + \|v\|_{L^r}\right) \quad \text{for } r>1, \tag{9}$$

we obtain

$$\frac{d}{dt}\|\Delta u\|_{L^2}^2 \leq C_1\|\Delta u\|_{L^2}^2 + C_1, \tag{10}$$

where C_1 is a constant depending on $M_{1,\infty}$, $\|\nabla u\|_{L^2}$ and ε. Then, by Gronwall's inequality and the relaton $\|u_{ij}\|_{L^r} \leq \|u_{ij}\|_{L^2}^{\frac{2}{r}}\|u_{ij}\|_{L^\infty}^{\frac{r-2}{r}}$, we deduce the boundness of $\|D^2u\|_{L^r}$ for all r in $[2,\infty]$.

(iii) Estimate for $\|D^3u(\cdot,t)\|_{L^\infty}$

(a) Estimate for $\|\nabla\Delta u(\cdot,t)\|_{L^r}$ $(2 \le r \le \infty)$

Multiplying $\Delta(\mathrm{P})^\varepsilon$ by $-\nabla\Big(|\nabla\Delta u|^{r-2}\nabla\Delta u\Big)$ and repeating the same type of arguments as above, we can deduce the following estimate:

$$\frac{d}{dt}\|\nabla\Delta u(t)\|_{L^r} \;\le\; C_2\Big(\|\nabla\Delta u(t)\|_{L^r}+1\Big), \tag{11}$$

where C_2 is a constant depending on $M_{2,\infty} = \sup_{0\le t\le T_1}\|u(t)\|_{W^{2,\infty}(\mathbf{R}^N)}$. Therefore, by Gronwall's inequality, we can derive a priori bound for $\|\nabla\Delta u\|_{L^r}$ on $[0,T_1]$. We can repeat much the same argument as above for higher derivatives of $u(t)$.

(b) Estimate for $\|\Delta^2 u(\cdot,t)\|_{L^r}$ $(2 \le r \le \infty)$

Next we derive the L^r-estimate for $\Delta^2 u$.
Multiplication of $\Delta^2(\mathrm{P})^\varepsilon$ by $|\Delta^2 u|^{r-2}\Delta^2 u$ gives

$$\begin{aligned}
&\|\Delta^2 u(t)\|_{L^r}^{r-1}\frac{d}{dt}\|\Delta^2 u(t)\|_{L^r}\\
&\quad= \int \Delta^2\Big((\varepsilon+u^\ell)\Delta u\Big)|\Delta^2 u|^{r-2}\Delta^2 u\;dx\\
&\qquad+\int \Delta^2(\ell u^{\ell-1}|\nabla u|^2)|\Delta^2 u|^{r-2}\Delta^2 u\;dx\\
&\quad\le\; I\;+\;J
\end{aligned}\tag{12}$$

Here we get

$$\begin{aligned}
I \;&\le\; -\int (r-1)(\varepsilon+u^\ell)|\Delta^2 u|^{r-2}|\nabla\Delta^2 u|^2 dx\\
&\qquad+\;M_2\|\Delta^2 u\|_{L^r}^r + M_2(1+\|\nabla\Delta u\|_{L^\infty})\|\nabla\Delta u\|_{L^r}\|\Delta^2 u\|_{L^r}^{r-1}\\
J \;&=\; 2\int u^{\ell-1}u_i\Delta^2 u_i\cdot|\Delta^2 u|^{r-2}\Delta^2 u\;dx\\
&\qquad+\ell\int\Big((\ell-1)u^{\ell-2}|\nabla u|^2\Delta^2 u + 4u^{\ell-1}D^2uD^2\Delta u + 6D_i(u^{\ell-1})u_j\Delta u_{ij}\Big)\\
&\qquad\qquad\qquad\times|\Delta^2 u|^{r-2}\Delta^2 u\;dx\\
&\qquad+\int\Big(2u^{\ell-1}D^3u\cdot D^3u + 2u^{\ell-1}(\Delta u_i)^2\Big)|\Delta^2 u|^{r-2}\Delta^2 u\;dx\\
&\qquad+\int a_{ijk}(u,u_i,u_{jk})u_{ijk}|\Delta^2 u|^{r-2}\Delta^2 u\;dx\\
&\qquad+\int a(u,\nabla u,u_{ij})|\Delta^2 u|^{r-2}\Delta^2 u\;dx\\
&\le\;\frac{r-1}{4}\int u^\ell|\Delta^2 u|^{r-2}|\nabla\Delta^2 u|^2\;dx\\
&\qquad+M_2\Big(\|\Delta^2 u\|_{L^r}^r + (\|\nabla\Delta u\|_{L^\infty}+1)\|\nabla\Delta u\|_{L^r}\|\Delta^2 u\|_{L^r}^{r-1}\Big),
\end{aligned}$$

