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EQUILIBRIA AND TRAVELING WAVES FOR BISTABLE EQUATIONS WITH NON-LOCAL AND DISCRETE DISSIPATION

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ABSTRACT. We prove the existence of stationary solutions or traveling waves for non-local continuum equations or lattice dynamical systems arising in the theory of phase transitions. The systems can also be viewed as nonlocal or discrete versions of a reaction-diffusion equation which include infinite-range coupling.

1. INTRODUCTION

We consider an Ising-like spin system on a lattice Λ at subcritical temperature. Including interaction between all pairs of particles, we may derive an expression for the total free energy of a spin field $\{u(r)\in\mathbb{R}:r\in\Lambda\}$ of the form

$$E(u)=\frac{1}{4\lambda}\sum_{r,r'\in\Lambda}J(r-r')(u(r)-u(r'))^{2}+\sum_{r\in\Lambda}W(u(r)), \tag{1.1}$$

where λ is a constant for normalizing J, J/λ is the (translationally-invariant) interaction coefficient, and W is a smooth double-well potential having minima at the values $u=\pm 1$ (see [4] for details). A continuum version of this free energy is

$$E(u)=\frac{1}{4\lambda}\int_{\mathbb{R}^{n}}\int_{\mathbb{R}^{n}}J(x-y)(u(x)-u(y))^{2}dxdy+\int_{\mathbb{R}^{n}}W(u(x))dx. \tag{1.2}$$

In ferromagnetic systems one has (i) $\lambda>0$ and $\int_{\mathbb{R}^{n}}J(x)dx=1$ (or $\sum_{r\in\Lambda}J(r)=1$ in the lattice case), and (ii) $J\geq 0$. Here we shall always assume (i) but shall not assume (ii) for some of our results.

With these expressions for the free energy, one is led to questions about the existence and nature of equilibria and also the evolution of initial configurations. We take the simple point of view that, consistent with thermodynamics, a configuration state u_0 will evolve according to the negative gradient of the free energy of the state at that point in

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time. That is, we postulate

$$\tau u_t = -\text{grad } E(u), \tag{1.3}$$

where τ is a relaxation time parameter.

There is of course some ambiguity about the meaning of "grad", since the metric in the state space has not been selected. Indeed, for different choices of metric, different evolution laws result from (1.3), each decreasing the free energy along trajectories. Here we consider perhaps the most natural metric, that associated with ℓ^2 for the lattice and L^2 for the continuum case. Thus, from (1.3) one derives the evolution equation

$$\tau u_t = \frac{1}{\lambda}\{J * u - u\} - W'(u), \tag{1.4}$$

where $*$ is convolution (discrete or continuous, depending on the setting).

Note that in the case $J \geq 0$, the operator $Lu \equiv \frac{1}{\lambda}\{J * u - u\}$ has some similarities with Δu, namely,

(i) L, acting in L^2 (or ℓ^2), is self-adjoint and $(Lu, u) \leq 0$ for all $u \in L^2$ (or ℓ^2).

(ii) Equation (1.4) has a comparison principle: If $u_0 \leq v_0$ then solutions with these initial data are so ordered. The proof of this useful fact is given in the Appendix.

Furthermore, in the case of a lattice, L may be thought of as a discretization of (a multiple) of the Laplacian, the nearest neighbor finite-difference operator being a special case (see [10]).

On the other hand, L is a bounded operator and so (1.4) may be solved locally backwards as well as forwards in time for general initial data. Hence, there is no smoothing associated with the forward solution map.

Perhaps important for applications, it is worth noting that anisotropy is included naturally in (1.4) through the interaction kernel J.

When L is replaced by Δ in (1.4), the result is the Allen-Cahn equation [1]

$$\tau u_t = \Delta u - W'(u), \tag{1.5}$$

about which much is known for traveling waves, one important reference being [19]. Nonlocal equations like (1.4) for the continuum have also received considerable attention recently, arising in several fields as diverse as neuroscience and elasticity. The interested reader is referred to [34], [17], [16], [21], [5], [9], [20], [26], [3], [11], and the references therein. These are almost all for the case $J \geq 0$.

Likewise, there have been many studies of lattice dynamical systems like (1.4) in the discrete case. A large body of work exists having an emphasis on applications to neural networks (see [15], [33], [14], [35] and [36] for instance). Other applications and theoretical developments may be found in [7], [38], [28], [25], [29], [39], [23], [22], [30], [13], [8], [31], [18], [4], and [6].

Here we start by outlining some of the principal results of [3] and [4] on continuum and lattice versions of (1.4), mainly for the case when λ is large. This includes results giving traveling or stationary waves for the discrete case when the lattice is $\mathbb{Z}$, the integers, and the interaction is ferromagnetic ($J \geq 0$) . Then we give some new results particularly in the case that λ is small and J is allowed to change sign. The limit $\lambda \to 0$ is important from a numerical analysis point of view, since $\frac{1}{\lambda}\{J * u - u\}$ may be viewed as a discretization of a multiple of the Laplacian. Allowing J to change sign may be important in modeling non-ferromagnetic materials or situations in which the combined inter-atomic forces change sign with separation distance.

2. Stationary Solutions for Small Interaction

In this section the results apply equally to both the continuum and lattice versions in any dimension. For simplicity of exposition, we consider only the continuum case. Renaming $W'(u)$ as $f(u)$, for stationary solutions to (1.4) we consider

$$J * u - u - \lambda f(u) = 0. \tag{2.1}$$

We first give results for the "elliptic" case, where $J(x) \geq 0$.

Assume

(J) $J \in W^{1,1}(\mathbb{R}^n), J(-x) = J(x) \geq 0$, and $\int_{\mathbb{R}^n} J(x)dx = 1$

and

(F) $f \in C^2(\mathbb{R})$ has exactly three zeros at ± 1 and $a \in (-1,1)$ and $f'(a) < 0 < f'(\pm 1)$.

Theorem 1. *Assume* **(J)** *and* **(F)**. *Let* $\alpha^- \in (-1, a)$ *and* $\alpha^+ \in (a, 1)$ *be such that*

$$f'(z) > 0 \qquad \textit{for} \;\; z \in [-1, \alpha^-] \cup [\alpha^+, 1].$$

Let Ω be a measurable set with complement Ω^c. Assume that λ is large enough such that

$$\lambda f(\alpha^-) \ \geq \ (1-\alpha^-)\sup_{x\in\Omega^c}\int_\Omega J(x-y)dy, \tag{2.2}$$

$$-\lambda f(\alpha^+) \ \geq \ (1+\alpha^+)\sup_{x\in\Omega}\int_{\Omega^c} J(x-y)dy. \tag{2.3}$$

Then there exists a unique solution $\hat{u}$ to (2.1), such that

$$\hat{u}(x)\begin{cases} \geq \alpha^+ & \text{for } x\in\Omega, \\ \leq \alpha^- & \text{for } x\in\Omega^c. \end{cases}$$

Moreover, $\hat{u}$ is C^0 on Ω and Ω^c, C^2 in the interior of Ω and Ω^c, and (locally) asymptotically stable in the $L^\infty(\mathbb{R}^n)$ norm.

