数理解析研究所講究録
1178 巻 2000 年 87-94

Pulse Dynamics of Reaction-Diffusion Systems

Shin-Ichiro Ei (Yokohama City University)
栄 伸一郎 (横浜市立大学)
joint work with M. Mimura (Hiroshima University)

1 Introduction

In nature, many kinds of spatial and/or temporal patterns are observed, some of them are simple and the others are complicated. To understand theoretically the dynamics of such patterns, many model equations have been proposed and analyzed. Among them, some sort of reaction- diffusion systems are one of the most familar classes.

Recently, several reaction-diffusion model equations have been known as examples exhibiting various complicated behaviors of solutions; self-replicating behavior of pulses ([6] and its references), refleciton of pulses ([3]), the behavior of pulses like elastic objects (e.g. [1], [4], [8], [7]).

In this report, we specially consider the particle like dynamics of pulses in two dimensional space and give a theoretical basis to it.

In [4], following reaction-diffuison systems which has a moving localized solution in two dimensional space was proposed:

$$(1.1)\qquad \left\{\begin{array}{rcl} \sigma u_t &=& \varepsilon\Delta u+\varepsilon^{-1}f(u,v,w),\\ v_t &=& \Delta v+u-v+h_1,\\ \tau w_t &=& d\Delta w+u-w+h_2, \end{array}\right.$$

where $f(u,v,w)=ru-u^3-k_1v-k_2w$. They showed numerically the existence of a moving localized solution, say *travelling spot* , under suitable conditions (Fig.1). They also showed numerically multi travelling spots interact like elastic objects (Fig.2).

In order to understand these phenomena, we first consider the existence of a travelling spot in two dimensional space under suitable conditions. To show the existence of such moving solutions, we assume the existence of stable (radially) symmetric stationary solutions and when it loses the stability, we construct a travelling spot as the bifurcating solutions from it.

Secondly, we analyze their interactions when there exist multiple travelling spots.

As a consequence, we can derive ODEs describing the particle like dynamics. The reduced ODEs show how pulses interact and reflection occur.

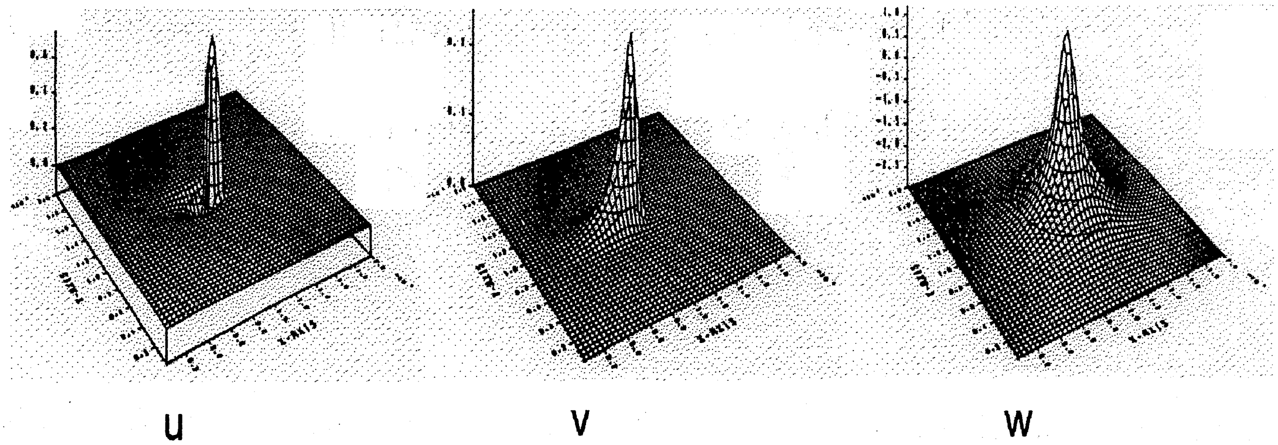

Figure 1: spatial profiles of a travelling spot. Parameter values are $\varepsilon = 0.1$, $\sigma = 0.04$, $r = 1.5$, $k_1 = 1.0$, $k_2 = 5.0$, $h_1 = 1.0$, $h_2 = 0.8$, $\tau = 0.01$, $d = 7.0$.

