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An Example of Convergent Star Product *

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Abstract An example of convergent star product is described. The Moyal product is considered for the linear Poisson algebra associated with the Heisenberg Lie algebra. The product is absolutely convergent in a certain class of entire functions. The critical exponent of entire functions for convergence is obtained.

1 Introduction

The purpose of this note is to give a concrete example of convergent deformation quantizations for certain Poisson algebras. This note is besed on the joint work of H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka [7, 8].

We have proposed the notion of the deformation quantization of a

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Fréchet-Poisson algebra in [7], which means convergenct star products for the Poisson algebras in the Fréchet categories. Similar notion has been studied in the C^* framework by Rieffel [9].

Different from formal deformations [1], if we consider their convergence of star product, we have met anomalous phenomena. In fact, we showed in [7] that the Moyal product converges for the entire functions on $\mathbb{C}^2$ with certain order; it makes sense as an associative product for order less than 2, but it failed the associative properties for order ≥ 2.

Subsequent to [7], we attempt this argument to the linear-Poisson structures of Heisenberg Lie algebras. The classes of Fréchet algebras will be also taken in the space of entire functions on complex $2n+1$-space. Introducing various families of semi-norms, we study the convergence of star products for the Fréchet-Poisson algebras in certain classes of entire functions. We also study some algebraic properties for the closure of free tensor algebras. The techniques in [7] proceeds to find a similar phenomena to [7] even in their paper. We have similar features and different properties from [7] according to the semi-norms, which are shown in the last section.

2 Fréchet Poisson algebras

2.1 Fréchet algebras of $\mathbb{C}^{n+1}$.

We first introduce a system of semi-norms on the set of entire functions to obtain Fréchet algebras. Let $\mathbb{C}^{n+1}$ $(n \geq 0)$ be a complex $(n+1)$-space with the complex coordinates $(x_0, x_1, \cdots, x_n)$, and $\mathcal{P}(\mathbb{C}^{n+1})$ the set of all polynomial functions on $\mathbb{C}^{n+1}$.

Let p, b denote $(n+1)$-tuples $p = (p_0, p_1, \cdots, p_n)$, $b = (b_0, b_1, \cdots, b_n)$ with $p_i > 0$ and $b_i > 0$ $(i = 0, \dots, n)$, respectively. By dropping the components p_0 and b_0, p_* and b_* denote $p_* = (p_1, \cdots, p_n)$, $b_* = (b_1, \cdots, b_n)$, respectively. Let r_0 and N_0 be positive real number and non negative integer. We define semi-norms $||\cdot||_{p,b}$, $||\cdot||_{p_*,b_*,r_0}$ and $||\cdot||_{p_*,b_*,N_0}$ on $\mathcal{P}(\mathbb{C}^{n+1})$ as follows:

Definition 2.1 *For $f(x_0, \dots, x_n) \in \mathcal{P}(\mathbb{C}^{n+1})$ we set*

$$||f||_{p,b} = \sup_{(x_0,\cdots,x_n)\in\mathbb{C}^{n+1}} |f| \exp(-\sum_{i=0}^{n} b_i |x_i|^{p_i}) \tag{2.1}$$

$$||f||_{p_*,b_*,r_0} = \sup_{|x_0|\leq r_0} \sup_{(x_1,\cdots,x_n)\in\mathbb{C}^n} |f| \exp(-\sum_{i=1}^{n} b_i|x_i|^{p_i}) \qquad (2.2)$$

$$||f||_{p_*,b_*,N_0} = \sum_{k=0}^{N_0} ||f_k(x_1,\cdots,x_n)||_{p_*,b_*}, \qquad (2.3)$$

where we expand f in x_0 variable:

$$f(x_0,x_1,\cdots,x_n) = \sum_k f_k(x_1,\cdots,x_n)x_0^k. \qquad (2.4)$$

For a fixed p, we consider the completions of $\mathcal{P}(\mathbb{C}^{n+1})$ under the systems of seminorms $\{||\cdot||_{p,b}\}_b$, $\{||\cdot||_{p_*,b_*,r_0}\}_{b_*,r_0}$ and $\{||\cdot||_{p_*,b_*,N_0}\}_{b_*,N_0}$, respectively. We set the completions:

(E.1) $\mathcal{E}_p(\mathbb{C}^{n+1})$ with respect to $\{||\cdot||_{p,b}\}_b$,

(E.2) $\mathcal{E}_{\mathrm{Hol},p_*}(\mathbb{C}^{n+1})$ with respect to $\{||\cdot||_{p_*,b_*,r_0}\}_{b_*,r_0}$

(E.3) $\mathcal{E}_{\infty,p_*}(\mathbb{C}^{n+1})$ with respect to $\{||\cdot||_{p_*,b_*,N_0}\}_{b_*,N_0}$.

Let $\mathcal{E}(\mathbb{C}^{n+1})$ denote the space of all entire functions on $\mathbb{C}^{n+1}$. Then, it is easy to see $\mathcal{E}_p(\mathbb{C}^{n+1})$ (resp. $\mathcal{E}_{\mathrm{Hol},p_*}(\mathbb{C}^{n+1})$) consists of $f \in \mathcal{E}(\mathbb{C}^{n+1})$ such that $||f||_{p,b} < \infty$ for $\forall b$ (resp. $||f||_{p_*,b_*,r_0} < \infty$ for $\forall b_*, r_0$). But as to the third space we have

$$\mathcal{E}_{\infty,p_*}(\mathbb{C}^{n+1}) = \mathcal{E}_{p_*}(\mathbb{C}^n)[[x_0]] \qquad (2.5)$$

as topological spaces where the topology of the right hand side is given as the space of formal power series in x_0.

For simplicity we denote by $\mathcal{E}_{Ej}(\mathbb{C}^{n+1})$ any one of the spaces given in (E.1) -(E.3) according to $j = 1,2,3$ in the sequel.

