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Recent Results on the Onset of Superconductivity: Domains with Corners

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Abstract

We will survey recent results on the eigenvalue problem describing the onset of superconductivity in the presence of large magnetic fields. We will then focus on a setting in which new results have been obtained: two-dimensional samples with corners. In all of the studies mentioned, the Ginzburg-Landau model is used to describe the physical setting.

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1 Introduction

The phenomenon of superconductivity is characterized by a loss of resistivity and the expulsion of applied magnetic fields. In this paper, I will consider the setting in which a superconducting sample is subjected to an applied magnetic field. It is well-known that sufficiently large magnetic fields tend to destroy superconductivity. Alternatively, and more in keeping with experimental work on the subject, one can destroy superconductivity by applying

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a field at a fixed level and raising the temperature to a sufficiently high level. (See e.g. [6, 7, 21].) It is this second experiment that is the starting point for the results I will shortly survey, though all of the discussion could equally well be carried out for the first setting as well.

For the purposes of capturing this critical temperature below which superconductivity is first observed in a sample, the Ginzburg-Landau model is extremely effective (cf. [11, 13, 14]). For most of this discussion, I will focus on the case of a thin cylindrical sample with two-dimensional cross-section denoted by Ω. I will take the direction of the applied field $\mathbf{H}$ to be orthogonal to this cross-section and the magnitude of the applied field will be taken as a constant denoted by h.

Within the Ginzburg-Landau theory, physically realizable states are then characterized as stable critical points of the energy

$$\begin{aligned} G(\Psi, \mathbf{A}) = \; & \textstyle\int_\Omega \frac{1}{2} \left|(i\nabla + \mathbf{A})\Psi\right|^2 + \frac{\lambda}{4}(|\Psi|^2 - 1)^2 \, dx\, dy \\ & + \textstyle\int_{\mathbf{R}^2} \frac{\kappa^2}{\lambda} \left|\nabla \times \mathbf{A} - \mathbf{H}\right|^2 \, dx\, dy. \end{aligned} \tag{1.1}$$

Here we have used a characteristic length R of the sample to non-dimensionalize the energy, so that Ω should be viewed as a bounded domain of unit diameter. The function $\Psi : \Omega \to \mathbf{C}$ is an order parameter with $|\Psi|^2$ corresponding to the superconducting electron density, while the other dependent variable $\mathbf{A} : \mathbf{R}^2 \to \mathbf{R}^2$ denotes the magnetic potential so that $\nabla \times \mathbf{A}$ is the effective magnetic field. As mentioned above, we will take the applied field $\mathbf{H}$ to be given by $\mathbf{H} = h\hat{z}$ for some constant $h > 0$. The constant κ is the Ginzburg-Landau parameter, a dimensionless ratio of two relevant length-scales, while

$$\lambda = \lambda(T) = \frac{R^2}{\xi_0^2}\left(\frac{T_c - T}{T_c(0)}\right) \tag{1.2}$$

where T is the temperature, T_c is the transition temperature in the absence of any applied field, and ξ_0 denotes the so-called coherence length at $T = 0$, a material-dependent parameter (cf. [3]).

Taking variations of (1.1), we get the Ginzburg-Landau equations

$$(i\nabla + \mathbf{A})^2\Psi - \lambda\Psi + \lambda\,|\Psi|^2\,\Psi = 0 \text{ in } \Omega, \tag{1.3}$$

$$\nabla \times \nabla \times \mathbf{A} + \left(\tfrac{i\lambda}{2\kappa^2}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) + |\Psi^2|\,\mathbf{A}\right)\chi_\Omega = 0 \text{ in } \mathbf{R}^2, \tag{1.4}$$

along with the boundary conditions

$$\mathbf{n} \cdot (i\nabla + \mathbf{A})\Psi = 0 \quad \text{on } \partial\Omega, \qquad \nabla \times \mathbf{A} \to h\hat{z} \quad \text{as } (x, y) \to \infty. \tag{1.5}$$

Here $\mathbf{n}$ is the unit normal to $\partial\Omega$, $(\cdot)^*$ denotes complex conjugation, and χ_Ω denotes the characteristic function of the set Ω.