Proof. Let $h>0$ be small enough such that

$$\lambda h f'(u) < (1-h) \text{ for } u\in[-1,\alpha^-]\cup[\alpha^+,1].$$

Then the mapping

$$Tu(x)\equiv u(x)+h[J*u(x)-u(x)-\lambda f(u(x))]$$

is a contraction on

$$B=\{u\in L^\infty(\mathbb{R}^n):\ u(x)\in[\alpha^+,1] \text{ for } x\in\Omega \text{ and } u(x)\in[-1,\alpha^-] \text{ for } x\in\Omega^c\}.$$

The proofs of stability and regularity may be found in [3]. □

Remark. *The idea is that with J nonnegative, a certain monotonicity holds. Furthermore, all operators are bounded. It can be seen that (2.2) and (2.3) can only hold if $g(u)\equiv u+\lambda f(u)$ is non-monotone, consistent with the results in* [5, 9] *giving discontinuous stationary states in the one-dimensional case.*

To interpret the result, we can view (2.2) and (2.3) as ensuring that if $u(x)$ takes on a value in the "domain of attraction" of -1 or of 1, according to the kinetics, then the influence of interaction through J with neighbors of x where u takes values in the other domain of attraction, is not sufficiently strong to cause $u(x)$ to change to the other domain of attraction. If a set Ω badly fails to satisfy (2.2) and (2.3), as might happen if a component of Ω or Ω^c is too small and isolated or if there is a cusp or sharp corner, then initial data $u_0(x)$ taking values in $[\alpha^+,1]$ over Ω and values in $[-1,\alpha^-]$ in Ω^c, may evolve so that $\Omega_\infty\equiv u(\cdot,\infty)^{-1}([\alpha^+,1])$ satisfies these conditions, or some local version of them.

That is, phase domains will evolve to form stationary shapes according to the strength of the anisotropic interaction.

It is interesting to note that if λ is sufficiently large, then any measurable set Ω satisfies (2.2) and (2.3), and so supports a stable stationary solution which is discontinuous across its boundary. Such a weak interaction may not be of practical interest however.

The above observation about large λ and the fact that $u \to J * u - u$ is a bounded operator suggests that, even if J changes sign, there should be similar stationary solutions when λ is large. This is indeed the case. We now relax the nonnegativity assumption on J and instead require only

(J)^N $J \in W^{1,1}(\mathbb{R}^n), J(-x) = J(x)$, and $\int_{\mathbb{R}^n} J(x)dx = 1$.

We assume that $\lambda > 0$ is so large that for some

$$b > \int_{\mathbb{R}^n} |J|, \tag{2.4}$$

the following holds

$$\{z \,:\, z + \lambda f(z) \in [-b, b]\} = \mathrm{I}_1 \cup \mathrm{I}_2 \cup \mathrm{I}_3, \tag{2.5}$$

$$|1 + \lambda f'(z)| > \textstyle\int_{\mathbb{R}^n} |J| \qquad \text{on } \mathrm{I}_1 \cup \mathrm{I}_2 \cup \mathrm{I}_3, \tag{2.6}$$

$$-1 \in I_1 = (u_1, u_2), \qquad a \in \mathrm{I}_2 = (u_3, u_4), \qquad 1 \in \mathrm{I}_3 = (u_5, u_6), \tag{2.7}$$

One can prove that, under the above assumptions, stationary solutions to (2.1) are *a priori* bounded and in fact

Proposition 1. *Assume that* **(J)**^{N}*,* **(F)***, and (2.4)–(2.7) hold. Then any* $L^\infty(\mathbb{R}^n)$ *solution* u *to (2.1) satisfies*

$$u(x) \in I_1 \cup I_2 \cup I_3 \text{ for all } x \in \mathbb{R}^n. \tag{2.8}$$

We now state a theorem which gives stationary solutions which take values in each of these intervals on prescribed measurable sets.

Theorem 2. *(Existence) Under the above assumptions all solutions of (1.4) are characterized as follows.*

Let Ω_1 and Ω_2 be any two disjoint measurable sets. Then there exists a unique solution u to (2.1), such that $u(x) \in I_1$ for $x \in \Omega_1$, $u(x) \in I_2$ for $x \in \Omega_2$ and $u(x) \in I_3$ for $x \in (\Omega_1 \cup \Omega_2)^c$. Moreover, u is C^2 on $\operatorname{int}(\Omega_1)$, $\operatorname{int}(\Omega_2)$ and $\operatorname{int}(\Omega_1 \cup \Omega_2)^c$.

As one might guess from linearization at ± 1 and at a, we also have

Theorem 3. *(Stability) Let $u(x)$ be a solution of (2.1). Then*

1. *If $u(x) \in I_1 \cup I_3$ for all $x \in \mathbb{R}^n$, u is (locally) exponentially asymptotically stable in the $L^\infty(\mathbb{R}^n)$ norm.*
2. *If $\{x : u(x) \in I_2\}$ has a positive measure, u is unstable in the $L^\infty(\mathbb{R}^n)$ norm.*

Both theorems are proved using the Implicit Function Theorem by continuation from $\lambda = \infty$. Note that dividing by λ and setting $\lambda = \infty$ results in the equation $f(u(x)) = 0$, which has as a solution any function u whose range is contained in $\{a, \pm 1\}$. The nondegeneracy of these zeros allows continuation to $\lambda < \infty$. The stability and instability results rely on invariant manifold theory. See [3] for details.

3. Waves on the Integer Lattice with $J \geq 0$

We now restrict our attention to the discrete, one-dimensional case where $\Lambda = \mathbb{Z}$. Thus, we study the following infinite system of coupled semilinear evolution equations

$$\dot{u}_n = (J * u)_n - u_n - \lambda f(u_n), \; n \in \mathbb{Z}, \tag{3.1}$$

where $(J * u)_n \equiv \sum_{i \in \mathbb{Z}\setminus\{0\}} J(i) u_{n-i}$, $\sum_{i \in \mathbb{Z}\setminus\{0\}} J(i) = 1$, and f is as before.

We further restrict our attention to traveling or stationary waves for (3.1) by setting $u_n(t) = u(n - ct)$. It is convenient to define $J_\delta(x) = \sum_{|i| \geq 1} J(i)\delta(x-i)$, and with $x = n - ct$, write the traveling wave equation as

$$c\, u'(x) + (J_\delta * u)(x) - u(x) - \lambda f(u(x)) = 0, \; x \in \mathbb{R}, \tag{3.2}$$

together with the boundary conditions

$$u(-\infty) = -1, \; u(+\infty) = 1. \tag{3.3}$$

The results in this section are for the "elliptic" case where $J \geq 0$. Included is the case where the second spatial derivative in the Allen-Cahn equation is replaced by the usual finite difference operator on a uniform grid. Such equations have been studied for many years and it has been noted that "pinning" may occur. That is, even if one of the two stable wells are not balanced (i.e., $\int_{-1}^{1} f(s)ds \neq 0$) a wave form may fail to propagate (see [27], [29], [33], and [18] for instance). We hope to shed some light on this phenomenon.

We assume that J satisfies

$(\mathbf{J})^L$ $J(i) = J(-i) \geq 0$ for all $i \in \mathbb{Z}$, $\sum_{|i|\geq 1} J(i) = 1$ and $\sum_{|i|\geq 1} |i| J(i) < \infty$.

Define $g(u) \equiv u + \lambda f(u)$. For simplicity, we make the following assumptions: g has at most three intervals of monotonicity, i.e., for some β and γ with $-1 < \beta \leq \gamma < 1$,

$$g' > 0 \text{ on } [-1, \beta) \cup (\gamma, 1], \qquad g' \leq 0 \text{ on } (\beta, \gamma).$$

In the case $\beta < \gamma$, for any number

$$k \in K \equiv [\max\{-1, g(\gamma)\}, \min\{1, g(\beta)\}$$

define $g_k(u)$ to be the continuous nondecreasing function obtained by modifying g to be the constant value k between the ascending branches of g.

In the case $\beta = \gamma$, k can be chosen to be any number in $[-1, 1]$, and $g_k(u) \equiv g(u)$.

Theorem 4. *(Existence of monotone traveling waves) Assume that* $(\mathbf{J})^L$ *and* $(\mathbf{F})$ *hold. There exists a monotone traveling wave solution* $u_n(t) = u(n - ct)$ *of (3.1), such that* $u(-\infty) = -1$ *and* $u(+\infty) = 1$. *Moreover,*

A. *If there exists* k *such that* $\int_{-1}^{1} g_k(u)du = 0$, *then* $c = 0$.

B. *If the condition in A does not hold, it is still possible that* $c = 0$. *Nevetheless, if* $c \neq 0$, *then* $\operatorname{sgn} c = \operatorname{sgn} \int_{-1}^{1} f(u)du$.

