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GLOBAL CONTINUATION BEYOND SINGULARITY ON THE BOUNDARY

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1. INTRODUCTION

We consider problems of the form

$$\begin{aligned} &u_t = u_{xx}, && 0<x<1, \quad 0<t<T,\\ &u_x(0,t)=0, && 0<t<T,\\ &u_x(1,t)=f(u(1,t)), && 0<t<T,\\ &u(x,0)=u_0(x)>0, && 0\le x\le 1, \end{aligned}$$

where $f(u) = -u^{-p}$, $p > 0$, or $f(u) = u^p$, $p > 1$. We shall call them Problem (Q) and Problem (B), respectively. We discuss them separately.

1.1. Problem (Q) ($f(u) = -u^{-p}$). This problem was studied before by Fila & Levine(1993) where it was shown that that every solution quenches in a finite time $T = T(u_0)$ in the sense that $u > 0$ in $[0,1] \times [0,T)$ and $u(1,t) \to 0$ as $t \to T$. The behavior of u near $(1,T)$ for $t \le T$ was also studied.

The question whether it is possible to continue the solution beyond $t = T$ (in some suitable sense) was raised by Levine(1993). Since $u(\cdot,T) \in C([0,1])$ and $u(1,T) = 0$, an obvious possibility of continuing the solution is to extend it for $t > T$ by $\tilde{u}$ which solves

$$\begin{aligned} &\tilde{u}_t = \tilde{u}_{xx}, && 0<x<1, \quad t>T,\\ &\tilde{u}_x(0,t)=0, && t>T,\\ &\tilde{u}(1,t)=0, && t>T,\\ &\tilde{u}(x,T)=u(x,T), && 0\le x\le 1. \end{aligned}$$

We show that this continuation is natural since it can be obtained as a limit of a sequence of solutions of regularized problems. More precisely, if $\varepsilon > 0$ and $f_\varepsilon \in C^1([0,\infty))$ is such that $f_\varepsilon(0) = 0$ and

$$\begin{aligned} &f_\varepsilon(s) = -s^{-p} && \text{for} \quad s \ge \varepsilon,\\ &f(s) \le f_{\varepsilon_1}(s) \le f_{\varepsilon_2}(s) && \text{for} \quad s>0 \quad \text{and} \quad \varepsilon_1<\varepsilon_2, \end{aligned}$$

then the solutions of (Q_ε):

$$\begin{cases} u^\varepsilon_t = u^\varepsilon_{xx}, & 0 < x < 1, \quad 0 < t < \infty, \\ u^\varepsilon_x(0,t) = 0, & 0 < t < \infty, \\ u^\varepsilon_x(1,t) = f_\varepsilon(u^\varepsilon(1,t)), & 0 < t < \infty, \\ u^\varepsilon(x,0) = u_0(x), & 0 \le x \le 1, \end{cases}$$

converge to the extension of u by $\tilde{u}$.

The fact that solutions of Problem (Q) can be continued beyond $t = T$ for all $p > 0$ is in contrast with the situation when quenching occurs in the interior. Namely, for the problem

$$\begin{aligned} u_t &= u_{xx} - u^{-p}, && 0 < x < 1, \quad 0 < t < T, \\ u_x(0,t) &= 0, && 0 < t < T, \\ u(1,t) &= 1, && 0 < t < T, \\ u(x,0) &= u_0(x), && 0 \le x \le 1, \end{aligned}$$

solutions can be continued beyond quenching if and only if $0 < p < 1$ (cf. Phillips(1987), Galaktionov & Vazquez(1995)).

Let us also mention here that a similar phenomenon when the continuation beyond gradient blow-up does not satisfy the original boundary condition was observed by Fila & Lieberman(1994).

1.2. Problem (B) ($f(u) = u^p$)**.** The study of blow-up of solutions of the heat equation with a nonlinear boundary condition was initiated by Levine & Payne(1974) and it has attracted considerable attention (see a survey paper of Fila & Filo(1996)). It was shown by Fila(1989) that every solution of Problem (B) blows up in a finite time $T = T(u_0)$ and it is also known (cf. López Gómez, Márquez, & Wolanski(1991)) that the only blow-up point is $x = 1$.

(By a blow-up point we mean a point $a \in [0,1]$ such that there are $\{x_n\} \subset [0,1]$ and $t_n \to T$ such that $x_n \to a$ and $u(x_n, t_n) \to \infty$ as $n \to \infty$.)

We show that for Problem (B) blow-up is always complete in the following sense. If

$$f^n(s) = \min\{s^p, n^p\}, \quad s \ge 0, \quad n \in \mathbb{N}, \tag{1.1}$$

and u^n is the solution of (B^n):

$$\begin{cases} u^n_t = u^n_{xx}, & 0 < x < 1, \quad 0 < t < \infty, \\ u^n_x(0,t) = 0, & 0 < t < \infty, \\ u^n_x(1,t) = f^n(u^n(1,t)), & 0 < t < \infty, \\ u^n(x,0) = u_0(x), & 0 \le x \le 1, \end{cases}$$

then $u^n(x,t) \to \infty$ for $(x,t) \in [0,1] \times (T, \infty)$.

For results on complete blow-up for the problem when the nonlinearity occurs in the equation we refer to the papers of Baras & Cohen(1987), Lacey & Tzanetis(1988), Galaktionov & Vazquez(1995, 1997), Martel(1998), etc.