We note here that the problem is invariant under the 'gauge transformation' $(\Psi, \mathbf{A}) \to (\Psi', \mathbf{A}')$ where

$$\Psi' \equiv \Psi e^{i\phi}, \quad \mathbf{A}' \equiv \mathbf{A} + \nabla\phi$$

for an arbitrary smooth real-valued function ϕ. Throughout this article it will be convenient to impose the additional conditions

$$\operatorname{div} \mathbf{A} = 0 \text{ in } \Omega, \quad \mathbf{A} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega.$$

This amounts to choosing a gauge, thus eliminating the degeneracy associated with the gauge invariance of our problem.

As mentioned earlier, one does not expect to see a stable superconducting state at sufficiently high temperatures and in the presence of an applied magnetic field, one expects the critical temperature to be even higher (cf. [15]). In light of (1.2), this corresponds to the fact that for $\lambda > 0$ sufficiently close to zero, the so-called normal state given by the conditions

$$\Psi \equiv 0 \text{ in } \Omega, \quad \nabla \times \mathbf{A} \equiv h\hat{z} \text{ in } \mathbf{R}^2$$

is a stable critical point. (Note that whatever the value of λ, the normal state is always a critical point, i.e. a solution of (1.3), (1.4), (1.5).) For convenience, we introduce now the vector field $\mathbf{a_N}$ given by

$$\nabla \times \mathbf{a_N} = \hat{z}, \quad \operatorname{div} \mathbf{a_N} = 0 \text{ in } \Omega, \quad \mathbf{a_N} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega, \tag{1.6}$$

so that the magnetic potential corresponding to the normal state is given by $h\mathbf{a_N}$.

We will study the onset of superconductivity as a bifurcation off of the normal state. Phrasing the problem variationally, one calculates:

$$\frac{d^2}{d\varepsilon^2}\Big|_{\varepsilon=0} G(0 + \varepsilon\Psi, h\mathbf{a_N}) =$$

$$\int_\Omega |(i\nabla + h\mathbf{a_N})\Psi|^2 - \lambda |\Psi|^2 \, dx$$

so that instability of the normal state $(0, h\mathbf{a_N})$ occurs whenever λ exceeds the lowest eigenvalue $\mu_\Omega(h)$ given by

$$\mu_\Omega(h) \equiv \inf_{\Psi \in H^1(\Omega)} \frac{\int_\Omega |(i\nabla + h\mathbf{a_N})\Psi|^2 \, dx\, dy}{\int_\Omega |\Psi|^2 \, dx\, dy}. \tag{1.7}$$

This eigenvalue problem is the focus of our investigation. In addition to gaining an understanding of the dependence of μ (and hence, of temperature via (1.2) on the field strength h, we are particularly interested in understanding the dependence of μ on the topology and geometry of the sample

Ω. We note here that a first eigenfunction $\Psi^{(1)}$ for (1.7), should it exist, would satisfy the elliptic problem

$$(i\nabla + h\mathbf{a_N})^2\Psi^{(1)} = \mu_\Omega(h)\Psi^{(1)} \quad \text{in } \Omega, \quad \nabla\Psi^{(1)} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega. \tag{1.8}$$

In Section 2, I will survey the known results on the subject, starting with unbounded domains and then progressing to more recent results on smooth, bounded planar domains. In Section 3, I will discuss in more detail the work of my student, Hala Jadallah, on planar domains with a corner.

2 Survey of Known Results on Onset in 2-D

Onset in the Plane

For the case where $\Omega = \mathbf{R}^2$, one finds that $\mathbf{a_N} = 1/2(-y, x)$ satisfies (1.6). Then by a rescaling of space, one readily finds that

$$\mu_{\mathbf{R}^2}(h) = h\mu_{\mathbf{R}^2}(1).$$

Furthermore, by writing any competitor Ψ in (1.7) in a Fourier series one can argue that $\mu_{\mathbf{R}^2}(1) = 1$. This infimum is achieved by infinitely many functions, but in particular, the function $e^{-\frac{(x^2+y^2)}{4}}$ is a first eigenfunction. See [18] for details.