C. $c_\delta \neq 0$ *if one of the following conditions holds:*

(a) $\int_{-1}^{1} f(u)du \neq 0$ *and* $0 < \lambda \leq \lambda(f)$, *where* $\lambda(f)$ *is small enough;*

(b) g *is monotone and there exists* $u^* \in (-1, 1)$ *such that* $|\lambda f(u^*)| > 1$ *;*

(c) g *is nonmonotone and there exists* u^* *such that (i)* $\lambda f(u^*) < -1$ *and* $u^* \in (-1, \beta)$, *or (ii)* $\lambda f(u^*) < -1$, $u^* \in (\gamma, 1)$ *and* $g(\beta) < g(u^*)$, *or analogous conditions when* $\lambda f(u^*) > 1$.

Remark. *The wave is strictly monotone if supp*(J) *contains* 1 *or contains two relatively prime integers. A condition such as this is natural, since with only second-nearest-neighbor* *interaction for instance, the system decouples into one with odd subscripts and one with even subscripts. The two independent solutions can then be translated in any way to provide a composite non-monotone solution. Even if we specify a sign change at $n = 0$, two copies of the same solution can be combined to produce a single solution which is not strictly monotone. Similar considerations apply if the interaction is such that the system can be decoupled into several independent systems.*

The proof of the theorem is rather technical and so is omitted here. For details one may refer to [4]. Suffice it to say that for existence and parts A. and B., one first approximates J_δ by a smooth, compactly supported function J_m and then applies the results found in [5], in particular,

Proposition 2. *With J_δ in (3.2) replaced by J_m, there exists a solution (u_m, c_m) of (3.2) such that u_m is (strictly) monotone, $u_m(-\infty) = -1$ and $u_m(+\infty) = 1$. Moreover,*

$$c_m = 0 \text{ if and only if there exists } k \text{ such that } \int_{-1}^{1} g_k(u)du = 0, \tag{3.4}$$

and otherwise, sgn c_m = sgn $\int_{-1}^{1} f(u)du$.

Our solution is found by passing to the limit as $J_m \to J_\delta$ in the appropriate sense. Note that A., which follows from (3.4), gives a very simple sufficient condition for pinning. In [4] we show that, unlike for the continuum case, the existence of such a k is not necessary for pinning. Conclusion C. (a) of the above theorem, which says that the wave moves if f is not balanced and the interaction is sufficiently strong, is included as a special case of the main result in the next section. Conclusions C. (b) and (c) give motion when the interaction is not so strong but f is sufficiently unbalanced in some sense. On the other hand, the requirement in C. (b), for instance, allows $\lambda \int_{-1}^{1} f$ to be arbitrarily close to 0 and still have a wave of nonzero speed, so the sense in which f is unbalanced has more to do with its shape than with the difference between the depths of the two wells.

As was noted at the start of this section, these results are for the case that $J \geq 0$, relying as they do on the above proposition. This is because, to prove the proposition, a comparison principle is used to show strict monotonicity of waves. In fact, the proof involves a homotopy argument, deforming the traveling wave for the Allen-Cahn equation

(1.5) to our equation and strict monotonicity of the waves obtained throughout the homotopy allows the use of the Implicit Function Theorem to continue the solution branch. In the next section, we drop the comparison principle entirely by allowing J to change sign. Thus, the proofs are quite different and we are unable to show that waves are monotone.

4. Indefinite J and Small λ

We again consider the discrete, one-dimensional case where $\Lambda = \mathbb{Z}$ but now replace the positivity condition on J by one in transform space, allowing J to change sign but retaining $\lambda > 0$ and $\sum_{n\neq 0} J(n) = 1$. Most of the results here are for large λ and so we rewrite equation (3.1) as

$$\dot{u}_n = \frac{1}{\varepsilon^2}\sum_{k=-\infty}^{\infty} \alpha_k u_{n-k} - f(u_n),\ n \in \mathbb{Z}, \tag{4.1}$$

where $\varepsilon > 0$ is small, $\sum_k \alpha_k = 0$, $\alpha_0 < 0$, and $\alpha_{-k} = \alpha_k$. We shall not assume $\alpha_k > 0$ for all $k \neq 0$, but we assume that $\sum_{k>0} \alpha_k k^2 = 1$.

Setting $u_n(t) = u(\varepsilon n + ct)$ and using the properties of the α_k's, we may write the traveling wave equation for (4.1) in variable $x = \varepsilon n + ct$ as

$$0 = cu' - \sum_{k>0} \alpha_k k^2 \frac{u(x+k\varepsilon) + u(x-k\varepsilon) - 2u(x)}{(k\varepsilon)^2} + f(u). \tag{4.2}$$

We seek a solution u such that $u(\pm\infty) = \pm 1$.

Formally, as $\varepsilon \to 0$, we obtain the traveling wave equation for the Allen-Cahn equation:

$$cu' - u'' + f(u) = 0 \ \text{ in } \mathbb{R}, \qquad u(\pm\infty) = \pm 1. \tag{4.3}$$

It is well-known that (4.3) has a unique (up to translation) traveling wave profile, u_0, and wavespeed, c_0. Furthermore, $u_0' > 0$ and the operator obtained by setting $c = c_0$ and linearizing the left hand side of (4.3) at u_0, has 0 as a simple isolated eigenvalue, the remaining spectrum being in the open right half-plane. It is therefore natural to hope that for $\varepsilon > 0$ and sufficiently small, (4.2) also has a traveling wave $(u_\varepsilon, c_\varepsilon)$ close to (u_0, c_0). We prove that this is the case, first under the assumption that $c_0 \neq 0$, that is, $\int_{-1}^{1} f(u)du \neq 0$. We then consider stationary waves in the case that f is balanced.

We uniquely determine u_0 by requiring $u_0(0) = 0$. For brevity and to suggest what is to follow we introduce the notation

$$\Delta_\varepsilon u := \frac{1}{\varepsilon^2}\sum_{k>0}\alpha_k(u(x+\varepsilon k)+u(x-\varepsilon k)-2u(x)). \tag{4.4}$$

Thus, we study

$$c_\varepsilon u_\varepsilon{}' - \Delta_\varepsilon u_\varepsilon + f(u_\varepsilon) = 0 \quad \text{on } \mathbb{R}, \quad u_\varepsilon(\pm\infty) = \pm 1. \tag{4.5}$$

Instead of positivity of the coefficients, we shall assume

(A1) $\sum_{k>0}\alpha_k k^2 = 1$, $\sum_{k>0}|\alpha_k|k^2 < \infty$, and $\sum_{k>0}\alpha_k(1-\cos(kz)) \geq 0$ for all $z \in [0, 2\pi]$.

As an example, with only nearest and second nearest neighbor interaction, $\alpha_1 \geq 0$, $\alpha_2 = (1-\alpha_1)/4 \leq 1/4$, and $\alpha_2 < 0$ if and only if $\alpha_1 > 1$.

