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Complex Gibbs measures for complex dynamical systems and eigen-hyperfunctions of complex Ruelle operator

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0. Introduction

In this note, we compute the eigen-functions to the complex Ruelle operator and the eigen-hyperfunctions to the dual of the Ruelle operator applied on the space of pre-hyperfunctions and hyperfunctions respectively supported on the Julia set of a complex dynamical system. In order to examine the structure of the eigen-functions, we consider a most simple and non-trivial case, *i.e.*, the case of postcritically finite quadratic polynomial $R(z)=z^{2}+i$. The critical point, $z=0$, is postcritically finite, since $R(0)=i$, $R(i)=i-1$, $R(i-1)=-i$, and $R(-i)=i-1$. As mentioned in [10], the Fredholm determinant of the complex Ruelle operator is a rational function in the postcritically finite case. It can be explicitely computed. For more detailed definition of the space of prehyperfunctions and the complex Ruelle operator operating on the prehyperfunctions, see [10].

1. Prehyperfunctions supported on the Julia set and complex Ruelle operator

In this section, we briefly recall the formulation of prehyperfunctions defined in a neighborhood of the Julia set. In this note, for the sake of simplicity, we consider only the case of the postcritically finite quadratic function $R(z)=z^{2}+i$. The infinity is a superattractive fixed point of R. Let $F=F(R)$ denote the Fatou set of R, and let $J=J(R)$ denote the Julia set of R. In our case, F is the attractive basin of the infinity and J is a dendrite and they are both connected. In order to avoid confusion we set $i=\sqrt{-1}$ and will not use i as an index variable. The origin $z=0$ is the critical point in the Julia set. The postcritical set

$P = P(R)$ consists of three points $\{i, i-1, -i\}$. Note that we denote by R_n the n-th iterate of R instead of R^n or $R^{\circ n}$, since we have to treat their derivatives. In the backward iterarion case, we denote also R^{-k} in place of R_{-k} to emphasize it is backward.

Let $\mathcal{O}(J)$ denote the space of germs of functions $g : J \to \mathbb{C}$ which can be extended holomorphically to some neighborhood of J. The topology if this space of functions is defined as follows : a sequence of functions $\{g_n\}$ in $\mathcal{O}(J)$ converges to some function g_∞ in $\mathcal{O}(J)$ if there exists a neighborhood of J such that $\{g_n\}$ are extendable to this neighborhood and the sequence converges to g_∞ uniformly in this neighborhood.

Let $\mathcal{O}(F)$ denote the space of holomorphic functions $f : F \to \mathbb{C}$ with the topology of local uniform convergence. We denote by $\mathcal{O}_0(F)$ the set of holomorphic functions $f \in \mathcal{O}(F)$ satisfying $f(\infty) = 0$.

The space of prehyperfunctions $\mathcal{H}(J)$ supported on J is defined by a direct sum :

$$\mathcal{H}(J) = \mathcal{O}(J) \oplus \mathcal{O}_0(F).$$

This space is a Fréchet space.

For $\varphi \in \mathcal{H}(J)$, let $\varphi = \varphi_J \oplus \varphi_F$ with $\varphi_J \in \mathcal{O}(J)$ and $\varphi_F \in \mathcal{O}_0(F)$. A prehyperfunction φ defines a function holomorphic in a deleted neighborhood of the Julia set. Conversely, a function, defined in a deleted neighborhood, say $U \setminus J$, of the Julia set, and holomorphic in $U \setminus J$, can be decomposed uniquely into a direct sum by the following integrations.