2 Construction of travelling spot

Let us consider general types of reaction-diffusion systems with bifurcation parameter k;

(2.1) $$\boldsymbol{u}_t = \mathcal{L}(\boldsymbol{u}; k),\ x \in \boldsymbol{R}^2,\ t > 0,$$

where $\mathcal{L}(\boldsymbol{u}; k) = D\Delta \boldsymbol{u} + F(\boldsymbol{u}; k)$, $\boldsymbol{u} \in \boldsymbol{R}^N$ and D is a diagonal matrix with elements $\{d_j\}$ $(j = 1, 2, \cdots, N)$. We assume following assumptions.

H1) There exist a radially symmetric standing solution $S(x)$ such that $\mathcal{L}(S(x); k) = \mathbf{0}$ and $S(x) \to \mathbf{0}$ as $|x| \to \infty$, where $\mathbf{0} = (0, \cdots, 0) \in \boldsymbol{R}^N$.

Let $X = \{L^2(\boldsymbol{R}^2)\}^N$ and let $L(k) = \mathcal{L}'(S(x); k)$ be the linearized operator of (2.1) with respect to $S(x)$ and $\Sigma(k)$ be the spectrum of $L(k)$. Note that $L(k)S_j = 0$ $(j = 1, 2)$ hold and 0 is necessarily eigenvalue of $L(k)$, where $S_j = \dfrac{\partial S}{\partial x_j}$ for $x = (x_1, x_2)$.

H2) There exists $k = k_c$ such that $\Sigma_c = \Sigma(k_c)$ consists of two sets $\Sigma_0 = \{0\}$ and $\Sigma_1 \subset \{z \in \boldsymbol{C};\ Re(z) < -\gamma_0\}$ for positive constant γ_0. The generalized eigenspace corresponding to Σ_0, say X_0, is given by $X_0 = span\{S_j, \Psi_j\}$ $(j = 1, 2)$, where Ψ_j are functions satisfying $L_c\Psi_j = -S_j$ $(j = 1, 2)$.

Let Q_c and R_c be projections at $k = k_c$ with respect to L_c corresponding to the spectral sets Σ_0 and Σ_1, respectively. Define a function $U(x; P, \boldsymbol{\zeta}) = S(x - P) + \sum_{j=1}^{2} \zeta_j \Psi_j$ for P, $\boldsymbol{\zeta} = (\zeta_1, \zeta_2) \in \boldsymbol{R}^2$ and a set $\mathcal{M} = \{S(x - P);\ P \in \boldsymbol{R}^2\}$.

We consider (2.1) in the neighborhood of the parameter $k = k_c$. To do so, we put $k = k_c + \eta$ and rewrite (2.1) as

(2.2) $$\boldsymbol{u}_t = \mathcal{L}_c(\boldsymbol{u}) + \eta g(\boldsymbol{u}),$$