Theorem 5. *Suppose $c_0 \neq 0$. Assume that f satisfies* **(F)** *and $\{\alpha_k\}$ satisfy* **(A1)**. *Then there exists a positive constant ε^* such that for every $\varepsilon \in (0, \varepsilon^*)$, problem (4.5) admits at least one solution, $(c_\varepsilon, u_\varepsilon)$, which is locally unique and has the property that*

$$\lim_{\varepsilon\searrow 0}(c_\varepsilon, u_\varepsilon) = (c_0, u_0) \quad \text{in } \mathbb{R}\times H^1(\mathbb{R}).$$

The outline of the proof is as follows. We write

$$u_\varepsilon = u_0 + \phi_\varepsilon, \qquad \phi_\varepsilon \in H^1(\mathbb{R}).$$

Then the traveling wave problem (4.5) is equivalent to finding $(c_\varepsilon, \phi_\varepsilon) \in \mathbb{R}\times H^1(\mathbb{R})$ such that

$$\mathcal{L}^+_{\varepsilon,\delta}\phi_\varepsilon = \mathcal{R}(c_\varepsilon, \phi_\varepsilon), \tag{4.6}$$

where

$$\begin{aligned}
\mathcal{L}^\pm_{\varepsilon,\delta}\phi &= \{\pm c_0\tfrac{d}{dx} - \Delta_\varepsilon + f_u(u_0(x)) + \delta\}\phi, && (4.7)\\
\mathcal{R}(c,\phi) &= (c_0 - c)(u_0' + \phi') + (\Delta_\varepsilon - \tfrac{d^2}{dx^2})u_0 + \delta\,\phi - \mathbf{N}(u_0,\phi), && (4.8)\\
\mathbf{N}(u_0,\phi) &= f(u_0+\phi) - f(u_0) - f_u(u_0)\phi && (4.9)
\end{aligned}$$

and $\delta > 0$ is a small positive constant chosen at our convenience. The operator $\mathcal{L}^-_{\varepsilon,\delta}$ is introduced since it is the adjoint of $\mathcal{L}^+_{\varepsilon,\delta}$ and we use the Fredholm Alternative Theorem.

The most difficult part of the proof involves showing that $\mathcal{L}^+_{\varepsilon,\delta}$ has a bounded inverse, $(\mathcal{L}^+_{\varepsilon,\delta})^{-1}$, from $L^2(\mathbb{R})$ to $H^1(\mathbb{R})$, and that when restricted to the orthogonal complement of $u_0'e^{-c_0x}$ (this generates the kernel of the limit of the adjoint operator as $\varepsilon \to 0$ and $\delta = 0$), $(\mathcal{L}^+_{\varepsilon,\delta})^{-1}$ is bounded independent of ε and δ (for sufficiently small positive ε). Hence, for every small $\phi \in H^1(\mathbb{R})$, we choose $c_\varepsilon = c_\varepsilon(\phi)$ such that $\mathcal{R}(c_\varepsilon(\phi), \phi)$ is orthogonal to $u_0'e^{-c_0x}$. Then we define $\tilde{\phi} = (\mathcal{L}^+_{\varepsilon,\delta})^{-1}\mathcal{R}(c_\varepsilon(\phi), \phi)$. One can show that the mapping $\phi \to \tilde{\phi}$ is a contraction and thus possesses a fixed point, in some small ball in $H^1(\mathbb{R})$, thereby establishing the existence of a solution to (4.5). Details may be found in [2].

The following lemma encapsulates the essential properties of Δ_ε which are useful in the above analysis and may be of independent interest.

Lemma 4.1. *Let Δ_ε be defined as in (4.4), where $\{\alpha_k\}$ satisfy* **(A1)**. *Then*

(1) *for any $\phi \in L^\infty(\mathbb{R})$ with $\phi'' \in L^2(\mathbb{R})$, $\|\Delta_\varepsilon\phi - \phi''\|_{L^2} \to 0$ as $\varepsilon \searrow 0$;*

(2) *for any $\phi \in H^1(\mathbb{R})$, $(\Delta_\varepsilon\phi, \phi') = 0$;*

(3) *for any $\phi, \psi \in L^2(\mathbb{R})$, $(\Delta_\varepsilon\phi, \psi) = (\phi, \Delta_\varepsilon\psi)$ and $(\Delta_\varepsilon\phi, \phi) \le 0$.*

The proof, which uses Fourier transforms, is straightforward and is omitted.

To bound the inverse of $\mathcal{L}^+_{\varepsilon,\delta}$ we begin by studying the $\varepsilon \to 0$ limit case, where Δ_ε becomes $\frac{d^2}{dx^2}$. Hence, we introduce operators $\mathcal{L}_0^\pm$ and functions $\phi_0^\pm$ by

$$\mathcal{L}_0^\pm\phi := \pm c_0\phi' - \phi'' + f_u(u_0)\phi\,, \tag{4.10}$$
$$\phi_0^+ = u_0' \,/\, \|u_0'\|_{L^2}, \quad \phi_0^- = u_0'e^{-c_0x} \,/\, \|u_0'e^{-c_0x}\|_{L^2}\,.$$

Lemma 4.2. *Let $\mathcal{L}_0^\pm$ and $\phi_0^\pm$ be as in (4.10). The following hold.*

(1) $\phi_0^\pm \in H^2(\mathbb{R})$ and $\mathcal{L}_0^\pm\phi_0^\pm = 0$.

(2) For every $\psi \in L^2(\mathbb{R})$, the problem

$$\mathcal{L}_0^\pm\phi = \psi, \qquad \phi \in H^2 \ \text{ with } \ \phi \perp \phi_0^\pm$$

has a unique solution ϕ if and only if $\psi \perp \phi_0^\mp$. In addition, there exists a positive constant C_1, which depends only on f, such that

$$\|\phi\|_{H^2} \le C_1\|\,\mathcal{L}_0^\pm\phi\,\|_{L^2} \qquad \text{for all } \ \phi \in H^2(\mathbb{R}) \ \text{ satisfying } \ \phi \perp \phi_0^\pm.$$

(3) There exists a positive constant C_2, depending only on f, such that for every $\delta > 0$,

$$\|\phi\|_{H^2} \le C_2\Big\{\|\psi\|_{L^2} + \frac{1}{\delta}|(\psi, \phi_0^\mp)|\Big\} \qquad \text{for all } \ \phi \in H^2(\mathbb{R}), \text{where } \ \psi = \mathcal{L}_0^\pm\phi + \delta\phi. \tag{4.11}$$

The proof is fairly straightforward and is omitted. We only mention that for part (3) the proof is separated into the cases where δ is large, small and of intermediate size.

For every positive δ and ε, we define

$$(4.12)\quad \Lambda^{\pm}(\varepsilon,\delta)=\inf_{\|\phi\|_{H^1}=1}\Big\{\|\mathcal{L}^{\pm}_{\varepsilon,\delta}\phi\|_{L^2}+\frac{1}{\delta}|(\mathcal{L}^{\pm}_{\varepsilon,\delta}\phi\,,\,\phi_0^{\mp})|\Big\},\quad \Lambda^{\pm}(\delta)=\liminf_{\varepsilon\searrow 0}\Lambda^{\pm}(\varepsilon,\delta).$$

Lemma 4.3. *There exists a positive constant C_0 such that $\Lambda^{\pm}(\delta)>C_0$ for all $\delta>0$.*

Proof. Let $\delta>0$ be any fixed positive constant. By the definition of $\Lambda^{\pm}(\delta)$, there exists a sequence $\{(\varepsilon_j,\phi_j)\}_{j=1}^{\infty}$ in $(0,1)\times H^1(\mathbb{R})$ such that $\lim_{j\to\infty}\varepsilon_j=0$, $\|\phi_j\|_{H^1}=1$ for all j, and $\psi_j:=\mathcal{L}^{\pm}_{\varepsilon_j,\delta}\phi_j$ satisfies

$$\lim_{j\to\infty}\{\|\psi_j\|_{L^2}+\tfrac{1}{\delta}|(\psi_j,\,\phi_0^{\pm})|\}=\Lambda^{\pm}(\delta).$$

By taking a subsequence if necessary, we can assume that there exist functions $\phi\in H^1$ and $\psi\in L^2$ such that, as $j\to\infty$,

$$\begin{aligned}\phi_j &\longrightarrow \phi &&\text{in } L^2_{\rm loc}(\mathbb{R}) \text{ and weakly in } H^1(\mathbb{R})\,,\\ \psi_j &\longrightarrow \psi &&\text{weakly in } L^2(\mathbb{R}).\end{aligned}$$

By the weak lower semi–continuity of the $L^2(\mathbb{R})$ norm, $\|\psi\|_{L^2}+\frac{1}{\delta}|(\psi,\phi_0^{\mp})|\le\Lambda^{\pm}(\delta)$.