Lemma 2.1 *Each space $\mathcal{E}_{Ej}(\mathbb{C}^{n+1})$ for $j = 1,2,3$ becomes a commutative Fréchet algebra by the usual multiplication of functions.*

2.2 Fréchet Poisson algebras $\mathcal{E}_{Ej}(\mathbb{C}^{n+1})$.

Let $\mathcal{F}$ be a commutative associative Fréchet algebra over $\mathbb{C}$, i.e., $\mathcal{F}$ has a metrizable complete topology defined by a system of semi-norms, and a product dented by the dott $\cdot$ is smooth.

Definition 2.2 *$\mathcal{F}$ is called a Fréchet Poisson algebra if $\mathcal{F}$ has a continuous bilinear operation $\{\ ,\ \}: \mathcal{F}\times\mathcal{F}\to\mathcal{F}$ (called a Poisson bracket on $\mathcal{F}$) such that for any $f,g,h\in\mathcal{F}$,*

$$
\begin{array}{ll}
(P.1) & \{f,g\} = -\{g,f\}, \\
(P.2) & \sum_{cyclic}\{f,\{g,h\}\} = 0, \\
(P.3) & \{f, g\cdot h\} = \{f,g\}\cdot h + g\cdot\{f,h\}.
\end{array}
$$

The Fréchet Poisson algebras we discuss in this paper are as follows: Consider the complex $(2n+1)$-space $\mathbb{C}^{2n+1}$. For convenience in notation, set $(x_0, x_1, \dots, x_{2n}) = (z, x, y)$, where $x_0 = z$, $x = (x_1, \dots, x_n)$ and $y = (x_{n+1}, \dots, x_{2n})$. We now define the following Poisson bracket on $\mathbb{C}^{2n+1}$:

$$\{f,g\}_H = z(f(\overleftarrow{\partial_x}\cdot\overrightarrow{\partial_y} - \overleftarrow{\partial_y}\cdot\overrightarrow{\partial_x})g) \tag{2.6}$$

for every functions $f{=}f(z,x,y)$ and $g{=}g(z,x,y)$, where $\overleftarrow{\partial_x}\cdot\overrightarrow{\partial_y} - \overleftarrow{\partial_y}\cdot\overrightarrow{\partial_x}$ stands for a bidifferential operator:

$$f(\overleftarrow{\partial_x}\cdot\overrightarrow{\partial_y} - \overleftarrow{\partial_y}\cdot\overrightarrow{\partial_x})g = \sum_i \partial_{x_i}f\,\partial_{y_i}g - \partial_{y_i}f\,\partial_{x_i}g. \tag{2.7}$$

It is easily seen that $\{,\}_H$ gives a Poisson bracket on $\mathcal{P}(\mathbb{C}^{2n+1})$, called of *Heisenberg type.* We have

$$\{z, x_i\}_H = 0, \quad \{z, y_i\}_H = 0, \quad \{x_i, y_j\}_H = z\delta_{ij} \tag{2.8}$$

which gives a linear Poisson structure on $\mathbb{C}^{2n+1}$ associated with the Heisenberg Lie algebra.

By definition, $\mathcal{P}(\mathbb{C}^{2n+1})$ is dense in each $\mathcal{E}_{Ej}(\mathbb{C}^{2n+1})$, hence we have

Lemma 2.2 *Let $\mathcal{E}_{Ej}(\mathbb{C}^{2n+1})$ be one of (E1)-(E3). Then $\mathcal{E}_{Ej}(\mathbb{C}^{2n+1})$ is a Fréchet Poisson algebra with Poisson structure $\{\ ,\ \}_H$.*

2.3 Deformation quantization of $\mathcal{E}_{Ej}(\mathbb{C}^{2n+1})$.

Now, we consider a noncommutative product on $\mathcal{P}(\mathbb{C}^{2n+1})$:

$$f*g = \sum_{k=0}^{\infty}\frac{(iz)^k}{2^k k!}\, f(\overleftarrow{\partial_x}\cdot\overrightarrow{\partial_y} - \overleftarrow{\partial_y}\cdot\overrightarrow{\partial_x})^k g, \tag{2.9}$$

where $(\overleftarrow{\partial_x} \cdot \overrightarrow{\partial_y} - \overleftarrow{\partial_y} \cdot \overrightarrow{\partial_x})^k$ denotes the k-th power of the bidifferential operator (2.6). As for $f{=}f(z,x,y)$, $g{=}g(z,x,y) \in \mathcal{P}(\mathbb{C}^{2n+1})$, the product (2.9) gives an associative product since $*$ is given by the Moyal product formula. We focus to a question how (2.9) extends to Fréchet Poisson algebra $\mathcal{E}_{Ej}(\mathbb{C}^{2n+1})$. In this section, we are concentrated with the case of the weights

$$p = (p_0, p_1, \dots, p_1), \quad b = (b_0, b_1, \dots, b_1). \tag{2.10}$$

One of goals in this note is to show the following:

Theorem 2.1 *Let $(\mathcal{E}_{Ej}(\mathbb{C}^{2n+1}), \cdot, \{,\}_H)$ be a Fréchet Poisson algebra given by Lemma 2.2. Assume that p satisfies (2.10). Then the following properties hold:*

(1) $(\mathcal{E}_{Ej}(\mathbb{C}^{2n+1}), *)$ is an associative Fréchet algebra if and only if

(A1) For (E.1), $0 < p_1 \leq \dfrac{2p_0}{p_0+1}$,
(A2) For (E.2), $0 < p_1 \leq 2$,
(A3) For (E.3), $0 < p_1$.