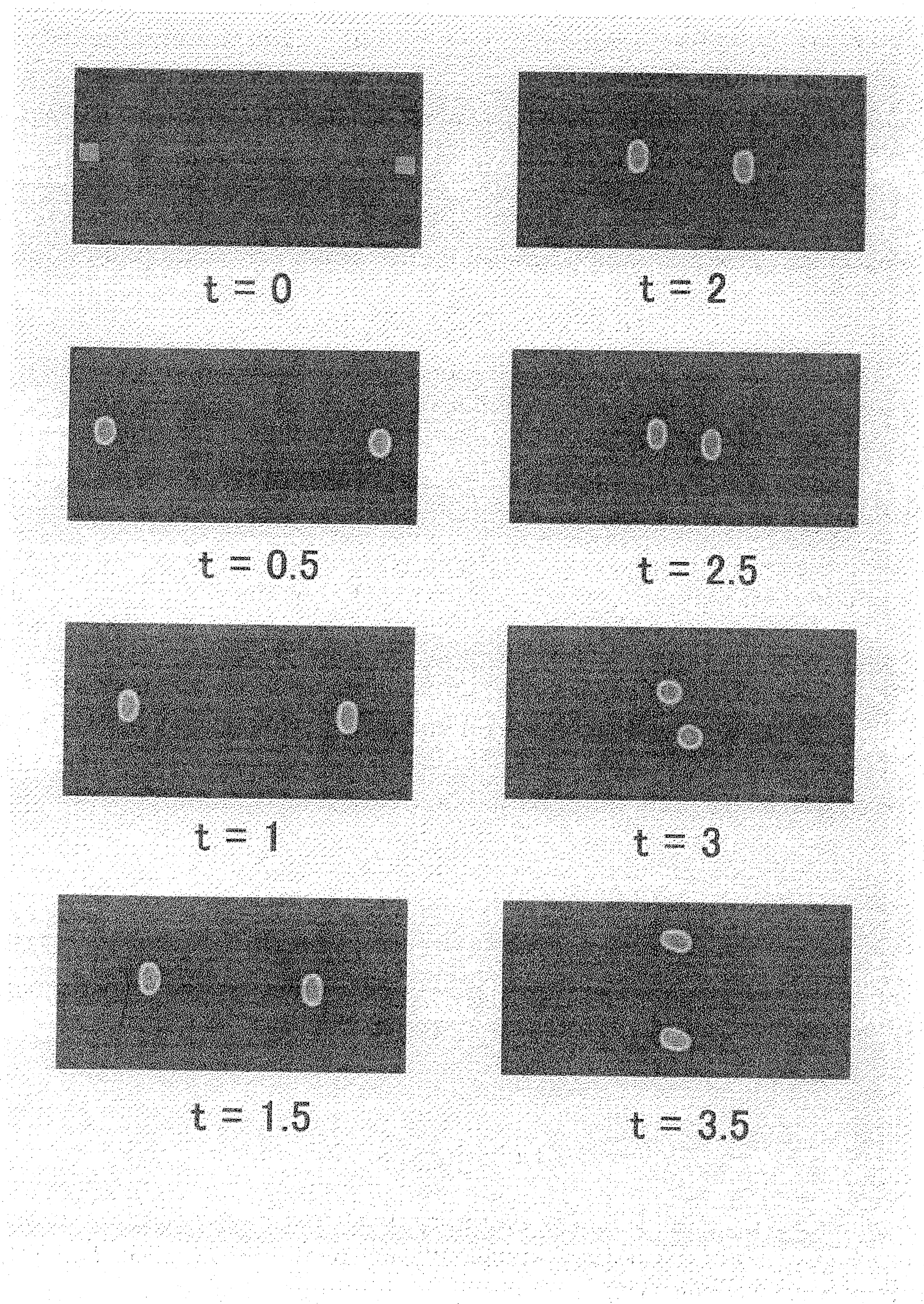

Figure 2: Particle like behavior of travelling spots. Each spot corresponds to the location of each travalling spot.

where $\mathcal{L}_c(\boldsymbol{u}) = \mathcal{L}(\boldsymbol{u}; k_c) = D\Delta\boldsymbol{u} + F(\boldsymbol{u}; k_c)$ and $\eta g(\boldsymbol{u}) = \eta g(\boldsymbol{u}; k) = \mathcal{L}(\boldsymbol{u}; k) - \mathcal{L}_c(\boldsymbol{u})$.

Then, we have the theorem:

Theorem 2.1 *If the initial data $\boldsymbol{u}(0)$ is in the neighborhood of $\mathcal{M}$ in $\{H^2(\boldsymbol{R}^2)\}^N$, then the solution $\boldsymbol{u}(t)$ of (2.2) satisfies*

$$\|\boldsymbol{u}(t) - U(\cdot, P(t), \boldsymbol{\zeta}(t))\|_\infty = O(|\boldsymbol{\zeta}(t)|^2 + |\eta|)$$

as long as $|\boldsymbol{\zeta}| < \zeta^*$ and $|\eta| < \eta^*$ for constants $\zeta^* > 0$ and $\eta^* > 0$. P and $\boldsymbol{\zeta}$ are estimated by

$$\dot{P} = O(|\boldsymbol{\zeta}(t)| + |\eta|^2),\ \dot{\boldsymbol{\zeta}} = O(|\boldsymbol{\zeta}(t)|^2 + |\eta|^2).$$