For any test function $\zeta\in C_0^\infty(\mathbb{R})$, $(\psi_j,\zeta)=(\mathcal{L}^{\pm}_{\varepsilon_j,\delta}\phi_j,\,\zeta)=(\phi_j,\mathcal{L}^{\mp}_{\varepsilon_j,\delta}\zeta)$. Since $\lim_{\varepsilon\searrow 0}\|\Delta_\varepsilon\zeta-\zeta''\|_{L^2}=0$, sending $j\to\infty$ we obtain $(\psi,\zeta)=(\phi,\,(\mathcal{L}_0^{\mp}+\delta)\zeta)$ for all $\zeta\in C_0^\infty(\mathbb{R})$. That is, $\phi\in H^1(\mathbb{R})$ is a weak solution to $(\mathcal{L}_0^{\pm}+\delta)\phi=\psi$. An elliptic estimate then shows that ϕ is in $H^2(\mathbb{R})$. Consequently, by Lemma 4.2 (3),

$$(4.13)\qquad \|\phi\|_{H^2}\le C_2\{\|\psi\|_{L^2}+\tfrac{1}{\delta}|(\psi,\phi_0^{\mp})|\}\le C_2\Lambda^{\pm}(\delta).$$

It remains to find a positive lower bound of $\|\phi\|_{L^2}$.

First of all, using $(\mathcal{L}^{\pm}_{\varepsilon_j,\delta}\phi_j,\phi_j')=(\psi_j,\phi_j')$ and the identity $(\Delta_\varepsilon\phi_j,\phi_j')=0=(\phi_j,\phi_j')$, we obtain $c_0\|\phi_j'\|^2_{L^2}=(\psi_j,\phi_j')-(f_u(u_0)\phi_j,\phi_j')$. Cauchy's inequality then gives

$$\|f_u(u_0)\|_{L^\infty}\|\phi_j\|_{L^2}\ge|c_0|\;\|\phi_j'\|_{L^2}-\|\psi_j\|_{L^2}\,,$$

which implies

$$(4.14)\qquad 2\|f_u(u_0)\|^2_{L^\infty}\|\phi_j\|^2_{L^2}\ge c_0^2\|\phi_j'\|^2_{L^2}-2\|\psi_j\|^2_{L^2}.$$

Let m be a positive constant such that

$$0 < a := \tfrac{1}{2}\min\{f_u(1), f_u(-1)\} = \min_{|x|\geq m}\{f_u(u_0(x))\}.$$

Using $(\psi_j, \phi_j) = (\mathcal{L}^{\pm}_{\varepsilon_j,\delta}\phi_j, \phi_j)$, the identity $(\phi_j', \phi_j) = 0$, and the fact $(-\Delta_{\varepsilon_j}\phi_j, \phi_j) \geq 0$, we obtain

$$\begin{aligned}(\psi_j, \phi_j) &\geq (f_u(u_0)\phi_j, \phi_j) \geq \min_{|x|\geq m}\{f_u(u_0)\}\int_{|x|\geq m}\phi_j^2 - \|f_u(u_0)\|_{L^\infty}\int_{|x|\leq m}\phi_j^2 \\ &= a\|\phi_j\|_{L^2}^2 - (a + \|f_u(u_0)\|_{L^\infty})\int_{|x|\leq m}\phi_j^2.\end{aligned}$$

Therefore,

$$(4.15)\quad (a + \|f_u(u_0)\|_{L^\infty})\int_{|x|\leq m}\phi_j^2 \;\geq\; a\|\phi_j\|_{L^2}^2 - (\phi_j, \psi_j) \geq \tfrac{a}{2}\|\phi_j\|_{L^2}^2 - \tfrac{1}{2a}\|\psi_j\|_{L^2}^2.$$

Adding a multiple of (4.14), we see that there exist positive constants C_3 and C_4, which depend on $|c_0| > 0$ and f, such that

$$\int_{|x|\leq m}\phi_j^2 \geq C_3\|\phi_j\|_{H^1}^2 - C_4\|\psi_j\|_{L^2}^2 = C_3 - C_4\|\psi_j\|_{L^2}^2.$$

Sending $j \to \infty$ we then conclude that

$$(4.16)\qquad \int_{|x|\leq m}\phi^2 \geq C_3 - C_4\Lambda^{\pm}(\delta).$$

In view of (4.13), we then obtain $\Lambda^{\pm}(\delta) \geq C_3/(C_2 + C_4) = C_0$. This completes the proof. □

The proof of Theorem 5 now proceeds as outlined above and is omitted.

We now turn our attention to stationary solutions. The case $c_0 = 0$ is more difficult in some respects and our attack is different, using variational arguments.

Assume that f is balanced, that is, it is the derivative of a double equal-well potential. Since $f'(\pm 1) > 0$ we may write

$$(4.17)\qquad f(u) = F'(u) \quad \text{where} \quad F(\pm 1) = 0 \quad \text{and} \qquad F(u) \geq m_0(1 - u^2)^2$$

for a positive constant m_0 and $u \in [-2, 2]$. When u is outside $[-2, 2]$, we assume that F is bounded away from zero, and approaches infinity as $|u| \to \infty$. We look for a function u_ε such that

$$(4.18)\qquad -\Delta_\varepsilon u_\varepsilon + f(u_\varepsilon) = 0 \qquad \text{for all } x \in \mathbb{R}, \qquad \lim_{x\to\pm\infty} u_\varepsilon(x) = \pm 1.$$

Observe that if u_ε is a solution, then for every $x_0 \in \mathbb{R}$, if we define $u_n = u_\varepsilon(x_0 + n\varepsilon)$ for all $n \in \mathbb{Z}$, then $\{u_n\}_{n=-\infty}^{\infty}$ satisfies

$$(4.19)\qquad \begin{cases} -\varepsilon^{-2}\sum_{k>0}\alpha_k(u_{n+k}+u_{n-k}-2u_n)+f(u_n)=0 & \text{for all } n\in\mathbb{Z}, \\ \lim_{n\to\pm\infty} u_n = \pm 1. \end{cases}$$

On the other hand, if a sequence $\{u_n\}$ satisfies (4.19), then the function $u_\varepsilon = \sum_n u_n\chi_n^\varepsilon$ is a solution to (4.18) where χ_n^ε is the characteristic function of the set $(\varepsilon(n-1/2), \varepsilon(n+1/2)]$, i.e.,

$$(4.20)\qquad \chi_n^\varepsilon(x) = \begin{cases} 1 & \text{if } x\in(\varepsilon(n-1/2),\varepsilon(n+1/2], \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the solvability of (4.18) and (4.19) are equivalent.