(2) For any case (A1)-(A3), the algebra $(\mathcal{E}_{Ej}(\mathbb{C}^{2n+1}), *)$ has the following properties:

(i) $[z, \mathcal{E}_{Ej}(\mathbb{C}^{2n+1})] = 0$, *i.e. z is a central element.*
(ii) $[\mathcal{E}_{Ej}(\mathbb{C}^{2n+1}), \mathcal{E}_{Ej}(\mathbb{C}^{2n+1})]_* \subset z * \mathcal{E}_{Ej}(\mathbb{C}^{2n+1})$
(iii) $\mathcal{E}_{Ej}(\mathbb{C}^{2n+1}) = \mathcal{E}_{E1}(\mathbb{C}^{2n}) \oplus z * \mathcal{E}_{Ej}(\mathbb{C}^{2n+1})$ *(direct sum)*
(iv) (Self-similarity) *$z*$, and $*z$ are continuous linear isomorphisms of $\mathcal{E}_{Ej}(\mathbb{C}^{2n+1})$ onto $z*\mathcal{E}_{Ej}(\mathbb{C}^{2n+1})$.*
(v) *$a \to \bar{a}$ gives an involutive anti-automorphism of $(\mathcal{E}_{Ej}(\mathbb{C}^{2n+1}), *)$ such that $\bar{z} = z$.*
(vi) $\bigcap_{m \geq 0} z^m * \mathcal{E}_{Ej}(\mathbb{C}^{2n+1}) = \{0\}$

where $[\,,\,]$ is the commutator bracket with respect to the product $*$. Here $\mathcal{E}_{E1}(\mathbb{C}^{2n}) = \mathcal{E}_{p_*}(\mathbb{C}^{2n})$ consists of entire functions of $(x_1, \dots, x_n)$ variables satisfying the condition (2.1) given by $p_* = (p_1, \dots, p_1)$ and $b_* = (b_1, \dots, b_1)$.

If we use the notion of regulated algebras defined in [6], the properties (i)–(vi) above gives a special case of a regulated algebra: Replace (i) by (i') $[z, \mathcal{E}_{Ej}(\mathbb{C}^3)] \subset z * \mathcal{E}_{Ej}(\mathbb{C}^3) * z.$

Definition 2.3 *An associative (Fréchet) algebra $\mathcal{A}$ with the properties (i')-(v) is called a z-regulated (Fréchet) algebra (cf. [6]). If $\mathcal{A}$ satisfies (i), then $\mathcal{A}$ is called z-central. If $\mathcal{A}$ satisfying (vi) is called analytic (resp. formal) when every element is analytic with respect to z varible (resp. every element is a formal power series of z).*

We have several typical deformation quantization from the product $*$ as follows.

(i) By replacing z by $\hbar z$ in (2.9), we have a product $*_\hbar$.

Corollary 2.1 *Assume $(\mathcal{E}_p(\mathbb{C}^{2n+1}), *)$ satisfies (A 1) of Theorem 2.1. Then, $(\mathcal{E}_p(\mathbb{C}^{2n+1}), *_\hbar)$ is a deformation quantization of the Fréchet Poisson algebra $(\mathcal{E}_p(\mathbb{C}^{2n+1}), \cdot, \ \{,\}_H)$ absolutely convergent with respect to $\hbar \in \mathbb{C}$.*

(ii) For $f(x,y), g(x,y) \in \mathcal{E}_{p_*}(\mathbb{C}^{2n})$, we set a Poisson bracket by

$$\{f,g\}_0 = f(\overleftarrow{\partial_x} \cdot \overrightarrow{\partial_y} - \overleftarrow{\partial_y} \cdot \overrightarrow{\partial_x})g. \tag{2.11}$$

Then we have a Fréchet Poisson algebra $(\mathcal{E}_{p_*}(\mathbb{C}^{2n}), \cdot, \{\ ,\ \}_0)$ with the relation $\{x_i, y_j\}_0 = \delta_{ij}$.

Now consider the algebra $(\mathcal{E}_{\mathrm{Hol},p_*}(\mathbb{C}^{2n+1}), *)$ under the condition (A 2) of Theorem 2.1. Replacing z by $\hbar$ in $\mathcal{E}_{\mathrm{Hol},p_*}(\mathbb{C}^{2n+1})$, we get the space of analytic functions of $\hbar$ with values in $\mathcal{E}_{p_*}(\mathbb{C}^{2n})$.

Corollary 2.2 *Assume $(\mathcal{E}_{Hol,p_*}(\mathbb{C}^{2n+1}), *)$ satisfies (A 2) of Theorem 2.1. Then $(\mathcal{E}_{p_*}(\mathbb{C}^{2n}), *_\hbar)$ is a deformation quantization of $(\mathcal{E}_{p_*}(\mathbb{C}^{2n}), \cdot, \ \{,\}_0)$ absolutely convergent with respect to the parameter $\hbar \in \mathbb{C}$.*

(iii) As we have seen in (2.5), replacing z by $i\hbar$ in $\mathcal{E}_{\infty,p_*}(\mathbb{C}^{2n+1})$ gives the space $\mathcal{E}_{p_*}(\mathbb{C}^{2n})[[\hbar]]$. In the case (A 3) of Theorem 2.1, the product $*_\hbar$ implies a formal deformation quantization:

Corollary 2.3 *Assume $(\mathcal{E}_{\infty,p_*}(\mathbb{C}^{2n+1}), *)$ satisfies (A 3) of Theorem 2.1. Then $(\mathcal{E}_{p_*}(\mathbb{C}^{2n})[[\hbar]], *_\hbar)$ is a formal deformation quantization of the Fréchet Poisson algebra $(\mathcal{E}_{p_*}(\mathbb{C}^{2n}), \cdot, \ \{,\}_0)$.*

3 Free tensor algebra

3.1 Completion of Free tensor algebra

Let $\mathcal{T}$ be the free tensor algebra over a $(n+1)$-vector space V:

$$\mathcal{T} = \sum_{k=0}^{\infty} \oplus V^{\otimes k}, \quad \text{(finite sum)} \tag{3.1}$$

where $V^{\otimes 0} = \mathbb{C}$, $V^{\otimes k} = V \otimes \cdots \otimes V$ (k times).