To obtain more accurate dynamics of P and $\boldsymbol{\zeta}$, we have to know the explicit form of the projection Q_c. In fact, the equation governing P and $\boldsymbol{\zeta}$ is formally derived in the similar manner to [2] as

$$Q_c \frac{d}{dt} U = Q_c \mathcal{L}(U; k_c + \eta) + h.o.t., \tag{2.3}$$

which is in general very difficult to calculate in explicit way.

In the following, we obtain the explicit form of Q_c under suitable assumptions and show the dynamics of P and $\boldsymbol{\zeta}$.

Since the standing solution $S(x)$ is radially symmetric, we write it as $S(x) = S(r)$, where $r = |x|$. Define the functional space consisting of radially symmetric functions by $X_R = \{L^2(0,\infty)\}^N$ with the inner product $\langle\, \boldsymbol{u}, \boldsymbol{v}\, \rangle_R = \int_0^\infty r \langle\, \boldsymbol{u}(r), \boldsymbol{v}(r)\, \rangle\, dr$ for $\boldsymbol{u}$ and $\boldsymbol{v} \in X_R$.

Let $L_R(k)$ be the restriction of the linearized operator $L(k)$ on X_R, that is,

$$L_R(k)\boldsymbol{u} = D\{\boldsymbol{u}_{rr} + \frac{1}{r}\boldsymbol{u}_r\} + F'(S(r); k)\boldsymbol{u}$$

for $\boldsymbol{u} \in \mathcal{D}_R = \{\boldsymbol{u} \in H^2(0,\infty) \cap X_R;\ \boldsymbol{u}_r(0) = 0\}$.

H3) The spectrum of $L_R(k)$ in X_R is uniformly apart from the imaginary axis in the left hand side for the parameter k in the neighborhood of k_c.

Define an operator $\widetilde{L}(k)$ on X_R by

$$\widetilde{L}(k)\boldsymbol{u} = D\{\boldsymbol{u}_r + \frac{1}{r}\boldsymbol{u}\}_r + F'(S(r); k)\boldsymbol{u}$$

for $\boldsymbol{u} \in \widetilde{\mathcal{D}} = \{\boldsymbol{u} \in H^2(0,\infty) \cap X_R;\ \boldsymbol{u}(0) = \boldsymbol{0}\}$. Here, we note that $\widetilde{L}(k)S_r = \boldsymbol{0}$ holds while $L_R(k)S_r \neq \boldsymbol{0}$. This means 0 is necessarily an eigenvalue of $\widetilde{L}(k)$. Let $\widetilde{L}_c = \widetilde{L}(k_c)$ and $\widetilde{\Sigma}_c$ be the spectrum of $\widetilde{L}_c$.

H4) $\widetilde{\Sigma}_c$ consists of two sets $\widetilde{\Sigma}_0 = \{0\}$ and $\widetilde{\Sigma}_1 \subset \{z \in \boldsymbol{C};\ Re(z) < -\gamma_1\}$ for a positive constant γ_1. The generalized eigenspace corresponding to $\widetilde{\Sigma}_0$, say $\widetilde{X}_0$, is given by

$\widetilde{X}_0 = span\{S_r, \psi\}$, where ψ is a function satisfying $\widetilde{L}_c\psi = -S_r$.

Let $\widetilde{L}_c^*$ be the adjoint operator of $\widetilde{L}_c$ in X_R. Note that it is given by

$$\widetilde{L}_c^*\boldsymbol{u} = D\{\boldsymbol{u}_r + \frac{1}{r}\boldsymbol{u}\}_r + {}^tF'(S(r);k_c)\boldsymbol{u}.$$

$\widetilde{L}_c^*$ has also similar properties to $\widetilde{L}_c$, that is, there exist eigenfunctions ϕ^* and ψ^* in X_R satisfying $\widetilde{L}_c^*\phi^* = \mathbf{0}$ and $\widetilde{L}_c^*\psi^* = -\phi^*$.