To show that (4.18) or (4.19) admits a solution, we need stronger assumptions on Δ_ε than we needed for the non–stationary wave case. Let

$$(4.21)\qquad B(\zeta) := \sum_{k>0}\alpha_k\frac{\sin^2\frac{k\zeta}{2}}{\sin^2\frac{\zeta}{2}} = \sum_{k>0}\alpha_k\Big|\sum_{\ell=0}^{k-1}e^{\mathrm{i}\ell\zeta}\Big|^2, \quad \zeta\in\mathbb{R}.$$

Notice that $B(\cdot)$ is 2π-periodic and even. It can be expanded as a Fourier cosine series

$$(4.22)\ B(\zeta) = \frac{b_0}{2} + \sum_{\ell>0} b_\ell\cos(\ell\zeta) \ \text{ for } \zeta\in\mathbb{R} \ \text{ where } b_\ell = \frac{1}{\pi}\int_0^{2\pi} B(\zeta)\cos(\ell\zeta)\,d\zeta.$$

We assume the following:

(A2) $B(\zeta)$ is uniformly positive, bounded, and in $H^{1/2}([0,2\pi])$; that is, there exists a positive constant B_∞ such that

$$\frac{1}{B_\infty} \le B(\zeta) \le B_\infty \qquad \text{for all } \zeta\in\mathbb{R}$$

and

$$\sum_{\ell>0}\ell\, b_\ell^2 < \infty\,.$$

Theorem 6. *Assume that $f \in C^2$ satisfies (4.17) and $\{\alpha_k\}_{k=1}^{\infty}$ satisfies* **(A2)**. *Then for every $\varepsilon > 0$, problem (4.18) or problem (4.19) admits at least one solution.*

The theorem is proved via an energy minimization method. We define an energy $\mathbf{E}$ by

$$(4.23)\ \mathbf{E}[u] = \mathcal{E}_{ki}[u] + \mathcal{E}_{po}[u], \quad \mathcal{E}_{po}[u] = 2\int_{\mathbb{R}} F(u)\,dx, \quad \mathcal{E}_{ki}[u] = -\int_{\mathbb{R}} u\Delta_\varepsilon u\,dx,$$

where "po" stands for potential and "ki"" for kinetic. We show that $\mathbf{E}[\cdot]$ has a minimizer in the space

$$\mathbf{X} := \left\{ u = \sum_n u_n \chi_n^\varepsilon \ : \ \sum_{n=0}^{\infty} |1 - u_n|^2 + \sum_{n=-1}^{-\infty} |1 + u_n|^2 < \infty \right\}, \tag{4.24}$$

giving the desired solution.

First we write the kinetic energy $-(\Delta_\varepsilon u, u)$ in a convenient form for piecewise constant functions. Using Parseval's identity, one may prove

Lemma 4.4. *Let $u(x) = \sum_n u_n \chi_n^\varepsilon$ and $v(x) = \sum_n v_n \chi_n^\varepsilon$. Assume that $\sum_n (u_{n+1} - u_n)^2 < \infty$ and $\sum_n (v_{n+1} - v_n)^2 < \infty$. Then,*

$$(-\Delta_\varepsilon u, v) = \frac{1}{2\pi\varepsilon} \int_0^{2\pi} B(\zeta)\phi(\zeta)\overline{\psi(\zeta)}\, d\zeta, \tag{4.25}$$

where $B(\zeta)$ is as in (4.21),

$$\phi(\zeta) = \sum_n (u_{n+1} - u_n) e^{\mathrm{i}n\zeta} \quad \text{and} \quad \psi(\zeta) = \sum_m (v_{m+1} - v_m) e^{\mathrm{i}m\zeta}. \tag{4.26}$$

Consequently,

$$\mathcal{E}_{ki}[u] = \frac{1}{2\pi\varepsilon} \int_0^{2\pi} B(\zeta) \Big|\phi(\zeta)\Big|^2 d\zeta.$$

When all α_k's for $k > 0$ are non–negative, the energy of any non–monotonic function can be decreased by removing the "bumps" of the function. In our current situation where some of the α_k's may be negative, we cannot use this modification. Indeed, an energy minimizer may not necessary be monotonic. Hence, to show that an energy minimizer satisfies needed asymptotic behavior as $x \to \pm\infty$, we need extra care.

For convenience we redefine m_0 so that

$$f_u > m_0 \qquad \text{in} \quad (-1 - m_0, -1 + m_0) \cup (1 - m_0, 1 + m_0). \tag{4.27}$$

By making careful comparisons for both potential and kinetic energies, one can prove

Lemma 4.5. *Let M be any fixed positive integer. Assume that $\mathbf{E}[u] < \infty$ and $|u+1| \le m_0$ on $(-\varepsilon/2, \varepsilon(M + 1/2)]$. Then for any $\eta \in (0, 1)$,*

$$\mathbf{E}[u^r] + \mathbf{E}[u^l] \le \tfrac{(1+\bar{b}_M B_\infty)}{1-\eta} \mathbf{E}[u] + \tfrac{2B_\infty}{\eta(1-\eta)} \left\{ \frac{\max_{0 \le n < M} |u_n + 1|^2}{\varepsilon M} + \sum_{0 \le n < M} \frac{(u_{n+1} - u_n)^2}{\varepsilon} \right\},$$

where

$$u^r = -1 + \theta(1+u), \quad u^l = -1 + (1-\theta)(1+u), \quad \textit{and} \ \ \theta = \sum_{n\geq 0} \min\{\frac{n}{M}, 1\}\chi_n^\varepsilon.$$

Notice that $\theta = 0$ for $x \le \varepsilon/2$ and $\theta = 1$ for $x \ge (M - 1/2)\varepsilon$. It then follows that $u^r = -1, u^l = u$ for $x \le \varepsilon/2$ and $u^r = u, u^l = -1$ for $x > (M - 1/2)\varepsilon$. This lemma shows that if u is in a "resting" state for a large interval, i.e., both $\varepsilon^{-1}\sum_{0<n<M}|u_{n+1} - u_n|^2$ and $\max_{0\le n<M}|u_n + 1|$ are small, then the energy of u can be decomposed as the sum of the energy of u^r and that of u^l. In particular, it eliminates the possibility of energy minimizers having transition layers in "remote" locations. This property is crucial in our proof of the existence of an energy minimizer with required asymptotics at $x = \pm\infty$.

One can also easily provide an energy lower bound for states which change sign.

Lemma 4.6. *There exists a positive constant e_0 such that $\mathbf{E}[u] \ge e_0$ if u changes sign at least once, i.e., $u_n u_{n+1} \le 0$ for some $n \in \mathbb{Z}$, where $u = \sum_n u_n \chi_n^\varepsilon$.*

In fact, we may take

$$e_0 = \min\{\ 2\varepsilon \min_{|s|\le 1/2}\{F(s)\},\ 1/(4B_\infty\varepsilon)\ \}.$$

The existence of a minimizer for each $\varepsilon > 0$ (not necessarily small) now may be established by taking a minimizing sequence in $\mathbf{X}$ (translated to have a change of sign at $n = 0$), using the coercivity of F to extract a subsequence convergent on each integer, and then using the above lemmas to show that the limit is in fact an energy minimizer in $\mathbf{X}$. Details may be found in [2].

Even though this result is for all $\varepsilon > 0$, when $\varepsilon \to 0$ one again obtains convergence of the minimizer to u_0:

Theorem 7. *Let $\{u^\varepsilon\}_{\varepsilon>0}$ be the energy minimizing solutions to (4.19) obtained above, translated so that $x = \varepsilon/2$ is the first place where $u^\varepsilon = \sum_n u_n^\varepsilon \chi_n^\varepsilon$ experiences a sign change; i.e., $u_n^\varepsilon < 0$ for all $n \le 0$ and $u_1^\varepsilon \ge 0$. Then,*

$$\lim_{\varepsilon\searrow 0} u_\varepsilon = u_0 \qquad \textit{in}\ \ L^\infty(\mathbb{R}),$$

$$\lim_{\varepsilon\searrow 0}(\tilde{u}_\varepsilon - u_0) = 0 \qquad \textit{in}\ \ H^1(\mathbb{R})$$

where $\tilde{u}^\varepsilon$ is the "companion" of u^ε obtained by a linear interpolation of the node values of u^ε at εn, $n \in \mathbb{Z}$:

$$\tilde{u}^\varepsilon(x) = \sum_n \left\{u_n^\varepsilon + \tfrac{(x-\varepsilon n)}{\varepsilon}(u_{n+1}^\varepsilon - u_n^\varepsilon)\right\}\chi_n^\varepsilon(x - \varepsilon/2).$$

The proof requires an upper bound on the minimal energy and proceeds by first establishing uniform convergence on compact subintervals of $\mathbb{R}$. Getting convergence of the tails is more delicate (see [2] for details).