We introduce a system of semi-norms in $\mathcal{T}$. Similarly as in §2.1, we use the notation : τ and s denote $(n+1)$-tuples $\tau = (\tau_0, \tau_1, \cdots, \tau_n)$ and $s = (s_0, s_1, \cdots, s_n)$, where $\tau_i > 0, s_i > 0$, $(i = 0, \ldots, n)$. By forgetting τ_0 and s_0, τ_* and s_* denote $\tau_* = (\tau_1, \cdots, \tau_n)$ and $s_* = (s_1, s_2, \cdots, s_n)$. Also t and N_0 denote a positive real number and a nonnegative integer.

Let us fix a basis $X_0, X_1, \cdots, X_n$ of V. Then an element of $\mathcal{T}$ is given by a finite sum

$$T = \sum_w c_w X_w, \quad (c_w \in \mathbb{C}), \tag{3.2}$$

where $X_w = X_{w_1} \otimes \cdots \otimes X_{w_k}$, $w = (w_1, \ldots, w_k)$ are words. For a word X_w, we denote by $m_i(w)$ the number of X_i in X_w, $(i = 0, \ldots, n)$, and set $m(w) = (m_0(w), \cdots, m_n(w))$ and $m_*(w) = (m_1(w), \cdots, m_n(w))$.

Using these notations, we set

$$|X_w|_{\tau,s} = \Pi_{i=0}^{n} (\tau_i\, m_i(w))^{\tau_i m_i(w)} s_i^{\tau_i m_i(w)} \tag{3.3}$$

$$|X_w|_{\tau_*,s_*} = \Pi_{i=1}^{n} (\tau_i\, m_i(w))^{\tau_i m_i(w)} s_i^{\tau_i m_i(w)}. \tag{3.4}$$

Definition 3.1 *For an element $T = \sum_w c_w X_w \in \mathcal{T}$, we set semi-norms:*

$$||T||_{\tau,s} = \sum_w |c_w| \cdot |X_w|_{\tau,s} \tag{3.5}$$

and

$$||T||_{\tau_*,s_*} = \sum_w |c_w| \cdot |X_w|_{\tau_*,s_*} \tag{3.6}$$

Using the second semi-norm we also set

$$||T||_{\tau_*,s_*,t} = \sum_j ||T_j||_{\tau_*,s_*} t^j \tag{3.7}$$

$$||T||_{\tau_*,s_*,N_0} = \sum_{j=0}^{N_0} ||T_j||_{\tau_*,s_*} \tag{3.8}$$

Here, T_j is the component of T which contains X_0 j times in X_w, and we may write as $T = \sum_j T_j,\ T_j \in \mathcal{T}$.

The following inequality is useful

Lemma 3.1 *Let $u_1, \dots, u_l > 0$. Then, we have*

$$u_1^{u_1} \dots u_l^{u_l} \le (u_1 + \dots + u_l)^{u_1+\dots+u_l} \le l^{u_1+\dots+u_l} u_1^{u_1} \dots u_l^{u_l}.$$

It is easy by using Lemma 3.1 $|X_w \otimes X_{w'}|_{\tau,s} \le |X_w|_{\tau,2s}\, |X_{w'}|_{\tau,2s}$ which yields

Lemma 3.2

$$||T_1 \otimes T_2||_{\tau,s} \le ||T_1||_{\tau,2s} ||T_2||_{\tau,2s} \tag{3.9}$$

$$||T_1 \otimes T_2||_{\tau_*,s_*,t} \le ||T_1||_{\tau_*,2s_*,t} ||T_2||_{\tau_*,2s_*,t} \tag{3.10}$$

$$||T_1 \otimes T_2||_{\tau_*,s_*,N_0} \le ||T_1||_{\tau_*,2s_*,N_0} ||T_2||_{\tau_*,2s_*,N_0} \tag{3.11}$$

For a fixed τ, consider the system of semi-norms $\{||\cdot||_{\tau,s}\}_s$, $\{||\cdot||_{\tau_*,s_*,t}\}_{s_*,t}$ and $\{||\cdot||_{\tau_*,s_*,N_0}\}_{s_*,N_0}$, where $s = (s_0, s_1, \cdots, s_n)$ such that $s_i > 0$ $(i = 0, \dots, n)$ and $t > 0$, $N_0 \in \mathbb{Z}_+$. By taking completions of $\mathcal{T}$ with respect to the above semi-norms, we introduce the following Fréchet spaces:

Definition 3.2 *Under the above notation, we set*

$$\mathcal{T}_\tau = \{T \in \mathcal{T} \mid ||T||_{\tau,s} < \infty \textit{ for } \forall s\} \tag{T.1}$$

$$\mathcal{T}_{Hol,\tau_*} = \{T \in \mathcal{T} \mid ||T||_{\tau_*,s_*,t} < \infty \quad \textit{for } \forall s_*,\ \forall t > 0\} \tag{T.2}$$

$$\mathcal{T}_{\infty,\tau_*} = \{T \in \mathcal{T} \mid ||T||_{\tau_*,s_*,N_0} < \infty \quad \textit{for } \forall s_*,\ \forall N_0 \in \mathbb{Z}_+\} \tag{T.3}$$

$\mathcal{T}_{Tj}$ denotes any one of (T.1)-(T.3) according to $j = 1, 2, 3$.