$$\varphi_J(x) = \frac{1}{2\pi i}\int_{\gamma_J} \frac{\varphi(\tau)}{\tau - x} d\tau, \quad \text{for } x \in U,$$

and

$$\varphi_F(x) = \frac{1}{2\pi i}\int_{\gamma_F} \frac{\varphi(\tau)}{\tau - x} d\tau, \quad \text{for } x \in F,$$

where the intagration path $\gamma_J \subset U \setminus J$ turns once aroud the Julia set J in the counterclockwise direction passing near the boudary of U so that x belongs to the inside of the integration path, and the integration path $\gamma_F \subset U \setminus J$ turns once around the Julia set J in the clockwise direction passing near the Julia set J so that x belongs to the outside of the integration path. The integration paths are "ideal", or the integration should be considered as some limit. This defines functions $\varphi_J \in \mathcal{O}(U)$ and $\varphi_F \in \mathcal{O}_0(F)$. Moreover, we have $\varphi = \varphi_J + \varphi_F$ in $U \setminus J$. Here, $\varphi_J + \varphi_F$ means the usual sum of functions, and we don't distinguish the prehyperfunction and the function defined by φ in $U \setminus J$. Note that the decomposition is unique, since a function belonging to $\mathcal{O}(U) \cap \mathcal{O}_0(F)$ is holomorphic on the Riemann sphere and vanishes at the infinity, hence it is identically zero.

Let us define the Ruelle's transfer operator for our prehyperfunctions.

DEFINITION 1.1 Complex Ruelle operator $L : \mathcal{H}(J) \to \mathcal{H}(J)$ is defined by

$$(L\varphi)(x) = \sum_{y \in R^{-1}(x)} \frac{\varphi(y)}{(R'(y))^2}, \quad \varphi \in \mathcal{H}(J), \quad x \in U \setminus J.$$

This operator can be rewritten in an "integral operator form" as follows.

$$(L\varphi)(x) = \frac{1}{2\pi i} \int_{\gamma_J + \gamma_F} \frac{\varphi(\tau)}{R'(\tau)(R(\tau) - x)} d\tau,$$

where the integration path γ_J and γ_F are taken as before. This formula can be verified immediately by applying the residue formula. For each $x \in U \setminus J$, this formula defines the value $(L\varphi)(x)$ by choosing the integration path γ_J running sufficiently near the boudary ∂U, and by choosing the integration path γ_F running sufficiently near J.

The space of prehyperfunctions $\mathcal{H}(J)$ has a natural decomposition $\mathcal{H}(J) = \mathcal{O}(J) \oplus \mathcal{O}_0(F)$. This natural decomposition induces a natural decomposition of the complex Ruelle operator $L : \mathcal{O}(J) \oplus \mathcal{O}_0(F) \to \mathcal{O}(J) \oplus \mathcal{O}_0(F)$ as

$$L = \begin{pmatrix} L_{JJ} & L_{JF} \\ L_{FJ} & L_{FF} \end{pmatrix}.$$

In our case, these components are computed explicitly as follows.

$$(L_{JJ}\varphi_J)(x) = \sum_{y \in R^{-1}(x)} \frac{\varphi_J(y)}{(R'(y))^2} + \frac{\varphi_J(0)}{R''(0)(R(0) - x)},$$

$$(L_{JF}\varphi_F)(x) = 0,$$

$$(L_{FJ}\varphi_J)(x) = -\frac{\varphi_J(0)}{R''(0)(R(0) - x)},$$

$$(L_{FF}\varphi_F)(x) = \sum_{y \in R^{-1}(x)} \frac{\varphi_F(y)}{(R'(y))^2}$$

Note that in our case, or more generally, in the case of polynomial dynamical systems case with all finite critical points are included in the Julia set, the component L_{JF} vanishes and the complex Ruelle operator becomes a lower triangular matrix type. This fact simplifies our eigenvalue problem.