Onset in the Half-Plane

For the case where $\Omega = \mathbf{R}^2_+$ is the half-plane $\{(x, y) : x > 0\}$, one finds that $\mathbf{a_N} = (0, x)$ satisfies (1.6). One again finds through a rescaling that

$$\mu_{\mathbf{R}^2_+}(h) = h\mu_{\mathbf{R}^2_+}(1).$$

Saint James and deGennes [22] found a solution to (1.8) in this setting via separation of variables. That is, they formally sought $\Psi^{(1)}$ in the form

$$\Psi^{(1)}(x, y) = \psi_1(x)e^{i\beta^* y} \tag{2.1}$$

where $\mu_{\mathbf{R}^2_+}(1)$, the parameter β^* and the real-valued function ψ_1 are determined through the double minimization problem:

$$\mu_{\mathbf{R}^2_+}(1) = \inf_{\beta} \inf_{f \in H^1([0,\infty))} \frac{\int_0^\infty (f')^2 + (x-\beta)^2 f^2 \, dx}{\int_0^\infty f^2 \, dx} \tag{2.2}$$

It can be shown that a unique value of β, denoted by β^*, achieves this infimum ([5, 10]). One can carry out a numerical approximation to find $\mu_{\mathbf{R}^2_+}(1) \approx 0.59$, but in particular one can prove that

$$\mu_{\mathbf{R}^2_+}(1) < \mu_{\mathbf{R}^2}(1) = 1. \tag{2.3}$$

A more careful formal analysis of this problem can be found in [8, 9], and for rigorous results along these lines see [18, 12].

There are two important things to note here, however. First, observe that $\Psi^{(1)}$ given by (2.1) is not in $H^1(\mathbf{R}^2_+)$ and is therefore certainly not a minimizer of (1.7). Indeed it is shown in [18] that no minimizer exists. Second, the function ψ_1 is known to decay exponentially as $x \to \infty$, making the solution $\Psi^{(1)}$ exponentially localized along the boundary of the half-plane–a confirmation of the phenomenon known as "surface superconductivity" (cf. [11]) in which onset is first observed along the boundary of the sample.

Onset in a Smooth Bounded Domain

¿For the case where Ω is a smooth, bounded planar domain, it has been shown formally in [4] that

$$\mu_\Omega(h) \sim \mu_{\mathbf{R}^2}(1)h - \frac{\kappa_{\max}}{3I_0}h^{1/2} \text{ for } h >> 1 \tag{2.4}$$

where $\kappa_{\max}$ denotes the maximum of curvature of $\partial\Omega$ and the constant I_0 is the first moment of ψ_1 on the interval $0 < x < \infty$. Furthermore, one formally finds that any corresponding first eigenfunction must concentrate with an exponentially small tail away from the point(s) of maximum curvature of the boundary. For example, suppose $\partial\Omega$ possesses exactly one point of maximum curvature. Denoting by s arclength along $\partial\Omega$ with $s = 0$ corresponding to this point of maximum curvature, and denoting by η the distance to $\partial\Omega$, one finds in [4] that

$$\left|\Psi^{(1)}(s,\eta)\right| \leq e^{-h^{1/4}s^2}e^{-h^{1/2}\eta}$$

for s and η corresponding to a neighborhood of the point of maxmimum curvature.

We should note that proving existence of a first eigenfunction in this bounded case is easily accomplished by applying the direct method in the calculus of variations to (1.7).