Lemma 3.2 gives

Proposition 3.1 *Let $\mathcal{T}_{Tj}$ be any one of (T.1)-(T.3) in Definition 3.2. Then, $(\mathcal{T}_{Tj}, \otimes)$ is a Fréchet algebra.*

3.2 Subspace of symmetric elements

We first introduce a symmetric product

$$F \circ G = \frac{1}{2}(F \otimes G + G \otimes F)$$

in $\mathcal{T}$ and set (cf. [4])

$$(F\circ)^k \cdot H = F \circ (F \circ (\cdots (F \circ H) \cdots)), \qquad (k\text{-times}) \tag{3.12}$$

$$(F\circ)^k \cdot (G\circ)^l \cdot H = (F\circ)^k \cdot \Big((G\circ)^l \cdot H \Big). \tag{3.13}$$

Using these notations, we define a linear subspace $\mathcal{S}$ of $\mathcal{T}$ as follows. Let us fix a basis $\{X_0, X_1, \ldots, X_n\}$ of V. For a multi-index $\alpha = (\alpha_0, \ldots, \alpha_n)$ we set a term X^α as

$$X_\odot^\alpha = X_0^{\alpha_0} \odot X_1^{\alpha_1} \odot \ldots \odot X_n^{\alpha_n} = (X_0\circ)^{\alpha_0} \cdots (X_n\circ)^{\alpha_n} \cdot 1. \tag{3.14}$$

Then we set a linear subspace

$$\mathcal{S} = \{F \in \mathcal{T} \mid F = \sum_\alpha c_\alpha X_\odot^\alpha\} \tag{3.15}$$

Putting a commutative product $\odot$ for monomials as

$$\begin{aligned} &((X_0\circ)^{\alpha_0} \cdots (X_n\circ)^{\alpha_n} \cdot 1) \odot ((X_0\circ)^{\beta_0} \cdots (X_n\circ)^{\beta_n} \cdot 1) \\ &= (X_0\circ)^{\alpha_0+\beta_0} \cdots (X_n\circ)^{\alpha_n+\beta_n} \cdot 1, \end{aligned} \tag{3.16}$$

we extend $\odot$ on $\mathcal{S}$. Thus, $(\mathcal{S}, \odot)$ is a commutative associative algebra. Denote by $\mathcal{S}_{Tj}$ the closure of $\mathcal{S}$ in $\mathcal{T}_{Tj}$, that is, the topological space $\mathcal{S}_{Tj}$ is the completion of $\mathcal{S}$ with respect to the system of semi-norms given in Defintion 3.2.

Lemma 3.3 *We have for a word* $X_w = X_{w_1} \otimes \cdots \otimes X_{w_k}$

$$\|(X_0\circ)^{m_0(w)} \cdot (X_1\circ)^{m_1(w)} \cdot \ldots \cdot (X_n\circ)^{m_n(w)} \cdot 1\|_{\tau,s} = |X_w|_{\tau,s} \tag{3.17}$$

Hence, by Lemma 3.3, we see

$$\|F_1 \odot F_2\|_{\tau,s} \le \|F_1\|_{\tau,2s} \|F_2\|_{\tau,2s} \tag{3.18}$$

$$\|F_1 \odot F_2\|_{\tau_*,s_*,t} \le \|F_1\|_{\tau_*,2s_*,t} \|F_2\|_{\tau_*,2s_*,t} \tag{3.19}$$

$$\|F_1 \odot F_2\|_{\tau_*,s_*,N_0} \le \|F_1\|_{\tau_*,2s_*,N_0} \|F_2\|_{\tau_*,2s_*,N_0} \tag{3.20}$$

for $F_1, F_2 \in \mathcal{S}_{Tj}$.

We show the following: For $\tau = (\tau_0, \tau_1, \cdots, \tau_n)$, $\tau_i > 0$, put wieghts $\tau^{-1} = (\tau_0^{-1}, \tau_1^{-1}, \cdots, \tau_n^{-1})$ and $\tau_*^{-1} = (\tau_1^{-1}, \cdots, \tau_n^{-1})$.

Proposition 3.2 *We have the following isomorphism as a commutative Fréchet algebra:*

(i) For $(T.1)$, *we have* $(S_\tau, \odot) = \mathcal{E}_{\tau^{-1}}(\mathbb{C}^{n+1})$.
(ii) For $(T.2)$, *we have* $(S_{Hol,\tau_*}, \odot) = \mathcal{E}_{Hol,\tau_*}(\mathbb{C}^{n+1})$.
(iii) For $(T.3)$, *we have* $(S_{\infty,\tau_*}, \odot) = (S_{\tau_*}[[x_0]], \odot) = \mathcal{E}_{\infty,\tau^{-1}}(\mathbb{C}^{n+1})$.

Proof. We give a proof for the case of 1-variable, since the multi-variable cases are direct consequences of 1-variable case. We first show the case (i). Put $Z = X_0$ and consider an element $a = \sum_{n=0}^{\infty} a_n Z^n \in S_\tau$. Then by definition, for every $s > 0$ there exists a constant $C > 0$ such that

$$\|a\|_{\tau,s} \leq C \sum_{n\geq 0} |a_n| (\tau n)^{\tau n} s^{\tau n},$$

hence $|a_0| \leq C$ and $|a_n| \leq C(\tau n)^{-\tau n} s^{-\tau n}$ for all $n = 1, 2, \ldots$.

Now we will show the power series $f = \sum a_n z^n$ defines an element of $\mathcal{E}_p(\mathbb{C})$ for $p = \tau^{-1}$. The estimate for a_n above yields

$$\sum_n |a_n||z|^n \leq C + C \sum_{n=1}^{\infty} (\tau n)^{-\tau n} \left(\frac{|z|^{\frac{1}{\tau}}}{s} \right)^{\tau n}, \quad (z \in \mathbb{C}).$$