2. Eigenvalue problem and the Fredholm determinant

In this section, we consider the eigenvalue problem

$$\lambda L\varphi = \varphi, \qquad \lambda \in \mathbb{C}, \quad \varphi \in \mathcal{H}(J)$$

of the Ruelle operator $L : \mathcal{H}(J) \to \mathcal{H}(J)$. Note that the eigenvalues in the usual sense is the inverses of the zeros of the Fredholm determinant. In order to aviod confusions, a zero of the Fredholm determinant will be called a singular value of the operator. As computed in [10], the Fredholm determinant of L is given by the trace formula.

$$D(\lambda) = \det(I - \lambda L) = \exp\left(-\sum_{m=1}^{\infty} \frac{\lambda^m}{m} \mathrm{tr}[L^m]\right)$$

In our case $R(z) = z^2 + i$, the Fredholm determinant $D(\lambda)$ of the transfer operator L is directly computed as follows.

$$\begin{aligned} D(\lambda) &= 1 + \sum_{k=1}^{\infty} \frac{\lambda^k}{R''(0)R'_{k-1}(i)R_k(0)} \\ &= (1 - \frac{\lambda}{2})(1 + \frac{1-i}{2}\lambda)\frac{1}{(1 - \frac{\lambda^2}{4(1+i)})} \end{aligned}$$

This shows that the Fredholm determinant $D(\lambda)$ is rational and it is holomorphic for $|\lambda| < 2\sqrt[4]{2}$. It has poles at $\lambda = \pm 2\sqrt{1+i}$. Note that the absolute value $|\lambda|$ of the pole is related to the Collet-Eckmann condition, since it is given by the eigenvalue of the repelling periodic point in the postcritical set. $D(\lambda)$ has zeros at $\lambda = 2$ and $\lambda = -(1+i)$.

DEFINITION 2.1 Function $\chi_\zeta(z) = \frac{1}{z-\zeta}$ is called the *unit pole* at ζ. For each $\zeta \in F$, $\chi_\zeta \in \mathcal{O}(J)$, and for each $\zeta \in J$, $\chi_\zeta \in \mathcal{O}_0(F)$.

Let U denote the space of functions, spanned by unit poles at postcritical set, of the following form.

$$u = u_1\chi_i + u_2\chi_{i-1} + u_3\chi_{-i}, \qquad u_k \in \mathbb{C}, k = 1, 2, 3.$$

U is an invariant 3-dimensional complex vector space. The eigenfunction of L computed formally by the formula

$$f = \sum_{k=0}^{\infty} \frac{\lambda^k}{R'_k(R(0))} \chi_{R_k(R(0))}$$

is given by

$$f = -\frac{1+2i}{5}\left((2i-1)\chi_i - 2i\chi_{i-1} + \chi_{-i}\right).$$

The transfer operator restricted to this invarinat subspace U can be represented by the matrix

$$L_U = \begin{pmatrix} \frac{i}{2} & \frac{1+i}{4} & -\frac{i}{2} \\ -\frac{i}{2} & 0 & \frac{i}{2} \\ 0 & -\frac{1+i}{4} & 0 \end{pmatrix}$$

The characteristic polinomial of L_U is computed as follows.

$$\det(L_U - \lambda^{-1}I) = -\lambda^{-1}(\lambda^{-1} - \frac{1}{2})(\lambda^{-1} - \frac{i-1}{2}).$$

The eigenvector belonging to singular value $\lambda = -(1+i)$ is given by

$$\varphi_3(z) = (2i-1)\chi_i - 2i\chi_{i-1} + \chi_{-i} = -\frac{(4+2i)}{(z-i)(z-i+1)(z+i)},$$

which belongs to the same eigenspace as f above. Note that this function is of order of z^{-3} at the infinity. This is the reason why I denote it as φ_3.

The eigenfunction belonging to singular value $\lambda = 2$ is given by

$$\varphi_2(z) = \chi_i - (1+i)\chi_{i-1} + i\chi_{-i} = \frac{(3+i)z + 1 - i}{(z-i)(z-i+1)(z+i)},$$

and is of order z^{-2} at the infinity.

The eigenfunction belonging to the singular value $\lambda = \infty$ is given by

$$\varphi_1(z) = \chi_i + \chi_{-i} = \frac{2z}{z^2+1},$$

and is of order z^{-1} at the infinity.