Proposition 2.1 *Eigenfunctions ψ, ϕ^* and ψ^* are uniquely determined by the normalization*

$$\langle \psi, S_r \rangle_R = \langle \psi, \psi^* \rangle_R = 0,\ \langle S_r, \psi^* \rangle_R = 1.$$

We assume eigenfunctions are normalized according to the proposition. Put

$$\Psi(r) = \int_0^r \psi(r)dr - \int_0^\infty \psi(r)dr,\ \Phi^*(r) = \int_0^r \phi^*(r)dr - \int_0^\infty \phi^*(r)dr,$$

$$\Psi^*(r) = \int_0^r \psi^*(r)dr - \int_0^\infty \psi^*(r)dr.$$

Then, it is easily checked that

$$L_c\Psi_j = -S_j,\ L_c^*\Phi_j^* = \mathbf{0},\ L_c^*\Psi_j^* = -\Phi_j^*$$

hold for $j = 1, 2$, where $\Psi_j = \dfrac{\partial\Psi}{\partial x_j}$ and so on. By this, we have

Proposition 2.2 *The projection Q_c is given by*

$$\begin{aligned}\pi Q_c\boldsymbol{u} \;=\; & \int_0^{2\pi} \langle \boldsymbol{u}, \phi^* \rangle_R \cos\theta d\theta \cdot \Psi_1 + \int_0^{2\pi} \langle \boldsymbol{u}, \psi^* \rangle_R \cos\theta d\theta \cdot S_1 \\ & + \int_0^{2\pi} \langle \boldsymbol{u}, \phi^* \rangle_R \sin\theta d\theta \cdot \Psi_2 + \int_0^{2\pi} \langle \boldsymbol{u}, \psi^* \rangle_R \sin\theta d\theta \cdot S_2\end{aligned}$$

for $\boldsymbol{u} = \boldsymbol{u}(r,\theta) \in X$.

By using this expression of Q_c, we can obtain the explicit dynamics of P and $\boldsymbol{\zeta}$.

Theorem 2.2 *Under assumptions H1) - H4), $P(t)$ and $\boldsymbol{\zeta}(t)$ in theorem 2.1 satisfy*

$$\begin{cases} \dot{P} &= \boldsymbol{\zeta} + O(|\boldsymbol{\zeta}(t)|^3 + |\eta|^{\frac{3}{2}}), \\ \dot{\boldsymbol{\zeta}} &= -\nabla W + O(|\boldsymbol{\zeta}(t)|^4 + |\eta|^2) \end{cases}$$

as long as $|\boldsymbol{\zeta}(t)| < \zeta^*$ and $|\eta| < \eta^*$, where $W = W(\boldsymbol{\zeta}) = \frac{1}{4}M_1|\boldsymbol{\zeta}|^4 + \frac{1}{2}M_2\eta|\boldsymbol{\zeta}|^2$ for constants M_1 and M_2.

Remark 2.1 *The values of constants M_j in Theorem 2.2 are obtained in explicit forms while we will not show them in this report, which will be written in [1]. For (1.1), it is numerically checked that both M_1 and M_2 are positive.*

Remark 2.2 *Theorem 2.2 suggests that $\boldsymbol{\zeta}$ denotes the velocity of the spot S because P denotes the location of the spot. $\boldsymbol{\zeta}$ also stands for the deformation from radial symmetry of spot since the solution $\boldsymbol{u}(t,x)$ is close to the function $U(x;P(t),\boldsymbol{\zeta}(t))$ as in Theorem 2.1.*

Corollary 2.1 *Suppose M_1 and M_2 are positive. If $\eta > 0$, there exists a stable standing spot with profile $S(x)+O(|\eta|)$ while if $\eta < 0$, there exists a travelling spot with velocity $(|\boldsymbol{\zeta}(t)| =)\sqrt{\dfrac{-2M_2\eta}{M_1}}(1+o(1))$.*

3 Interaction of two spots

Let us consider how two travelling spots interact.