For the domain $|z| < s^\tau$, it holds $1 + \sum_{n=1}^{\infty} (\tau n)^{-\tau n} \left(|z|^{\frac{1}{\tau}}/s \right)^{\tau n} < M$ where $M = 1 + \sum_{n=1}^{\infty} (\tau n)^{-\tau n} < \infty$. On the domain $|z| \geq s^\tau$, we devide the summation into two parts ;

$$\sum_{n=1}^{\infty} (\tau n)^{-\tau n} \left(\frac{|z|^{\frac{1}{\tau}}}{s} \right)^{\tau n} = \sum_{n=1}^{n_0 - 1} (\tau n)^{-\tau n} \left(\frac{|z|^{\frac{1}{\tau}}}{s} \right)^{\tau n} + \sum_{n=n_0}^{\infty} (\tau n)^{-\tau n} \left(\frac{|z|^{\frac{1}{\tau}}}{s} \right)^{\tau n},$$

where n_0 is a poistive integer such that $(\tau n)^{-\tau n} \leq [\tau n]^{-[\tau n]}$ for all $n \geq n_0$ (say, $n_0 > (\tau e)^{-1}$). The first part is a polynomial of $|z|$ of degree $n_0 - 1$. For the second part, we estimate as

$$\sum_{n=n_0}^{\infty} (\tau n)^{-\tau n} u^{\tau n} \leq \sum_{n=n_0}^{\infty} u^{\delta_n} [\tau n]^{-[\tau n]} u^{[\tau n]}$$

where we put $u = |z|^{\frac{1}{\tau}}/s$ and $\delta_n = \tau n - [\tau n] < 1$. Using $[\tau n]! \leq [\tau n]^{[\tau n]}$ and $u^{\delta_n} \leq u$ for $u \geq 1$, we have $\sum_{n=n_0}^{\infty} (\tau n)^{-\tau n} u^{\tau n} \leq u e^u < e^{2u}$. Thus, the second part is bounded by the function $\exp\left(\frac{2}{s}|z|^{\frac{1}{\tau}}\right)$. Since the first part is a funciton of polynomial degree, the total summation is also bounded above by the function $C' \exp\left(\frac{2}{s}|z|^{\frac{1}{\tau}}\right)$ for certain positive constant C' depending on s. Then, we see the estimate on the whole complex plane

$$\sum_n |a_n||z|^n \le CC'' \exp\left(\frac{2}{s}|z|^{\frac{1}{\tau}}\right)$$

for certain constant C'', which yields $\|f\|_{p,2s^{-1}} < C''\|a\|_{\tau,s}$. Then S_τ is continuously embedded into the space $\mathcal{E}_p(\mathbb{C})$ for $p = \tau^{-1}$.

We show the converse. Assume $f \in \mathcal{E}_p(\mathbb{C})$, i.e., for every $b > 0$, f satisfies

$$\sup_{z\in\mathbb{C}} |f(z)| \exp(-b|z|^p) < \infty. \tag{3.21}$$

Putting $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and using the Cauchy integral formula on the contur of raduis R, we have estimates $|a_0| \le C$ and

$$|a_n| \le \min_{R\ge 0} C\left(\frac{e^{bR^p}}{R^n}\right) = C\left(\frac{e\,b\,p}{n}\right)^{\frac{n}{p}}, \quad (n = 1, 2, \dots), \tag{3.22}$$

where $C = \sup_{z\in\mathbb{C}} |f(z)| \exp(-b|z|^p)$. Now, we show the power series $a = \sum_{n=0}^{\infty} a_n Z^n$ defines an element of S_τ where $\tau = p^{-1}$. For an arbitrary $s > 0$, using the estimate (3.22) we calcalulate as

$$\|a\|_{\tau,s} = \sum_{n=0}^{\infty} |a_n| \left(\frac{n}{p}\right)^{\frac{n}{p}} s^{\frac{n}{p}} \le \sum_{n=0}^{\infty} C\left\{(e\,b\,s)^{\frac{1}{p}}\right\}^n = C\frac{1}{1-(e\,b\,s)^{\frac{1}{p}}} \tag{3.23}$$

by taking b small enough, say $b < 1/(e\,s)$, which indicates $\mathcal{E}_p(\mathbb{C})$ is continuously embedded into the space S_τ for $\tau = p^{-1}$. Thus, case (i) is obtained by extending the above arguments to multi-variable functions. To show the (ii), we remind estimate of the semi-norms for X_0 in S_{τ_*} and for x_0 in $\mathcal{E}_{p_*}$is same. Thus, the above argument also yields for case the (ii). Case (iii) seems rather trivial. Remark $S_{\infty,\tau_*} = S_{\tau_*}[[x_0]]$ and $\mathcal{E}_{\infty,p_*}(\mathbb{C}^{2n+1}) = \mathcal{E}_{p_*}(\mathbb{C}^n)[[x_0]]$, and their topology are the ones of the formal power series. Using the above observation for $\mathcal{E}_{p_*}(\mathbb{C}^n)$ and $S_{\tilde{\tau}_*}$, we have case the (iii).

3.3 $*$-product on S_{Tj}

In this subsection, we work with $(2n+1)$-generators $X_0, X_1, \dots, X_{2n}$. In what follows, we assume the weights

$$\tau = (\tau_0, \tau_1, \dots, \tau_1), \; s = (s_0, s_1, \dots, s_1). \tag{3.24}$$

For convienience, we write as $(X_0, X_1, \dots, X_{2n}) = (Z, X, Y)$ where $X = (X_1, \dots, X_n)$ and $Y = (X_{n+1}, \dots, X_{2n})$. We introduce an (commutative) associative product, denoted by $*$, on $S_{Tj} \subset \mathcal{T}_{Tj}$, where $\mathcal{T}_{Tj}$ is any one of Definition 3.2.