3. Backward expansion and the Fredholm determinant

In this section, we examine the relationship between the backward expansion coefficients and the Fredholm determinant. Theorem in this section holds for $R(z) = z^2 + c$. The backward expansion coefficients $\{b_k\}_{k=0}^{\infty}$ and the coefficients $\{\omega_k\}_{k=0}^{\infty}$ of the Fredholm determinant are defined as follows.

DEFINITION 3.1

$$b_0 = 1, \qquad b_k = \sum_{\eta \in R^{-k}(0)} \frac{1}{(R_k'(\eta))^2}, \qquad k = 1, 2, \cdots.$$

DEFINITION 3.2

$$\omega_0 = 1, \quad \omega_k = \frac{1}{R_k''(0)R_k(0)} = \frac{1}{R''(0)R_{k-1}'(R(0))R_k(0)}, \quad k = 1, 2, \cdots.$$

The Fredholm determinant $D(\lambda)$ is rewritten as

$$D(\lambda) \ = \ \sum_{k=0}^{\infty} \omega_k \lambda^k.$$

Let $B(\lambda)$ be the power series defined by

$$B(\lambda) \ = \ \sum_{k=0}^{\infty} b_k \lambda^k.$$

The following theorem shows that the backward complex expansion rate is directly related to the smallest singular value of the transfer operator.

THEOREM 3.3

$$D(\lambda)B(\lambda) \ = \ 1$$

holds as power series.

This theorem follows immediately from the following propositon.

PROPOSITION 3.4

$$\sum_{s=0}^{k} \omega_s b_{k-s} \ = \ 0, \qquad k = 1, 2, \cdots.$$

PROOF As $R_k(z)$ is a polynomial of degree 2^k, rational function $(R_k'(z)R_k(z))^{-1}$ has no residue at the infinity, for $k \geq 1$. Let $C(R_k)$ denote the set of all critical points of $R_k(z)$ in the complex plane. In our case of $R(z) = z^2 + c$, we have a decomposition of the set of critical points

$$C(R_k) \ = \ \bigcup_{s=1}^{k} R^{-(k-s)}(0).$$

As the sum of all resudues of this rational function vanishes. We have, for $k \geq 1$,

$$\begin{aligned} 0 \ &= \ \frac{1}{2\pi i} \int_{\gamma_J} \frac{d\tau}{R_k'(\tau)R_k(\tau)} \\ &= \ \sum_{\eta \in R^{-k}(0)} \frac{1}{(R_k'(\eta))^2} + \sum_{y \in C(R_k)} \frac{1}{R_k''(y)R_k(y)} \end{aligned}$$

$$
\begin{aligned}
&= b_k + \sum_{s=1}^{k} \frac{1}{R_s''(0)} \sum_{y \in R^{k-s}(0)} \frac{1}{(R_{k-s}'(y))^2 R_s(0)} \\
&= b_k + \sum_{s=1}^{k} \frac{1}{R_s''(0)(R_s(0)-0)} \sum_{y \in R^{k-s}(0)} \frac{1}{(R_{k-s}'(y))^2} \\
&= b_k + \sum_{s=1}^{k} b_{k-s}\omega_s \ = \ \sum_{s=0}^{k} b_{k-s}\omega_s .
\end{aligned}
$$

4. Dual Ruelle operator and its formal eigenhyperfunction

The dual operator of the complex Ruelle operator was defined in [10]. Here we recall some definitions and notaions. For the precise definitions, see [10].

DEFINITION 4.1 A complex linear functional $\Phi : \mathcal{O}(J) \to \mathbb{C}$ is said to be *holomorphic* if the value $\Phi[g_\mu]$ depends holomorphically upon μ for holomorphic family of functions g_μ.

DEFINITION 4.2 The *dual space* $\mathcal{O}^*(J)$ is the space of continuous, complex linear, and holomorphic functionals $\Phi : \mathcal{O}(J) \to \mathbb{C}$.

Representation of functionals as integral operators is given by the following propositions.