Here M_2 denotes a constant depending only on $M_{2,\infty}, \ell, N$ and r, and we used the elliptic estimate in L^r. Then, recalling the boundness of $\|\nabla\Delta u\|_{L^r}$ and the following embedding inequality

$$\|\nabla\Delta u\|_{L^\infty} \leq C_q(\|\nabla\Delta u\|_{L^q} + \|\Delta^2 u\|_{L^q}) \qquad \text{for} \quad q > N,$$

we deduce

$$\frac{d}{dt}\|\Delta^2 u\|_{L^r} \leq M_2\Big(\|\Delta^2 u\|_{L^r} + 1\Big) \quad \text{for all } r > N. \tag{13}$$

Thus we establish the boundness of $\|\Delta^2 u\|_{L^r}$ for all $r > N$, whence follows the boundness of $\|\nabla\Delta u\|_{L^\infty}$ and $\|D^3 u\|_{L^\infty}$.

(iv) Estimates for $\|D^{2n}u(\cdot,t)\|_{L^\infty}$ and $\|D^{2n+1}u(\cdot,t)\|_{L^\infty}$

For the higher derivatives of u, we can repeat the same argument. By using the following inequality:

$$\begin{aligned}\|D^{2n}u\|_{L^\infty} &\leq C_q(\|\nabla\Delta^n u\|_{L^q} + \|u\|_{L^q})\\ \|D^{2n+1}u\|_{L^\infty} &\leq C_q(\|\Delta^{n+1} u\|_{L^q} + \|u\|_{L^q}),\end{aligned}$$

we can establish the a priori bound for these norms if $u_0 \in W^{2n,\infty}(\mathbb{R}^1)$ or $u_0 \in W^{2n+1,\infty}(\mathbb{R}^1)$ respectively.
Finally we prove the convergence.

Convergence:

Theorem 3 *Let $u_0 \in H^{2k+1}(\mathbb{R}^N)$ with $k \in \mathbb{N}$ $(k \geq 2)$, then there exists a positive number T_0 depending only on $\ell, \|u_0\|_{W^{2,\infty}}$ such that* (P) *has a unique solution u belonging to* $\mathcal{C}^k_{T_0} := \{v \in C([0,T_0]; H^{2k}(\mathbb{R}^N)); v \in L^\infty(0,T_0; H^{2k+1}(\mathbb{R}^N)),$ $v_t \in L^2(0,T_0; H^{2k}(\mathbb{R}^N)),\ v^\ell\Delta v \in L^2(0,T_0; H^{2k}(\mathbb{R}^N))\}$ *such that*

$$\sup_{0\leq t\leq T_0} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq \|u_0\|_{L^\infty(\mathbb{R}^N)} \tag{14}$$

Moreover T_0 can be chosen as a monotone decreasing function of $\|u_0\|_{W^{2,\infty}}$ such that T_0 tends to 0 as $\|u_0\|_{W^{2,\infty}}$ tends to ∞.