Our method is different and it is restricted to one space dimension since we are using an intersection-comparison (or zero number(cf. [14])) argument.

2. Incomplete Quenching

In this section we show that if $u(x,t)$ is the solution of the problem

$$\begin{cases} u_t = u_{xx}, & 0 < x < 1, \quad 0 < t < T, \\ u_x(0,t) = 0, & 0 < t < T, \\ u_x(1,t) = -u^{-p}(1,t), & 0 < t < T, \\ u(x,0) = u_0(x) > 0, & 0 \le x \le 1, \end{cases} \tag{Q}$$

where $p > 0$ and T is the quenching time for u then there is a natural continuation of u beyond T. We shall assume that $u_0 \in C^1([0,1])$ and the compatibility conditions

$$u_0'(0) = 0, \qquad u_0'(1) = -u_0^{-p}(1)$$

are satisfied.

Assume that $0 < \varepsilon < u_0(1)$. Then there exists a unique global (in time) solution u^ε of (Q_ε) such that $u^\varepsilon \in C^{2,1}([0,1] \times [0,\tau])$ for any $\tau > 0$ and

(i) $u^\varepsilon > 0$ for $(x,t) \in [0,1] \times [0,\infty)$,
(ii) $u^{\varepsilon_1} \le u^{\varepsilon_2}$ for $0 < \varepsilon_1 < \varepsilon_2$ and $(x,t) \in [0,1] \times [0,\infty)$,
(iii) $u^\varepsilon \ge u$ for $(x,t) \in [0,1] \times [0,T)$.

Also, by the maximum principle, it is clear that

$$u^\varepsilon \le K \equiv \max_{0 \le x \le 1} u_0(x)$$

for all $\varepsilon > 0$.

Now, let

$$v(x,t) = \lim_{\varepsilon \to 0} u^\varepsilon(x,t), \qquad (x,t) \in [0,1] \times [0,\infty). \tag{2.1}$$

Then v is well-defined and $0 \le v \le K$ in $[0,1] \times [0,\infty)$. It follows from the regularity theory for parabolic equations that v satisfies the heat equation in $(0,1) \times (0,\infty)$. By the maximum principle, $v > 0$ in $(0,1) \times (0,\infty)$. Also, it is clear that $v_x(0,t) = 0$ for $t > 0$. Furthermore, if $t \in (0,T)$, then

$$v_x(1,t) = -v^{-p}(1,t).$$

It follows that v is a solution of (Q). By uniqueness, $v = u$ in $[0,1] \times [0,T)$. For the boundary condition for v on $\{x = 1, \ t > T\}$, it can be shown that $v(1,t) = 0$ for $t \ge T$.

We summarize the above results as follows:

Theorem 2.1[15]. *The function v defined by* (2.1) *satisfies*

$$\begin{aligned} v_t &= v_{xx}, && 0 < x < 1, \quad t > 0, \\ v_x(0,t) &= 0, && t > 0, \\ v_x(1,t) &= -v^{-p}(1,t), && 0 < t < T, \\ v(1,t) &= 0, && t \ge T, \\ v(x,0) &= u_0(x), && 0 \le x \le 1. \end{aligned}$$

It coincides with the solution u of Problem (Q) *for $t \le T$.*

3. Complete Blow-up

Consider the problem

$$\begin{cases} u_t = u_{xx}, & 0 < x < 1, \quad 0 < t < T, \\ u_x(0,t) = 0, & 0 < t < T, \\ u_x(1,t) = u^p(1,t), & 0 < t < T, \\ u(x,0) = u_0(x) > 0, & 0 \le x \le 1, \end{cases} \tag{B}$$

where $p > 1$, and T is the blow-up time for u. We assume further that $u_0'(0) = 0$ and $u_0'(1) = u_0^p(1)$.

Let $K = \max_{0 \le x \le 1} u_0(x)$. For any $n > K$, $n \in \mathbb{N}$, we define f^n as in (1.1). Note that f^n is Lipschitz and $u_0'(1) = f^n(u_0(1))$ if $n > K$. Hence, the solution of (B^n) is C^1 up to the boundary. We show that there exists a unique global (in time) solution u^n of (B^n) such that

(i) $u^n > 0$ for $(x,t) \in [0,1] \times [0,\infty)$,
(ii) $u^n \le u^{n+1}$ for $(x,t) \in [0,1] \times [0,\infty)$,
(iii) $u^n \le u$ for $(x,t) \in [0,1] \times [0,T)$.

Define

$$v(x,t) = \lim_{n \to \infty} u^n(x,t), \qquad 0 \le x \le 1, \quad t \ge 0. \tag{3.1}$$

Similarly, one can show that $v_x(1,t) = v^p(1,t)$ for $t \in (0,T)$. Then it is clear that $v(x,t) = u(x,t)$ for $0 < t < T$. Note that $v(1,T) = \infty$. Furthermore, there holds $v(1,t) = \infty$ for $t \ge T$.

This proves the following:

Theorem 3.1[15]. *The function v defined in* (3.1) *coincides with the solution u of Problem* (B) *for $t \le T$ and $v(x,t) = \infty$ for $(x,t) \in [0,1] \times (T,\infty)$.*

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