PROPOSITION 4.3 The dual space $\mathcal{O}^*(J)$ is isomorphic to $\mathcal{O}_0(F)$. More precisely, for $\Phi \in \mathcal{O}^*(J)$, $f(\zeta) = \Phi[\chi_\zeta]$ defines an $f \in \mathcal{O}_0(F)$, and we have

$$
\Phi[g] = \frac{1}{2\pi i} \int_{\gamma_F} f(\tau) g(\tau) d\tau, \qquad for \quad g \in \mathcal{O}(J).
$$

PROPOSITION 4.4 The dual space $\mathcal{O}_0^*(F)$ is isomorphic to $\mathcal{O}(J)$. More precisely, for $\Psi \in \mathcal{O}_0^*(F)$, $g(z) = \Psi[\chi_z]$ defines a $g \in \mathcal{O}(J)$, and we have

$$
\Psi[f] = \frac{1}{2\pi i} \int_{\gamma_J} g(\tau) f(\tau) d\tau, \quad \text{for} \quad f \in \mathcal{O}_0(F).
$$

Isomorphisms in Propositions 4.3 and 4.4 are called *Cauchy transformations*, since they are defined by the Chauchy kernel $\chi_\zeta(z)$.

DEFINITION 4.5 The *pairings* $\langle f, g \rangle_F$ and $\langle g, f \rangle_J$ are defined for $g \in \mathcal{O}(J)$ and $f \in \mathcal{O}_0(F)$ by

$$
\langle f, g \rangle_F = \frac{1}{2\pi i} \int_{\gamma_F} f(\tau) g(\tau) d\tau,
$$

and

$$\langle g, f\rangle_J = \frac{1}{2\pi i}\int_{\gamma_J} g(\tau)f(\tau)d\tau.$$

For $\varphi = \varphi_J \oplus \varphi_F \in \mathcal{H}(J)$ and $\psi = \psi^J \oplus \psi^F \in \mathcal{O}_0(F) \oplus \mathcal{O}(J) \simeq \mathcal{O}^*(J) \oplus \mathcal{O}_0^*(F) = \mathcal{H}^*(J)$, the pairing $\langle\psi, \varphi\rangle$ is defined by

$$\langle\psi, \varphi\rangle = \langle\psi^J, \varphi_J\rangle_F + \langle\psi^F, \varphi_F\rangle_J.$$

Let $L^* : \mathcal{H}^*(J) \to \mathcal{H}^*(J)$ denote the dual operator of the complex Ruelle operator $L : \mathcal{H}(J) \to \mathcal{H}(J)$. And let $\mathcal{L}^* : \mathcal{O}_0(F) \oplus \mathcal{O}(J) \to \mathcal{O}_0(F) \oplus \mathcal{O}(J)$ denote its representation via the Cauchy transformation. We call this operator $\mathcal{L}^*$ the *adjoint Ruelle operator.* The dual space of $\mathcal{H}(J)$ will be denoted by $\mathcal{H}^*(J)$, and we abuse this notation to denote the "adjoint" space $\mathcal{O}_0(F) \oplus \mathcal{O}(J)$, too. The components of $\mathcal{L}^*$ with respect to the natural decomposition will be denoted as

$$\mathcal{L}^* = \begin{pmatrix} \mathcal{L}^*_{JJ} & \mathcal{L}^*_{JF} \\ \mathcal{L}^*_{FJ} & \mathcal{L}^*_{FF} \end{pmatrix}$$

The explicit formula for the adjoint Ruelle operator of our case can be computed directly as follows.

PROPOSITION 4.6 For $\psi = \psi^J \oplus \psi^F$ with $\psi^J \in \mathcal{O}_0(F) \simeq \mathcal{O}^*(J)$, and $\psi^F \in \mathcal{O}(J) \simeq \mathcal{O}_0^*(F)$,

$$(\mathcal{L}^*\psi)(z) = \left(\frac{\psi^J(R(z))}{R'(z)} + \frac{\psi^F(R(0))}{R''(0)}\chi_0(z)\right)$$

$$\oplus \left(+\frac{\psi^F(R(z))}{R'(z)} - \frac{\psi^F(R(0))}{R''(0)}\chi_0(z)\right).$$

And in $U \setminus J$, where ψ defines a holomorphic function,

$$\mathcal{L}^*\psi = \frac{\psi \circ R}{R'}.$$

The proof is straightforward by direct computations applying the residue theorem. For more general cases and for detailed calculations, see [10].