Consider an element $F = \sum c_{k\alpha\beta} Z^k \odot X^\alpha \odot Y^\beta \in S_{Tj}$, where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$.

$$\begin{aligned} \partial_{X_i} F &= \sum c_{k\alpha\beta}\, \alpha_i Z^k \odot X^{\alpha - e_i} \odot Y^\beta \\ \partial_{Y_i} F &= \sum c_{k\alpha\beta}\, \beta_i Z^k \odot X^{\alpha} \odot Y^{\beta - e_i}. \end{aligned} \tag{3.25}$$

By a simple estiamte, we get $\partial_{X_i} F, \partial_{Y_i} F \in S_{Tj}$ and these operations are continuous. Similary, we define higher derivatives $\partial_X^{l_1} \partial_Y^{l_2} F$ as usual, and $\partial_X^{l_1} \partial_Y^{l_2} F \in S_{Tj}$. For $F_1, F_2 \in S_{Tj}$, we set

$$\{F_1, F_2\} = F_1 \left(Z \odot (\overleftarrow{\partial}_X \odot \overrightarrow{\partial}_Y - \overleftarrow{\partial}_Y \odot \overrightarrow{\partial}_X) \right) F_2. \tag{3.26}$$

Then, by Proposition 3.2, $(S_{Tj}, \odot, \{\,,\,\})$ is a Fréchet Poisson algebra which is isomorphic to $(\mathcal{E}_{Ej}(\mathbb{C}^{2n+1}), \cdot, \{\,,\,\}_H)$.

We also transform the formula given by (2.9) to S_{Tj} : for $F_1, F_2 \in S_{Tj}$, we set

$$F_1 * F_2 = \sum_{k=0}^{\infty} \frac{1}{2^k k!} F_1 \left(iZ \odot (\overleftarrow{\partial_X} \odot \overrightarrow{\partial_Y} - \overleftarrow{\partial_Y} \odot \overrightarrow{\partial_X}) \right)^k F_2 \tag{3.27}$$

which is an associative product on S_{Tj}.

Due to the identifications given in Proposition 3.2, Theorem 2.1 is induced from the following theorem.

Theorem 3.1 *Under the assumption (3.24) on τ, let Tj denote any one of $(T.1)$, $(T.2)$ or $(T.3)$ and consider Fréchet Poisson algebra $(S_{Tj}, \odot, \{\,,\,\})$.*

Assume

$$0 < \tau_0 \leq 2\tau_1 - 1 \quad for \quad (T.1). \tag{A'.1}$$

$$\tfrac{1}{2} \leq \tau_1 \quad for \quad (T.2). \tag{A'.2}$$

$$0 < \tau_1 \quad for \quad (T.3). \tag{A'.3}$$

Then, $(S_{Tj}, *)$ is a Z-central, Z-regulated Fréchet algebra according to $j = 1, 2, 3$, respectively. Further, it is analytic for (A'.1) and (A'.2), and formal for (A'.3).

4 Convergence of the product

In this section, we show the sufficiency part in Theorem 2.1.

4.1 Case $\mathcal{T}_\tau$

Let $\mathcal{T}_\tau$ and S_τ be as in §3. To show Theorem 2.1, we consider the following product on S^3_τ :

$$F_1 * F_2 = F_1 \exp \frac{iZ}{2} \odot \left(\overleftarrow{\partial X} \odot \overrightarrow{\partial Y} - \overleftarrow{\partial Y} \odot \overrightarrow{\partial X} \right) F_2 \tag{4.1}$$

for

$$F_1 = \sum a_{k_1 m_1 n_1} (Z\circ)^{k_1} (X\circ)^{m_1} (Y\circ)^{n_1} \cdot 1,$$
$$F_2 = \sum b_{k_2 m_2 n_2} (Z\circ)^{k_2} (X\circ)^{m_1} (Y\circ)^{n_2} \cdot 1 \in S_\tau.$$

In this section, we show the following:

Theorem 4.1 *Assume $\tau = (\tau_0, \tau_1, \dots, \tau_1), 0 < \tau_0 \leq 2\tau_1 - 1$. Then, $(S_\tau, *)$ is a Z-central, Z-regulated analytic Fréchet algebra.*

Proof. Let F_1, F_2 be as in (4.1). Computing

$$\begin{aligned} F_1 * F_2 \quad = \quad & \sum \frac{\sqrt{-1}^p}{2^p p!} \sum_{|i|+|j|=p} a_{k_1 m_1 n_1} b_{k_2 m_2 n_2} (-1)^{|j|} \\ & \times \frac{p!}{|i|!|j|!} \frac{m_1!}{(m_1 - i)!} \frac{n_1!}{(n_1 - j)!} \frac{m_2!}{(m_2 - j)!} \frac{n_2!}{(n_2 - i)!} \\ & \times (Z\circ)^{p+k_1+k_2} (X\circ)^{m_1+m_2-p} (Y\circ)^{n_1+n_2-p}. \end{aligned} \tag{4.2}$$

By using the definition of semi norms and inequality $\frac{m!}{(m-k)!} \leq m^k$, we have the following estimate:

$$\begin{aligned} ||F_1 * F_2||_{\tau,s} \leq & \sum |a_{k_1 m_1 n_1}| |b_{k_2 m_2 n_2}| \cdot \sum_{p \geq 0} \frac{1}{2^p p!} \sum_{i+j=p} \frac{p!}{i!j!} m_1^i n_1^j m_2^j n_2^i \\ & \times ||(Z\circ)^{p+k_1+k_2} (X\circ)^{m_1+m_2-p} (Y\circ)^{n_1+n_2-p}||_{\tau,s}. \end{aligned}$$

Remarking $m_1^i m_2^j n_1^i n_2^j \leq (|m_1| + |m_2| + |n_1| + |n_2|)^{2p}$, and using the inequality of Lemma 3.1, we have

$$\begin{aligned} ||F_1 * F_2||_{\tau,s} \leq & \sum |a_{k_1 m_1 n_1}| |b_{k_2 m_2 n_2}| \cdot \sum_{p=0} \frac{1}{p!} (|m_1| + |n_1| + |m_2| + |n_2|)^{2p} \\ & \times N_p^{N_p} s_0^{\tau_0(p+k_1+k_2)} s_1^{\tau_1(|m_1|+|m_2|+|n_1|+|n_2|-2p)}, \end{aligned} \tag{4.3}$$

where

$$N_p = \tau_0(p + k_1 + k_2) + \tau_1(|m_1| + |m_2| + |n_1| + |n_2| - 2p).$$

We have the inequality

$$N_p^{N_p} \le 2^{N_p} N_1^{N_1} N_2^{N_2}$$

where

$$N_1 = \tau_0(k_1 + k_2), \quad N_2 = (\tau_0 - 2\tau_1)p + \tau_1(|m_1| + |m_2| + |n_1| + |n_2|).$$