H5) The standing spot $S(x)$ has an aysmptotic form $S(r) \to \dfrac{1}{\sqrt{r}}e^{-\alpha r}\boldsymbol{a}$ $(r \to \infty)$ for a constant $\alpha > 0$ and a nonzero vector $\boldsymbol{a} \in \boldsymbol{R}^N$.

Remark 3.1 *The asymptotic form in H5) is true for many model equations in $\boldsymbol{R}^2$ such as the Gierer-Meinhardt model ([2]) and the Gray-Scott model ([9]).*

Define a function

$$U(x;P_1,P_2,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2) = \sum_{j=1}^{2}\{S(x-P_j) + \langle\, \boldsymbol{\zeta}_j, \nabla_x\Psi(x-P_j)\,\rangle\}$$

for P_j, $\boldsymbol{\zeta}_j \in \boldsymbol{R}^2$ and define a set

$$\mathcal{M}(h^*) = \{S(x-P_1)+S(x-P_2);\ |P_1-P_2| = h > h^*\}.$$

Theorem 3.1 *There exists a sufficiently large $h^* > 0$ such that if the initial data $\boldsymbol{u}(0)$ is sufficiently close to the set $\mathcal{M}(h^*)$, then the solution $\boldsymbol{u}(t)$ of (2.2) keeps close to $U(x;P_1,P_2,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2)$ with*

$$\boldsymbol{u}(t) = U(x;P_1,P_2,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2) + O(e^{-\alpha h} + |\boldsymbol{\zeta}_1|^2 + |\boldsymbol{\zeta}_2|^2 + |\eta|)$$

and for $j = 1,2$

$$(3.1)\quad \left\{\begin{array}{rcl} \dot{P}_j &=& \boldsymbol{\zeta}_j \mp M_0\dfrac{1}{\sqrt{h}}e^{-\alpha h}\boldsymbol{e} + O(e^{-2\alpha h} + |\boldsymbol{\zeta}_1|^3 + |\boldsymbol{\zeta}_2|^3 + |\eta|^{\frac{3}{2}}), \\ \dot{\boldsymbol{\zeta}}_j &=& -\nabla W(\boldsymbol{\zeta}_j) \mp \overline{M}_0\dfrac{1}{\sqrt{h}}e^{-\alpha h}\boldsymbol{e} + O(e^{-2\alpha h} + |\boldsymbol{\zeta}_1|^4 + |\boldsymbol{\zeta}_2|^4 + |\eta|^2) \end{array}\right.$$

hold as long as $h > h^*$, $|\boldsymbol{\zeta}_j(t)| < \zeta^*$ and $|\eta| < \eta^*$ for constants M_0 and $\overline{M}_0$, where $h = |P_2 - P_1|$ and $\boldsymbol{e} = \dfrac{1}{h}(P_2 - P_1)$.

Remark 3.2 *Constants M_0 and $\overline{M}_0$ are obtained in explicit way as constants M_1 and M_2 stated in Remark 2.1 while we will not show them in this report, which will be written in [1]. For (1.1), it is numerically checked that both M_0 and $\overline{M}_0$ are positive.*

In the rest of this report, we will intuitively consider the dynamics of P_j and $\boldsymbol{\zeta}_j$ in the case of $\eta < 0$ (the case of the existence of a travelling spot). Suppose both M_0 and $\overline{M}_0$ are positive. To understand the dynamics of $\boldsymbol{\zeta}_j$ intuitively, we consider a simplified ODE

$$\dot{\boldsymbol{\zeta}}_1 = -\nabla W(\boldsymbol{\zeta}_1) - K\boldsymbol{e} \tag{3.2}$$

for a positive constant K. Since the right hand side of (3.2) is written by $-\nabla W_1(\boldsymbol{\zeta}_1)$, where $W_1(\boldsymbol{\zeta}) = W(\boldsymbol{\zeta}) + K\langle\, \boldsymbol{\zeta}, \boldsymbol{e}\,\rangle$, (3.2) has one stable equilibrium with a form $-\beta\boldsymbol{e}$ for $\beta > 0$. Thus, $\boldsymbol{\zeta}_1$ is pushed toward the direction of $-\boldsymbol{e}$.