In our case $R(z) = z^2 + i$, and more generally, if the Fatou set contains no critical points (except the infinity), then the component $\mathcal{L}_{FJ}$ vanishes. In this case the adjoint operator becomes an upper triangular matrix form.

The eigenvalue problem for the adjoint Ruelle operator is formulated as

$$\lambda \mathcal{L}^* \psi = \psi, \qquad \lambda \in \mathbb{C}, \quad \psi \in \mathcal{H}^*(J).$$

In our case, the eigenfunction of $\mathcal{L}^*$ can be formally computed.

PROPOSITION 4.6 The image of a unit pole by the adjoint Ruelle operator is given by

$$\mathcal{L}^* \chi_y \;=\; \frac{\chi_y(R(0))}{R''(0)} \chi_0 + \sum_{\eta \in R^{-1}(y)} \frac{1}{(R'(\eta))^2} \chi_\eta .$$

The forward image of a unit pole at y consists of poles at its inverse image and a pole at the critical point. Hence, the linear combinations of poles at critical points and its backward images form an invariant subspace. In this space, we find an eigenfunction as a formal sum

$$\psi \;=\; \sum_{k=0}^{\infty} \sum_{y \in R^{-k}(0)} \frac{\lambda^k}{(R_k'(y))^2} \chi_y .$$

Unfortunately, however, this formal sum is divergent for singular values of λ, since $B(\lambda) = \infty$ exactly when $D(\lambda) = 0$. We have to look for the eigenfunctions in a larger space.

5. Dual Ruelle operator on a quotient space and hyperfunctions

Our purpose of studying the transfer operator is to find invariant measures and Gibbs measures supported on the Julia set, which are related to the eigen-functions.

DEFINITION 5.1 A *hyperfunction* supported on the Julia set is an element of the quotient space $\mathcal{H}(J)/\mathcal{O}(J)$.

What we are looking for are differential forms with hyperfunction coefficients. Since integration of a holomorphic differential form along a boudary of simply connected domain vanishes if the differential form is holomorphic in the domain, functions in $\mathcal{O}(J)$ do not contribute to the measure μ defined by

$$\mu(A \cap J) \;=\; \frac{1}{2\pi i} \int_{\partial A} \psi(\tau) d\tau$$

for open sets A included in a neighborhood of J, with appropriate measurability properties. That is, we look for eigen-hyperfunctions for the dual operator $\mathcal{L}^*$ instead of eigen-prehyperfunctions. For $k = 0, 1, \cdots$, let

$$\kappa_k(z) = \frac{1}{R_k'(z)R_k(z)}.$$

We see immediately that $\kappa_0 = \chi_0$ and

$$(\mathcal{L}^*\kappa_k) = \kappa_{k+1}, \quad \text{for} \quad k = 0, 1, \cdots.$$

PROPOSITION 5.2

$$\kappa_k = \sum_{y \in C(R_k)} \frac{1}{R_k''(y)R_k(y)}\chi_y + \sum_{y \in R^{-k}(0)} \frac{1}{(R_k'(y))^2}\chi_y.$$

The proof is immediate by decomposing κ_k into partial fractions. Note that κ_k belongs to $\mathcal{O}_0(F)$, since all poles of κ_k are in the Julia set.

For $\lambda \in \mathbb{C}$, let

$$\psi_\lambda = \sum_{k=0}^{\infty} \lambda^k \kappa_k.$$

We see immediately that $\psi_\lambda \in \mathcal{O}_0(F)$, since the sum converges uniformly on compact sets in the Fatou set. This function is almost an eigenfunction of $\mathcal{L}^*$. We have the following proposition.