We conclude by presenting perhaps an initially surprising result concerning the question of uniqueness of solutions:

Theorem 8. *There exists ε_2 such that for every $\varepsilon \in (0, \varepsilon_2]$, problem (4.19) admits at least two solutions, u_ε^1 and u_ε^2, which differ by more than translation, and as $\varepsilon \to 0$, $\|u_\varepsilon^i - u_0\|_{L^\infty(\mathbb{R})} \to 0$ for $i = 1, 2$.*

The proof requires some preparation. First we investigate the operator

$$\mathcal{L}_\varepsilon\phi \ := \ -\Delta_\varepsilon\phi + f_u(u_0)\phi \tag{4.28}$$

for functions lying in the space

$$\mathbf{X}_0 := \{\phi = \textstyle\sum_n \phi_n\chi_n^\varepsilon \ : \ \sum_n \phi_n^2 < \infty\},$$

which are clearly constant on every interval $(\varepsilon(n - 1/2), \varepsilon(n + 1/2)]$, $n \in \mathbb{Z}$. We define

$$\begin{aligned}\Lambda(\varepsilon) \ &:= \inf_{\phi\in\mathbf{X}_0,\|\phi\|_{L^2}=1,\phi(0)=0}\left\{\| - \Delta_\varepsilon\phi + f_u(u_0)\phi\|_{L^2(\mathbb{R}\setminus(-\varepsilon/2,\varepsilon/2])}\right\},\\ \Lambda_0 \ &:= \liminf_{\varepsilon\searrow 0}\Lambda(\varepsilon)\ .\end{aligned}$$

Lemma 4.7. $\Lambda_0 > 0$. *Consequently, $\Lambda(\varepsilon) > \Lambda_0/2$ for all small positive ε.*

Proof. By the definition of Λ_0 there is a sequence $\{\varepsilon_j, \phi_j, \psi_j\}$ such that $\lim_{j\to\infty}\varepsilon_j = 0$, $\lim_{j\to\infty}\|\psi_j\|_{L^2(\mathbb{R}\setminus(-\varepsilon_j/2,\varepsilon_j/2])} = \Lambda_0$, and for each $j \geq 1$, $\varepsilon_j > 0, \phi_j \in \mathbf{X}_0$, $\|\phi_j\|_{L^2} = 1$, $\phi_j = 0$ on $(-\varepsilon_j/2, \varepsilon_j/2]$, and $\psi_j = -\Delta_{\varepsilon_j}\phi_j + f_u(u_0)\phi_j$.

Using the identity $(-\Delta_{\varepsilon_j}\phi_j, \phi_j)+(f_u(u_0)\phi_j, \phi_j) = (\psi_j, \phi_j)$ and that $\phi_j = 0$ on $(-\varepsilon_j/2, \varepsilon_j/2]$, we obtain

$$\frac{1}{B_\infty}\|\tilde{\phi}_j'\|^2 + \int_{\mathbb{R}} f(u_0)\phi_j^2 \leq \|\phi_j\|_{L^2}\|\psi_j\|_{L^2(\mathbb{R}\setminus(-\varepsilon_j/2,\varepsilon_j/2])},$$

where $\tilde{\phi}_j$ is the linear interpolant of ϕ at node points. It then follows that $\|\tilde{\phi}'\|_{L^2}$ is uniformly bounded. Consequently, $\|\phi_j - \tilde{\phi}_j\|_{L^2(\mathbb{R})}$ is of size $O(\varepsilon_j^2)$. Thus, we can select a subsequence from $\{\tilde{\phi}_j, \psi_j\}$, still denoted by $\{\tilde{\phi}_j, \psi_j\}$, such that for some $\phi \in H^1(\mathbb{R})$ and $\psi \in L^2(\mathbb{R})$,

$$\begin{aligned} \tilde{\phi}_j &\longrightarrow \phi \qquad \text{in } L^2_{\rm loc}(\mathbb{R}) \text{ and weakly in } H^1(\mathbb{R}), \\ \psi_j &\longrightarrow \psi \qquad \text{weakly in } L^2_{\rm loc}(\mathbb{R} \setminus \{0\}). \end{aligned}$$

In addition, $\phi(0) = 0$. In the weak formulation, one can show that $-\phi'' + f(u_0)\phi = \psi$ in $\mathbb{R}\setminus\{0\}$. By an estimate similar to (4.15), (4.16), we conclude that $\Lambda_0 > 0$. This completes the proof. □

Now we consider a related constrained minimization problem. For every $\alpha \in (-1, 1)$ we define

$$\mathbf{X}_\alpha := \{u = \textstyle\sum_n u_n \chi_n^\varepsilon \;\; : \;\; u_0 = \alpha, \; \sum_{n>0} |1 - u_n|^2 + \sum_{n<0} |1 + u_n|^2 < \infty\}.$$

Define

$$E(\alpha, \varepsilon) := \inf_{u \in \mathbf{X}_\alpha} \mathbf{E}[u], \quad \mathrm{E}(\varepsilon) = \inf_{\alpha \in (-1,1)} E(\alpha, \varepsilon).$$

We note that $\mathrm{E}(\varepsilon)$ is the energy of the minimizer we discussed above.

It is not hard to prove the following:

Lemma 4.8. *There exists $\varepsilon_0 > 0$ such that for every $\alpha \in [-1/2, 1/2]$ and $\varepsilon \in (0, \varepsilon_0]$, there exists $u_\varepsilon^\alpha \in \mathbf{X}_\alpha$ such that $\mathbf{E}[u_\varepsilon^\alpha] = E(\alpha, \varepsilon)$.*

Now we compare this minimizer with u_0.

Lemma 4.9. *For every $\delta > 0$, there exists $\varepsilon_1(\delta) > 0$ such that if $\varepsilon \in (0, \varepsilon_1(\delta)], \alpha \in [-1/2, 1/2]$ and $u_\varepsilon^\alpha \in \mathbf{X}_\alpha$ is a minimizer of $\mathbf{E}[u]$ in $\mathbf{X}_\alpha$, then $\|u_\varepsilon^\alpha - u_0(z(\alpha) + \cdot)\|_{L^2 \cap L^\infty} \le \delta$, where $z(\alpha)$ is the point satisfying $u_0(z(\alpha)) = \alpha$.*

We omit the proof. Using this and the eigenvalue estimate above one can then show

Lemma 4.10. *There exists $\varepsilon_2 > 0$ such that for every $\varepsilon \in (0, \varepsilon_2]$ and $\alpha \in [-1/2, 1/2]$, the minimizer $u_\varepsilon^\alpha \in \mathbf{X}_\alpha$ for $\mathbf{E}[u]$ in $\mathbf{X}_\alpha$ is unique.*

This uniqueness implies continuity of u_ε^α in α. This allows one to prove

Lemma 4.11. *For every $\varepsilon \in (0, \varepsilon_2]$, $E(\alpha, \varepsilon)$ is continuously differentiable in $\alpha \in (-1/2, 1/2)$ and*

$$\frac{d}{d\alpha} E(\alpha, \varepsilon) = 2\varepsilon\{-\Delta_\varepsilon u_\varepsilon^\alpha + f(u_\varepsilon^\alpha)\}\Big|_{x=0}. \tag{4.29}$$

Consequently, u_ε^α solves (4.18) if and only if $\frac{d}{d\alpha} E(\alpha, \varepsilon) = 0$.

To prove the non-uniqueness theorem above let $(a, b) \subset [-1/2, 1/2]$ be an interval such that for some integers n_1 and n_2, $a = u_{n_1}^\varepsilon < b = u_{n_2}^\varepsilon$. Then consider the differentiable function $E(\alpha, \varepsilon)$ for $\alpha \in [a, b]$.

If $E(\alpha, \varepsilon)$ is a constant function, then every u_ε^α is a solution to (4.18) and hence we have a continuum of solutions to (4.19).