Aspects of these formally derived results have been made rigorous. For example, in the case of a disc, formula (2.4) was proven in [2] where then $\kappa_{\max}$ is replaced by the reciprocal of the disc's radius. In this case, $\Psi^{(1)}$ decays in the interior of the disc, but as in the half-plane case, it concentrates everywhere along the boundary. Capturing the first term in the asymptotic expansion (2.4) for a general smooth bounded domain was first accomplished in [19], as was interior decay. Exponential interior decay as well as rigorous evidence of tangential decay along $\partial\Omega$ can be found in [12].

3 Onset in a Domain with a Corner

In light of the sensitive dependence of both the leading eigenvalue and eigenfunction on the curvature of the sample boundary, it is natural to ask what happens in problem (1.7) when $\partial\Omega$ possesses one or more points of infinite curvature. To initiate this investigation, we focus on the case where Ω is a square and on the related case of a quarter-plane. The results I present here can be found in [16]. For convenience, we will denote by $\mathbf{Q}$ the quarter-plane $\{(x,y) : x > 0, y > 0\}$ and by $\mathbf{Q}_l$ the square $[0,l] \times [0,l]$. We note that again by rescaling one can argue that

$$\mu_{\mathbf{Q}_1}(h) = h\mu_{\mathbf{Q}_{\sqrt{h}}}(1) \tag{3.1}$$

and it is also not hard to show that

$$\lim_{h\to\infty} \frac{\mu_{\mathbf{Q}_1}(h)}{h} = \mu_{\mathbf{Q}}(1). \tag{3.2}$$

We begin with a crucial result which shows, in light of (3.2) that already at the leading order in the expansion for $\mu_{\mathbf{Q}_1}(h)$ in the large h regime, one has a departure from the expansion (2.4) valid in bounded smooth domains.

Theorem 3.1 [16] *There is an ordering to the first eigenvalues of (1.7) on the quarter-plane* $\mathbf{Q}$ *and the half-plane* $\mathbf{R}^2_+$ *¿as follows:*

$$\mu_{\mathbf{Q}}(1) < \mu_{\mathbf{R}^2_+}(1). \tag{3.3}$$

The proof of this theorem relies upon the use of a carefully employed truncation and perturbation of the first eigenfunction (2.1) for the half-plane. Specifically, one inserts the choice

$$\phi(x,y) = C(\epsilon)\psi_1(x)e^{i\beta^* y}(e^{-\epsilon y} + i\epsilon^{1/2}(x-\beta^*)e^{-y}), \tag{3.4}$$

into the Rayleigh quotients on $\mathbf{Q}$ and $\mathbf{R}^2_+$, where $\epsilon > 0$, $C(\epsilon)$ is a positive constant such that $\epsilon < C^2(\epsilon) < 2\epsilon$ and $\psi_1(x)$ is the first eigenfunction introduced in (2.1). This test function yields

$$\mu_{\mathbf{Q}}(1) \le \mu_{\mathbf{R}^2_+}(1) - 2\epsilon^{3/2}C_1 + \epsilon^2 C_2,$$

where C_1 and C_2 are positive real numbers independent of ϵ. The idea for this construction comes from a similar approach used in the study by Almog of (1.7) on rectangles, half-infinite strips and infinite strips found in [1].

Numerically, one can compute the two eigenvalues to illustrate both the validity of the theorem and the surprising closeness of the two eigenvalues; one finds

$$\mu_{\mathbf{Q}}(1) \approx 0.55 \text{ while } \mu_{\mathbf{R}^2_+}(1) \approx 0.59.$$

(cf. [17]).

The eigenvalue gap described above turns out to be critical in proving the following result yielding the exponential decay away from the corners for the first eigenfunction in a square.