PROPOSITION 5.3

$$\lambda \mathcal{L}^* \psi_\lambda = \psi_\lambda - \chi_0.$$

Functions κ_k has poles at inverse images of the critical point. The function ψ_λ does not have poles except at the critical point of R if λ is a singular value of the transfer operator.

PROPOSITION 5.4 If $D(\lambda) = 0$, then ψ_λ does not have poles in the backward orbit $O^-(0)$ of the critical point.

PROOF For $\ell \geq 1$ and $y \in R^{-\ell}(0)$, the residue of ψ_λ is

$$\frac{\lambda^\ell}{(R_\ell'(y))^2} + \sum_{t=1}^{\infty} \frac{\lambda^{\ell+t}}{R_t''(0)R_t(0)(R_\ell'(y))^2}$$

$$= \frac{\lambda^\ell}{(R_\ell'(y))^2}\left(1 + \sum_{t=1}^{\infty} \frac{\lambda^t}{R_t''(0)R_t(0)}\right) = \frac{\lambda^\ell}{(R_\ell'(y))^2}D(\lambda) = 0.$$

As we mentioned in the previous section, the operator $\mathcal{L}^* : \mathcal{O}_0(F) \oplus \mathcal{O}(J) \to \mathcal{O}_0(F) \oplus \mathcal{O}(J)$ is of upper triangular form and the subspace $\mathcal{O}_0(F)$ is mapped into itself by $\mathcal{L}^*$.

Let

$$\mathcal{V} = \{f \in \mathcal{O}(J) \mid f(i) = f(i-1) = f(-i) = 0\}$$

be the space of functions which vanish on the postcritical set. This space is a codimension 3 subspace of $\mathcal{O}(J)$. We see immediately that this subspace is mapped into itself by $\mathcal{L}^*$.

PROPOSITION 5.5

$$\mathcal{L}^*\mathcal{V} \subset \mathcal{V}.$$

PROOF For $f \in \mathcal{V}$, we can find $g \in \mathcal{O}(J)$ such that

$$f(z) = (z-i)(z-i+1)(z+i)g(z).$$

Then

$$(\mathcal{L}^*f)(z) = \frac{f(R(z))}{R'(z)} = \frac{1}{2}z(z-i)(z-i+1)(z+i)(z-1+1)g(z^2+i).$$

Hence, $\mathcal{L}^*f \in \mathcal{V}$.

Let $V = \mathcal{O}(J)/\mathcal{V}$ denote the quotient space. V is a vector space of complex dimension 3. We take a basis h_1, h_2, h_3 of V by

$$h_1(z) = -\frac{i}{2}(z+1-i)(z+i) = -\frac{i}{2}(z^2+z+1+i),$$

$$h_2(z) = \frac{1+2i}{5}(z^2+1),$$

$$h_3(z) = \frac{-2+i}{10}(z+1-i)(z-i) = \frac{-2+i}{10}(z^2+(1-2i)z-1-i).$$

These functions are determined by the following condition.

$$\begin{array}{lll} h_1(i) = 1, & h_1(i-1) = 0, & h_1(-i) = 0 \\ h_2(i) = 0, & h_2(i-1) = 1, & h_2(-i) = 0 \\ h_3(i) = 0, & h_3(i-1) = 0, & h_3(-i) = 1 \end{array}$$

Vector space spanned by these three functions is isomorphic to the qotient space V. We identify the quotient space V and the subspace of $\mathcal{O}(J)$ spanned by this basis. The canonical projection fron $\mathcal{O}(J)$ to V is given by

$$h = f(i)h_1 + f(i-1)h_2 + f(-i)h_3, \quad \text{for} \quad f \in \mathcal{O}(J),$$

modulo $\mathcal{V}$.