Proof of Theorem 3 : Let u_ε be the global solution of $(\mathrm{P})^\varepsilon$ belonging to $\mathcal{C}^k_T$. Then, we know that $\{u_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^\infty(0,T_0; H^{2k+1}(\mathbb{R}^N))$ and there exists a positive number T_0 depending only on $\ell, \|u_0\|_{W^{2,\infty}}$ such that the following inequalities hold.

$$\sup_{2\leq r\leq\infty} \sum_{j=0}^{2} \|D^j u_\varepsilon\|_{L^r} + \|D^4 u\|_{L^2} \leq L_4, \tag{15}$$

$$\varepsilon \int_0^{T_0} \|\Delta^{k+1} u_\varepsilon\|_{L^2}^2 \, dt + \int_0^{T_0} \int u_\varepsilon^\ell |\Delta^{k+1} u_\varepsilon(t)|^2 \, dx \, dt \ \leq \ L_{2k+1}, \tag{16}$$

$$\int_0^{T_0} \|(u_\varepsilon)_t\|_{H^{2k}}^2 \, dt \ \leq \ L_{2k+1}. \tag{17}$$

Here L_4 and L_{2k+1} are independent of ε and T_0 can be chosen as a monotone decreasing function of $\|u_0\|_{W^{2,\infty}}$ such that T_0 tends to 0 as $\|u_0\|_{W^{2,\infty}}$ tends to ∞. Now we are going to show below that $\{u_\varepsilon\}_{\varepsilon>0}$ forms a Cauchy sequence in $C([0,T_0]; H^2(\mathbb{R}^N))$. For any $\varepsilon_1 > 0, \varepsilon_2 > 0$, we denote $u_1 = u_{\varepsilon_1}, u_2 = u_{\varepsilon_2}$ and $w = u_1 - u_2$. Then w satisfies

$$w_t - \varepsilon_1 \Delta u_1 + \varepsilon_2 \Delta u_2 \ = \ \frac{1}{\ell+1} \Delta(d_{\ell+1} w) \tag{18}$$

$$= \ u_1^\ell \Delta w + d_\ell \Delta u_2 w + \ell u_1^{\ell-1} \nabla(u_1 + u_2) \nabla w + \ell |\nabla u_2|^2 d_{\ell-1} w, \tag{19}$$

where $d_\ell = u_1^{\ell-1} + u_1^{\ell-2} u_2 + \cdots + u_1 u_2^{\ell-2} + u_2^{\ell-1}$.
Multiplication of (18) by w gives

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 \ &\leq \ \big(\varepsilon_1 \|\Delta u_1\|_{L^2} + \varepsilon_2 \|\Delta u_2\|_{L^2}\big) \|w\|_{L^2} + \frac{1}{\ell+1} \int d_{\ell+1} w \Delta w \, dx \\ &\leq \ (\varepsilon_1 + \varepsilon_2) L_4 \|w\|_{L^2} + L_4^\ell \|w\|_{L^2} \|D^2 w\|_{L^2}. \end{aligned} \tag{20}$$

We differentiate (19) once and multiply it by $-\nabla \Delta w$. Then it is easy to see that there exists a constant C_ℓ depending only on ℓ such that

$$\frac{1}{2}\frac{d}{dt}\|\Delta w\|_{L^2}^2 \ \leq \ (\varepsilon_1 + \varepsilon_2) L_4 \|\Delta w\|_{L^2} \ + \ C_\ell L_4^\ell \|\Delta w\|_{L^2}^2 \tag{21}$$

Hence, by (20), (21) and Gronwall's inequality, we obtain

$$\|w\|_{H^2} \ \leq \ \|w\|_{L^2} \ + \ \|\Delta u\|_{L^2} \ \leq \ 2(\varepsilon_1 + \varepsilon_2) L_4 \ e^{(C_\ell+1) L_4^\ell t} \qquad \forall t \in [0, T_1].$$

