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Fat solenoidal attractors

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Abstract

We study dynamical systems generated by skew products

$$T : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}, \qquad T(x, y) = (\ell x, \lambda y + f(x))$$

where $\ell \geq 2$, $1/\ell < \lambda < 1$ and f is a C^2 function on S^1. We show that the SBR measure for T is absolutely continuous for almost every f.

1 Introduction

In this paper, we study a class of dynamical systems that stably admit an absolutely continuous ergodic measure (*acem*) with *a negative Lyapunov exponent.* It is well-known that expanding dynamical systems generally admit *acem*'s whose Lyapunov exponents are all positive. The aim of this paper is to study another kind of *acem*'s which is produced by a quite different mechanism: *overlap and sliding* in short.

We can find a typical example of such *acem*'s in a paper of Alxander and Yorke[1], where the so-called generalized baker's transformation is considered:

$$B : [-1, 1] \times [-1, 1] \circlearrowleft, \quad B(x, y) = \begin{cases} (2x - 1, \beta y + (1 - \beta)) & x \geq 0 \\ (2x + 1, \beta y - (1 - \beta)) & x < 0. \end{cases}$$

When $\beta = 1/2$, this map B is nothing but the ordinary baker's transformation. Alxander and Yorke studied the case $1/2 < \beta \leq 1$. In such case, the images of left and right halves of the domain, *i.e.*, $B([-1, 0] \times [-1, 1])$ and $B([0, 1] \times [-1, 1])$ overlap with some sliding. This makes the dynamical nature of the map B more complicated and interesting. They observed that the map B admits an *acem* if and only if the number β satisfies a delicate numerical condition: absolute continuity of the corresponding infinitely convoluted Bernoulli measure. As they noted, there are infinitely many numbers in $(1/2, 1]$ (*e.g.* $(\sqrt{5} - 1)/2$) for which B admits no *acem*'s, according to a result of Erdös[2]. On the other hand, B admits an *acem* for Lebesgue almost every β in $(1/2, 1]$ according to a more recent result of Solomyak[3].

In this paper, we consider a class of dynamical systems generated by maps

$$T : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}, \qquad T(x, y) = (\ell x, \lambda y + f(x)) \tag{1}$$

where $\ell \geq 2$ is an integer, $0 < \lambda < 1$ is a real number, and f is a C^2 function on $S^1 = \mathbb{R}/\mathbb{Z}$. We may regard this class of maps as a conceptual generalization of the generalized baker's transformations B in the sense that the translation in vertical direction depends smoothly on x.

The map T is a skew product on the expanding map $\tau : x \mapsto \ell x$ and it is uniformly contracting in the fiber direction. So T is an Anosov endomorphism. The ergodic property of T is rather simple: there exists an ergodic probability measure μ on $S^1 \times \mathbb{R}$, for which Lebesgue almost every point $\mathbf{x} \in S^1 \times \mathbb{R}$ is generic, that is,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i(\mathbf{x})} = \mu \quad \text{weakly.}$$

We will call this measure μ the *SBR measure* for T.

The question is smoothness of the SBR measure μ with respect to the Lebesgue measure on $S^1 \times \mathbb{R}$. In the case $\lambda\ell < 1$, the SBR measure is totally singular because T contracts area. The case $\lambda\ell > 1$, which corresponds to the case $\beta > 1/2$ for the generalized baker's transformations, is more interesting. We will focus on this case. First we give two examples in opposite directions.