The adjoint operator induces an operator on the quotient space $\mathcal{O}_0(F)\oplus V$. This operator is of an upper triangular form with respect to this splitting. The (V, V)-component of this operator is denoted as $\mathcal{L}_V^* : V \to V$. By a direct computation, we get the matix representation with respect to the basis h_1, h_2, h_3, as follws.

PROPOSITION 5.6

$$\mathcal{L}_V^* = \begin{pmatrix} \frac{i}{2} & -\frac{i}{2} & 0 \\ \frac{1+i}{4} & 0 & -\frac{1+i}{4} \\ -\frac{i}{2} & \frac{i}{2} & 0 \end{pmatrix}$$

This matrix is the transpose of the matrix L_U computed ine section 2. The eigenvalues of this matrix are $0, \frac{1}{2}$, and $-\frac{1}{1+i}$. Hence the singular values are 2 and $-(1+i)$ and ∞.

6. Eigenhyperfunctions and various measures on the Julia set

Singularvalues, eigenvectors, and eigenfunctions of $\mathcal{L}_V^*$ are, respectively,

$$\begin{array}{rcllr} \lambda & = & \infty, & {}^t(1,1,1) & \phi_0(z) = 1 \\ \lambda & = & 2, & {}^t(i, i-1, -i) & \phi_1(z) = z \\ \lambda & = & -(1+i), & {}^t(i,1,-i) & \phi_2(z) = (i-1)h_1 + (1+i)h_2 + (1-i)h_3. \end{array}$$

Let $\psi_0 \in \mathcal{O}(J)$ be a representative of $\phi_2 \in V$ given by

$$\psi_0(z) \;=\; \frac{1+7i}{5}(z^2+1) + (1+i)z.$$

And let $\theta_0 \in \mathcal{V}$ be defined by

$$\theta_0(z) \;=\; \frac{3-4i}{3}(z-i)(z+i)(z-i+1).$$

Further, deine a polynomial $\varpi \in \mathcal{O}(J)$ by

$$\varpi(z) \;=\; \frac{z}{2}(z-1+i).$$

And define functions $\theta_k \in \mathcal{V}$ for $k = 1, 2, \cdots$, by

$$\theta_k \;=\; (\mathcal{L}^*)^k \theta_0.$$

We see immediately that

$$\theta_k(z) \;=\; \theta_0(z) \prod_{j=0} k - 1\varpi(R_j(z)).$$

A direct computation shows the following.

PROPOSITION 6.1 For $\lambda = -(1+i)$, we have

$$\lambda \mathcal{L}^* \psi_0 = \psi_0 + \kappa_0 + \theta_0$$

and by setting

$$\Psi = \psi_0 + \sum_{k=0}^{\infty} \kappa_k + \sum_{k=0}^{\infty} \theta_k,$$
$$\lambda \mathcal{L}^* \Psi = \Psi$$

holds in a formal sense.

This formal series Ψ does not have a meaning as a prehyperfunction, since the holomophic part $\sum_{k=0}^{\infty} \theta_k$ diverges in the Fatou set. However, the limit is well defined in the quotient space $\mathcal{H}/\mathcal{V}$

THEOREM 6.2 Ψ is well defined in $\mathcal{H}/\mathcal{V}$ and represents an eigenhyperfunction of the adjoint Ruelle operator $\mathcal{L}^*$ considered on the quotient space.

THEOREM 6.3 The eigen-prehyperfunction $\varphi_3 \in \mathcal{H}(J)$ of the transfer operator $\mathcal{L}$ and the eigen-hyperfunction Ψ defines a hyperfunction $\varphi_3 \Psi \in \mathcal{H}(J)/\mathcal{O}(J)$ represented by

$$\varphi_3 \left(\psi_0 + \sum_{k=0}^{\infty} \kappa_k \right).$$

THEOREM 6.4 The hyperfunction $\varphi_3 \Psi$ defines an invariant measure supported on the Julia set, and the hyperfunction Ψ defines a complex Gibbs measure for complex potential $\log((R'(z))^2)$.

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