If $E(\alpha, \varepsilon)$ is not a constant function, then as it attains the global minimum $\mathrm{E}(\varepsilon)$ at $\alpha = a$ and $\alpha = b$, there exists at least a local maximum of $E(\cdot, \varepsilon)$ attained at some $c \in (a, b)$, at which $\frac{d}{d\alpha} E(c, \varepsilon) = 0$. Consequently, the local saddle $u_\varepsilon^c \in \mathbf{X}_c$ is a solution to (4.19). Clearly, u_ε^c is different from any translation of u_ε since their energies are different. Translating u_ε^c and using Lemma 4.9 completes the proof. □

Some interesting open questions arise. Consider for simplicity the case when the α_k 's are nonnegative for $k > 0$. By removing possible "bumps", we obtain nonuniqueness of increasing stationary waves for ε small enough. On the other hand, from [3] it follows that for ε large enough there is only one increasing wave. Hence the question: at which ε does the change from nonuniqueness to uniqueness take place? Also, is it true that for ε small enough there is actually a stationary solution to (4.5) of the form $u(x)$ for $x \in \mathbb{R}$, with u continuous, as one might conjecture?

5. An application

In this section, we comment on similarities and differences between (1.4) and another bistable equation which is currently of great interest, the Extended Fisher-Kolmogorov equation

$$u_t = -\gamma u^{iv} + u'' - f(u) \qquad (\gamma > 0). \tag{5.1}$$

This equation can also be thought of as being an L^2 gradient flow, with the underlying functional

$$E(u) = \frac{\gamma}{2}\int_{\mathbb{R}} |u''(x)|^2 dx + \frac{1}{2}\int_{\mathbb{R}} |u'(x)|^2 dx + \int_{\mathbb{R}} W(u(x))dx. \tag{5.2}$$

It has been argued that (5.1) is an "extension" of the Allen-Cahn equation (1.5); however, it is interesting to keep in mind that both these local equations can be obtained from appropriate "truncations" of (1.2). Namely, changing variables in (1.2) using $\eta = \frac{x-y}{2}$, $\xi = \frac{x+y}{2}$ and expanding $u(x) = u(\xi+\eta)$ and $u(y) = u(\xi-\eta)$ about ξ, we get the formal expression, for $E(u)$ in (5.2),

$$\int_{\mathbb{R}} (c_1(u'(\xi))^2 - c_2(u''(\xi))^2 + \dots)d\xi + \int_{\mathbb{R}} W(u(x))dx, \tag{5.3}$$

where $c_1 = \frac{2}{\lambda}\int_{\mathbb{R}} J(2\eta)\eta^2 d\eta$, $c_2 = \frac{2}{3\lambda}\int_{\mathbb{R}} J(\eta)\eta^4 d\eta$, etc. Assuming that J changes sign, c_2 can clearly be negative. With $c_1 > 0$, after truncating the summation in (5.3) and rescaling, one obtains (5.2). Of course one can also have a case in which $c_1 < 0$, which (after rescaling) results in the gradient flow

$$u_t = -\gamma u^{iv} - u'' - f(u),$$

an equation also of interest in pattern formation.

Some current results may suggest that actually there are some nontrivial similarities between (1.4) and (5.1).

Assume for simplicity that $f(u) = u(u^2-1)$. First, for $\gamma < \frac{1}{8}$, it has been shown that (5.2) admits local minimizers with any number of interfaces, located arbitrarily (in a sense) [24]. These may correspond to the pinned solutions constructed in [3] for (1.4). Actually, the stationary version of (5.1)

$$\gamma u^{iv} - u'' + f(u) = 0 \tag{5.4}$$

can be recast as a nonlocal equation. Write $f(u) = bu - S_b(u)$ and consider (5.4) with $S_b(u)$ neglected,

$$\gamma u^{iv} - u'' + bu = 0.$$

When $b \in (0, 1/4\gamma)$, this equation has four real solutions $e^{\pm m_1 x}$, $e^{\pm m_2 x}$ (with $m_1 > m_2 > 0$). After an elementary calculation, one sees that the above linear operator has the

Green's function $J(x-y)$ with

$$J_b(x) = \frac{1}{\gamma(m_1^2 - m_2^2)}\left[\frac{1}{m_2}e^{-m_2|x|} - \frac{1}{m_1}e^{-m_1|x|}\right] > 0.$$

Consequently, putting the $S_b(u)$ term back into the equation, (5.4) can be written as

$$J_b * S_b(u) = u. \tag{5.5}$$

Now if $\gamma < 1/8$, one can find $b \in (0, 1/4\gamma)$ such that S_b is increasing, so that, after the substitution $w = S_b(u)$ and dropping the subscripts b, (5.5) becomes

$$J * w - S^{-1}(w) = 0, \tag{5.6}$$

which has the form of (2.1). Observe that this in particular shows uniqueness of increasing kink (stationary wave) solutions of (5.4) (see [28] and [37] for other proofs), since [5] shows that such solutions are unique for (2.1). For $\gamma > \frac{1}{8}$, in order to obtain (5.5) with $J > 0$, J needs to be the Green's function of $\gamma u^{iv} - u'' + bu$, with $0 < b < \frac{1}{4\gamma}$. However, $S(u) = bu - f(u)$ is then nonmonotone, and (5.5) cannot be written as (5.6). The solvability of (5.5) with S nonmonotone remains an open problem.

Finally, another similarity between (1.4) and (5.1) arises in the occurrence of periodic minimizers, which, it should be pointed out, do not exist for the Allen-Cahn equation. The interested reader is referred to [32] and [12] for details.

6. Appendix

Here we present a proof of the comparison principle mentioned in the introduction. We only give the proof in the continuum case since the discrete case can be proved in the same way.

Theorem A. Assume that $g \in C^1(\mathbb{R})$, $J \geq 0$ and $\int_{\mathbb{R}^n} J = 1$. Let $u, v \in L^\infty(\mathbb{R}^n; C^1([0,T]))$ satisfy $u(x,0) \leq v(x,0)$ for all $x \in \mathbb{R}^n$ and

$$u_t - J * u - g(u) \geq v_t - J * v + g(v) \quad \text{for all } x \in \mathbb{R}^n \quad \text{and} \quad t \in (0,T].$$

Then

$$v(x,t) \leq u(x,t) \quad \text{for all } x \in \mathbb{R}^n \quad \text{and} \quad t \in (0,T].$$

Proof. Set $M = \|u\|_{L^\infty} + \|v\|_{L^\infty}$, $\beta = 3 + \max_{|z|\le M} g'(z)$, and $w = (u-v)e^{-\beta t}$. Then

$$\begin{aligned} w_t - J * w &= -\beta w + [u_t - J*u - v_t + J*v]e^{-\beta t} \\ &\ge -\beta w + [g(u) - g(v)]e^{-\beta t} = w[-\beta + g'(\theta)] \end{aligned} \tag{6.1}$$

for some (function) θ bounded between u and v.

Now suppose the assertion is not true. Then $M_0 := -\inf_{\mathbb{R}^n\times[0,T]} w > 0$ and there exists $(x_0,t_0) \in \mathbb{R}^N \times [0,T]$ such that $w(x_0,t_0) \le -M_0/2$. As $w(\cdot,0)\ge 0$, we have $t_0 > 0$. Since $w(x_0,\cdot)$ is C^1 on $[0,T]$, there exists $\hat{t}_0 \in (0,t_0]$ such that $w(x_0,\hat{t}_0) = \min_{t\in[0,t_0]} w(x_0,t)$. Now at $(x_0,\hat{t}_0)$, we have

$$w_t \le 0, \quad w \le -M_0/2, \quad J*w \ge -M_0, \quad w[-\beta + g'(\theta)] \ge -3w \ge 3M_0/2.$$

However, altogether these conclusions contradict (6.1). This contradiction shows that $w \ge 0$, i.e., $u \ge v$ on $\mathbb{R}^n \times [0,T]$.

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