Theorem 3.2 [16] *Let $\{\Psi^h\}$ be any sequence of eigenfunctions that minimize the Rayleigh quotient (1.7) in the unit square $\Omega = \mathbf{Q}_1$, normalized so that $\left\|\Psi^h\right\|_{L^\infty(\mathbf{Q}_1)} = 1$. Then there exists a constant $h_0 > 0$ and for every multi-index α , there exist positive constants c_1^α and c_2^α independent of h, such that*

$$\left|D^\alpha \Psi^h(z)\right| \leq (\sqrt{h})^{|\alpha|} c_1^\alpha e^{-c_2^\alpha \sqrt{h}\tilde{d}(z)} \text{ for all } z = (x,y) \in \mathbf{Q}_1 \text{ and } h \geq h_0, \quad (3.5)$$

$$\text{where} \quad \tilde{d}(z) = \min_{1\leq i\leq 4} dist(z, p_i) \quad \text{and} \quad p_i \in \{(0,0),(1,0),(0,1),(1,1)\}$$

Before sketching the idea of the proof, we note that this theorem represents the first rigorous confirmation of the notion formally advanced in [4], that the first eigenfunction will decay exponentially away from boundary points of maximum curvature (in this case, infinite curvature).

Sketch of proof. The argument follows the general idea of the exponential decay argument to be found in [12]. However, numerous subtleties and complications emerge that did not come forth in [12]. Note first that Ψ^h will satisfy the equation

$$(i\nabla + h\mathbf{a_N})^2 \Psi^h = \mu_{\mathbf{Q}_1}(h)\Psi^h \text{ in } \mathbf{Q}_1. \quad (3.6)$$

along with Neumann boundary conditions. Now denote

$$\Omega(k,h,R) = \{z \in \mathbf{Q}_1 : \tilde{d}(z) \geq \frac{kR}{\sqrt{h}}\}$$

for any integer $k > 0$ and any $h > 0$ and $R > 0$. The decay (3.2) follows readily from the claim:

There exists an $h_0 > 0$ and an $R_0 > 0$ such that

$$\left\|\Psi^h\right\|_{L^\infty(\Omega(k+1,h,R))} < \frac{1}{2}\left\|\Psi^h\right\|_{L^\infty(\Omega(k,h,R))} \quad (3.7)$$

for all $h \geq h_0$, all $R \geq R_0$ and all positive integers k.

The argument is by contradiction, for if (3.7) fails to hold, then there exist sequences $R_j \to \infty$, $h_j \to \infty$, a sequence of positive integers k_j and a sequence of points $z_j \in \Omega(k_j+1, h_j, R_j))$ such that

$$\left|\Psi^{h_j}(z_j)\right| \geq \frac{1}{2}\left|\Psi^{h_j}\right|_{L^\infty(\Omega(k_j,h_j,R_j))} \equiv \frac{1}{2}m_j \quad (3.8)$$

We shall refer to these points z_j as 'bad points.' The contradiction will come from a blow-up procedure about these bad points. There are two cases to consider:

$$\limsup_{h_j \to \infty} \sqrt{h_j} \mathrm{dist}(z_j, \partial \mathbf{Q}_1) = \infty \tag{3.9}$$

$$\limsup_{h_j \to \infty} \sqrt{h_j} \mathrm{dist}(z_j, \partial \mathbf{Q}_1) < \infty, \text{ but } \limsup_{h_j \to \infty} \sqrt{h_j} \tilde{d}(z_j) = \infty \tag{3.10}$$

In case (3.9) holds, one rescales Ψ^{h_j} as follows. We define the sequence of functions $f_j : B(0, R_j) \to \mathbf{C}$ by

$$f_j(x,y) = \frac{1}{m_j} \Psi^{h_j}\left(\frac{x}{\sqrt{h_j}} + x_j, \frac{y}{\sqrt{h_j}} + y_j\right) e^{-i\sqrt{h_j}\, x_j\, y},$$

where $z_j = (x_j, y_j)$ and $B(0, R_j)$ denotes the ball centered at the origin of radius R_j. Notice that by the contradiction hypothesis (3.8), we have

$$|f_j(0,0)| = \frac{1}{m_j} \left|\Psi^{h_j}(z_j)\right| \geq \frac{1}{2} \quad \text{and} \quad \|f_j\|_{B(0,R_j)} \leq 1.$$

Moreover, f_j solves the PDE :

$$(i\nabla + \mathbf{a_N})^2 f_j = \frac{\mu_{\mathbf{Q}_1}(h_j)}{h_j} f_j \text{ in } B(0, R_j). \tag{3.11}$$