Similarly in (3.1), $\boldsymbol{\zeta}_1$ is pushed toward the direction of $-\boldsymbol{e}$ and $\boldsymbol{\zeta}_2$ is done toward the direction of $\boldsymbol{e}$. As a consequence, approaching two spots push each other toward opposite directions and they eventually part from each other (Fig 3).

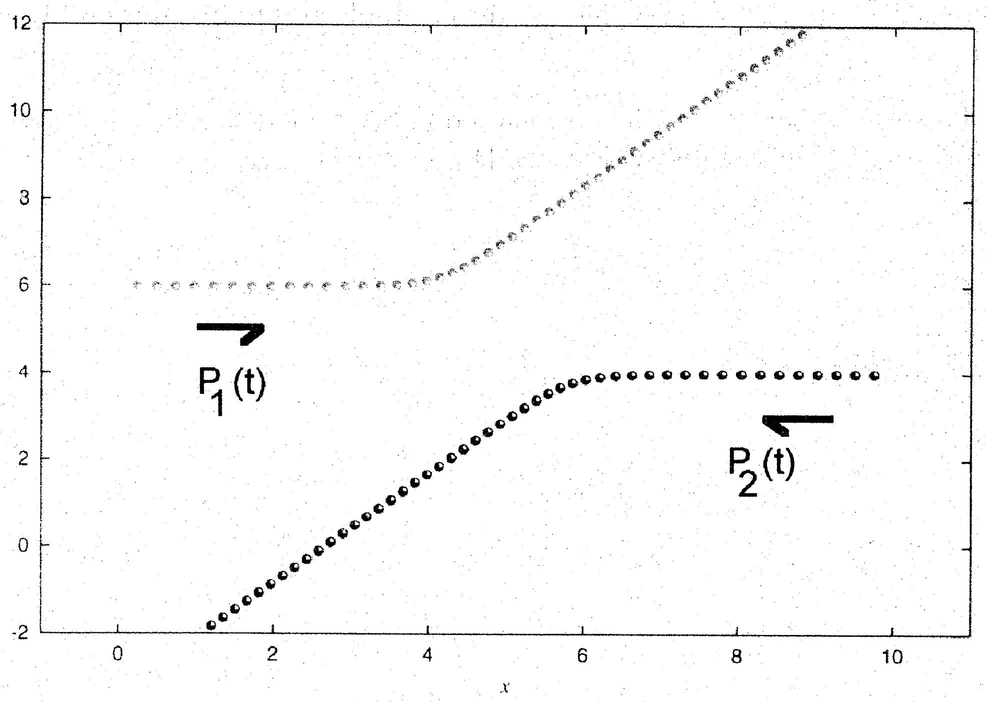

Figure 3: Movements of $P_1(t)$ and $P_2(t)$ which is the solution of ODE consisting of the principal parts of (3.1). Each dot stands for $P_j(t)$ in every time unit.

References

[1] S.-I. Ei and M. Mimura, in preparation.

[2] S.-I. Ei and J. Wei, Dynamics of metastable localized patterns and its application to the interaction of spike solutions for the Gierer-Meinhardt systems in two spatial dimension, preprint.

[3] S.-I. Ei, M. Mimura and M. Nagayama, Reflection of travelling pulses in reaction-diffusion systems, in preparation.

[4] S. Kawaguchi and M. Mimura, Collision of travelling waves in a reaction-diffusion system with global coupling effect, SIAM J. Appl. Math., 59 (1999), 920-941.

[5] M. Mimura and M. Nagayama, Nonannihilation dynamics in an exothermic reaction-diffusion system with mono-stable excitability, Chaos **7**(4), 1997, 817–826.

[6] Y. Nishiura and D. Ueyama, A skeleton structure of self-replicating dynamics, Physica D, 130 (1999), 73-104.

[7] T. Ohta, Pulse dynamics in a reaction-diffusion system, to appear in Physica D.

[8] T. Ohta and T. Ito, Pulse dynamics in an excitable reaction diffusion system, preprint.

[9] J. Wei, Pattern formation two-dimensional Gray-Scott model: existence of single spot solutions and their stability, Physica D., to appear.