Example 1 *Let $\ell = 2$, $0.5 < \lambda \leq 0.51$ and $f(x) = \sin 2\pi x$. Then the SBR measure μ for T is absolutely continuous with respect to the Lebesgue measure of $S^1 \times \mathbb{R}$. (See figure 1.)*

Example 2 *If $f(x) = \varphi(\tau(x)) - \lambda\varphi(x)$ for some measurable function φ on S^1, the SBR measure for T is supported on the graph of φ and totally singular.*

We claim that the SBR measure is absolutely continuous for almost every T and, moreover, that the absolute continuity is robust. Fix an integer $\ell \geq 2$. Let $\mathcal{D} \subset (0,1) \times C^2(S^1, \mathbb{R})$ be the set of combinations (λ, f) for which the SBR measure is absolutely continuous w.r.t. the Lebesgue measure on $S^1 \times \mathbb{R}$. We consider the interior $\mathcal{D}^\circ$ of $\mathcal{D}$ with respect to the topology that is defined as the product of the canonical topology on $(0,1)$ and C^2-topology on $C^2(S^1, \mathbb{R})$. The main result of this paper is the following.

Theorem 1 *Let $\ell^{-1} < \lambda < 1$. There exists a finite collection of C^∞ functions $\varphi_i : S^1 \to \mathbb{R}$, $i = 1, 2, \cdots, m$, such that , for any C^2 function $g \in C^2(S^1, \mathbb{R})$, the subset of $\mathbb{R}^m$,*

$$\left\{ (t_1, t_2, \cdots, t_m) \in \mathbb{R}^m \;\middle|\; \left(\lambda,\ g(x) + \sum_{i=1}^{m} t_i \varphi_i(x)\right) \notin \mathcal{D}^\circ \right\},$$

is a null set with respect to the Lebesgue mesure on $\mathbb{R}^m$.

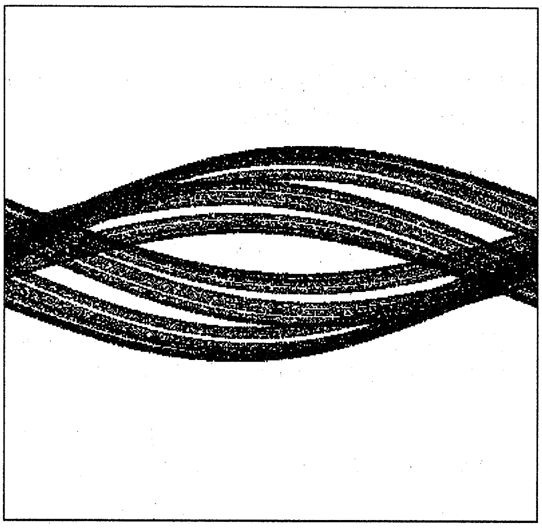

Figure 1: The orbit of the point $(0.1, 0)$ up to time 100000 when $\ell = 2$, $\lambda = 0.51$ and $f(x) = \sin(x)$.

As simple consequences, we obtain

Corollary 2 *$\mathcal{D}$ contains an open and dense subset of* $(1/\ell, 1) \times C^2(S^1, \mathbb{R})$.

Corollary 3 *For $\ell^{-1} < \lambda < 1$ and $2 \le r \le \infty$, the set of functions*

$$\mathcal{D}_\lambda^r = \{f \in C^r(S^1, \mathbb{R}) \mid (\lambda, f) \in \mathcal{D}^\circ\}$$

is an open and dense subset of $C^r(S^1, \mathbb{R})$.

Moreover, the claim of theorem 1 implies that the subset $\mathcal{D}_\lambda^r$ above occupies *almost everywhere* in $C^r(S^1, \mathbb{R})$. In fact, if $C^r(S^1, \mathbb{R})$ were a finite dimensional Euclidean space, the claim would imply that the subset $\mathcal{D}_\lambda^r$ had full measure with respect to the 'Lebesgue measure' on $C^r(S^1, \mathbb{R})$. See [5] and [6] for discussions about measure-theoretical conditions that imply "almost everywhere" for subsets in infinite dimensional spaces.

The proof of theorem 1 is based on an idea that transversality of the unstable manifolds leads to absolute continuity of the SBR measure. We took this idea from a paper of Solomyak and Peres[4] where the authors gave a simplified proof of the above mentioned result of Solomyak.

One can download the full paper at

`http://www.math.sci.hokudai.ac.jp/~tsujii/index.html`

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