Utilizing standard elliptic estimates, one can extract a C^2-convergent subsequence of $\{f_j\}$, and passing to the limit in (3.11), one obtains a limiting function f satisfying the problem

$$(i\nabla + \mathbf{a_N})^2 f = \lim_{h_{j_k} \to \infty} \frac{\mu_{\mathbf{Q}_1}(h_{j_k})}{h_{j_k}} f \text{ in } \mathbf{R}^2. \tag{3.12}$$

There can be no such solution on $\mathbf{R}^2$ in light of (2.3), (3.2) and (3.3). This completes the contradiction argument in case the bad points are accumulating in the interior of the square (cf. (3.9)).

The proof in case (3.10) is similar except that now one defines the blow-ups f_j on increasing half-balls instead of balls. The contradiction then comes ¿from obtaining a limiting function f satisfying the P.D.E. (3.12) on $\mathbf{R}^2_+$ subject to Neumann boundary conditions. Again, (3.2) and (3.3) mean that f represents an eigenfunction with corresponding eigenvalue too low.

This completes a sketch of the argument for (3.5) when $\alpha = 0$. The decay of higher derivatives comes from standard elliptic theory in which one uses the P.D.E. to estimate the magnitude of higher derivatives in terms of $\left|\Psi^h\right|^2$.

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We conclude this examination of the eigenvalue problem (1.7) on domains with corners with a discussion of how one proves the existence of an eigenfunction on the quarter-plane. Recall that for the case of a half-plane, there is no L^2 eigenfunction. In particular, the function given by (2.1) fails to decay in the direction tangential to the boundary. However, it turns out that for the quarter-plane, there is a first L^2 eigenfunction and this function decays exponentially away from the origin.

Theorem 3.3 [16] *There exists a function $\Psi_{\mathbf{Q}}$ minimizing the Rayleigh quotient (1.7) in the case $\Omega = \mathbf{Q}$. Furthermore, normalizing $\Psi_{\mathbf{Q}}$ so that $\|\Psi_{\mathbf{Q}}\|_{L^\infty(\mathbf{Q})} = 1$, for every multi-index α there exist positive constants c_1^α and c_2^α such that*

$$|D^\alpha \Psi_{\mathbf{Q}}(z)| \leq c_1^\alpha e^{-c_2^\alpha |z|} \quad \textit{for all } z \in \mathbf{Q}. \tag{3.13}$$

Idea of Proof. The approach in [16] hinges on the construction of a minimizing sequence for the eigenvalue problem on the quarter-plane. The minimizing sequence is then shown to be compact, with subsequential limit $\Psi_{\mathbf{Q}}$.

The construction relies on using eigenfunctions for the unit square $\mathbf{Q}_1$, and then rescaling by $z \to \sqrt{h}z$ so as to obtain eigenfunctions on expanding squares $\mathbf{Q}_{\sqrt{h}}$. However, as we have seen in the previous theorem, the eigenfunctions on the unit square may concentrate on all four corners, whereas the eigenfunction we are trying to obtain for the quarter-plane should only concentrate at the origin. This observation forces one to slightly alter problem (1.7) on the unit square in building the minimizing sequence. Specifically, one minimizes the Rayleigh quotient on $\mathbf{Q}_1$ amongst competitors which satisfy zero Dirichlet data on the two sides $[0,1] \times \{1\}$ and $\{1\} \times [0,1]$.

The resulting minimizers are then shown to decay exponentially away ¿from the origin–not at all four corners–using a method similar to but technically more complicated than the one invoked earlier. After rescaling to obtain eigenfunctions on $\mathbf{Q}_{\sqrt{h}}$, the uniformity of the decay rate in h allows for the needed compactness and a function $\Psi_{\mathbf{Q}}$ arising as a subsequential limit of these eigenfunctions on expanding squares proves to be the first eigenfunction on the quarter-plane satisfying (3.13). ∎

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