Thus $\{u_\varepsilon\}_{\varepsilon>0}$ forms a Cauchy sequence in $C([0,T_0]; H^2(\mathbb{R}^N))$.
Here we note that $u^\ell \Delta u$ and $\ell u^{\ell-1}(Du)^2$ are also bounded in $L^2(0,T_0; H^{2k}(\mathbb{R}^N))$ since u_ε is bounded in $L^\infty(0,T_0; H^{2k+1}(\mathbb{R}^N))$ and satisfies (16). Therefore, in view of (15)-(17), we find that there exists a sequence $\varepsilon_n \to 0$ such that $\{u_n\} = \{u_{\varepsilon_n}\}$ satisfies

$$\begin{array}{llll} u_n \to u & \text{strongly} & \text{in} & C([0,T_0]; H^2(\mathbb{R}^N)), \\ u_n \to u & \text{strongly} & \text{in} & L^\infty([0,T_0]; L^\infty_{loc}(\mathbb{R}^N)), \\ u_n \rightharpoonup u & \text{weakly} & \text{in} & L^2(0,T_0; H^{2k+1}(\mathbb{R}^N)), \\ & \text{and weakly star} & \text{in} & L^\infty(0,T_0; H^{2k+1}(\mathbb{R}^N)), \\ (u_n)_t \rightharpoonup u_t & \text{weakly} & \text{in} & L^2(0,T_0; H^{2k}(\mathbb{R}^N)), \\ u_n^\ell \Delta u_n \rightharpoonup g & \text{weakly} & \text{in} & L^2(0,T_0; H^{2k}(\mathbb{R}^N)), \\ \ell u_n^{\ell-1}(Du_n)^2 \rightharpoonup \chi & \text{weakly} & \text{in} & L^2(0,T_0; H^{2k}(\mathbb{R}^N)), \\ \varepsilon_n \Delta u_n \to 0 & \text{strongly} & \text{in} & L^2(0,T_0; H^{2k}(\mathbb{R}^N)). \end{array}$$

On the other hand, since the convergence of u_n to u in $C([0,T_0];H^2(\mathbb{R}^N))$ and in $L^\infty(0,T_0;L^\infty_{loc}(\mathbb{R}^N))$ implies

$$\begin{array}{lll} u_n^\ell \Delta u_n \rightharpoonup u^\ell \Delta u & \text{weakly in} & L^2(0,T_0;L^2(\mathbb{R}^N)), \\ \ell u_n^{\ell-1}(Du_n)^2 \rightharpoonup \ell u^{\ell-1}(Du)^2 & \text{weakly in} & L^2(0,T_0;L^2(\mathbb{R}^N)), \end{array}$$

whence follow $g = u^\ell \Delta u$ and $\chi = \ell u^{\ell-1}(D^2 u)$. Then u turns out to be the desired solution in Theorem 2. □

Now we are ready to prove our main theorem.

Proof of Theorem 2 : Since $u_0 \in \bigcap_{m=0}^\infty H^m(\mathbb{R}^N)$, Theorem 2 says that solution u belongs to $\mathcal{C}^k_{T_0}$ for all k. Therefore $u_t \in L^2(0,T_0;H^m(\mathbb{R}^N))$ for all $m \in \mathbb{N}$. Noting that $u_{tt} = \Delta(u^\ell u_t)$, we know $u_{tt} \in L^2(0,T_0;H^m(\mathbb{R}^N))$ for all $m \in \mathbb{N}$, which implies $u_t \in C([0,T_0];H^m(\mathbb{R}^N))$ for all $m \in \mathbb{N}$. Repeating this procedure, we easily find that $D_t^j u \in C([0,T_0];H^m(\mathbb{R}^N))$ for all $j,m \in \mathbb{N}$. Then the standard argument assures that $u \in C^\infty([0,T_0]\times\mathbb{R}^N)$. □

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