Using the assumption $\tau_0 \le 2\tau_1 - 1$, we have

$$N_2^{N_2} \le (\tau_1(|m_1| + |m_2| + |n_1| + |n_2|))^{N_2}.$$

Hence we see

$$(|m_1| + |n_1| + |m_2| + |n_2|)^p N_p^{N_p} \le 2^{N_p} N_1^{N_1} \tau_1^{\tau_1 M} M^{(\tau_0 - 2\tau_1 + 1)p + \tau_1 M}$$

where

$$M = |m_1| + |n_1| + |m_2| + |n_2|.$$

Using the assumption $\tau_0 \le 2\tau_1 - 1$ again, we have

$$\begin{aligned}&\sum_{p=0} \frac{1}{p!}(|m_1| + |n_1| + |m_2| + |n_2|)^{2p} \\ &\qquad \times N_p^{N_p} s_0^{\tau_0(p+k_1+k_2)} s_1^{\tau_1(|m_1|+|m_2|+|n_1|+|n_2|-2p)} \\ &\qquad \le \sum_{p=0} \frac{1}{p!}(M s_0^{\tau_0} s_1^{-2\tau_1})^p N_1^{N_1} (\tau_1 M)^{\tau_1 M} (2s_0)^{N_1} (2s_1)^{\tau_1 M}\end{aligned}$$

Thus, we have

$$\begin{aligned}||F_1 * F_2||_{\tau,s} &\le \sum |a_{k_1 m_1 n_1}||b_{k_2 m_2 n_2}| \cdot \exp[(m_1 + n_1 + m_2 + n_2) s_0^{\tau_0} s_1^{-2\tau_1}] \\ &\quad (\tau_0(k_1 + k_2))^{\tau_0(k_1+k_2)} (\tau_1(m_1 + m_2 + n_1 + n_2))^{\tau_1(m_1+m_2+n_1+n_2))} \\ &\quad \times (2s_0)^{\tau_0(k_1+k_2)} (2s_1)^{\tau_1(m_1+m_2+n_1+n_2)} \\ &\le \sum |a_{k_1 m_1 n_1}||b_{k_2 m_2 n_2}| \\ &\quad \times (\tau_0(k_1 + k_2))^{\tau_0(k_1+k_2)} (\tau_1(m_1 + m_2)^{\tau_1(m_1+m_2)}) (\tau_1(n_1 + n_2))^{\tau_1(n_1+n_2))} \\ &\quad \times (2s_0)^{\tau_0(k_1+k_2)} (4e^{s_1^{-2\tau_1} s_0^{\tau_0} \tau_1^{-1}} s_1)^{\tau_1(m_1+m_2+n_1+n_2)}.\end{aligned}$$

By the definition of semi-norms, we remind the following identity:

$$\begin{aligned}&||F_1 \odot F_2||_{\tau,\sigma} \qquad (4.4) \\ = &\sum |a_{k_1 m_1 n_1}||b_{k_2 m_2 n_2}| (\tau_0(k_1 + k_2))^{\tau_0(k_1+k_2)} (\tau_1(m_1 + m_2))^{\tau_1(m_1+m_2)} \\ &\qquad \times (\tau_1(n_1 + n_2))^{\tau_1(n_1+n_2)} \sigma_0^{\tau_0(k_1+k_2)} \sigma_1^{\tau_1(m_1+m_2+n_1+n_2)}\end{aligned}$$

Therefore, we have

$$||F_1 * F_2||_{\tau,s} \leq ||F_1 \odot F_2||_{\tau,\sigma}, \tag{4.5}$$

where $\sigma = (2s_0, 4e^{s_1^{-2\tau_1} s_0^{\tau_0} \tau_1^{-1}} s_1)$.

The properties (i)–(vi) in Theorem 2.1 are easily obtained.

4.2 Case $\mathcal{T}_{Hol,\tau_*}$, $\tau_1 \geq \frac{1}{2}$

Theorem 4.2 *Assume* $\tau_* = (\tau_0, \tau_1, \ldots, \tau_1)$, $\tau_1 \geq \frac{1}{2}$. *Then,* $(S_{Hol,\tau_*}, *)$ *is a Z-central, Z-regulated Fréchet analytic algebra.*

Proof. By following the computations as in §4.1, we see Theorem 4.2. In particular, we put $\tau_0 = 0$ in (4.3). Then, the same computation gives the following estimates:

$$||F_1 * F_2||_{\tau_*,s_*,t} \leq ||F_1 \odot F_2||_{\tau_*,s_{1*}} \tag{4.6}$$

where $s_{1*} = (\exp \tau_1 s_1^{-2a}) s_1$.

4.3 Case $\mathcal{T}_{\infty,\tau_*}$, $\tau_* > 0$

By the definition of (3.27), the product $*$ is well-defined for any $F_1, F_2 \in \mathcal{T}_{\infty,\tau_*}$. Then, we have

Theorem 4.3 *Assume* $\tau_* = (\tau_1, \ldots, \tau_1)$, $\tau_1 > 0$. *Then,* $(S_{\infty,\tau_*}, *)$ *is a Z-central, Z-regulated Fréchet analytic algebra.*

4.4 Remarks on the star product.

We remark the assumption in Theorem 4.1 is the best possible in the following sense, which give the necessity part in Theorem 2.1.

Proposition 4.1 *Assume* $\tau = (\tau_0, \tau_1, \ldots, \tau_1)$ *with* $\tau_0 > 0, \tau_1 > 0, \tau_0 > 2\tau_1 - 1$. *Then,* $*$ *does not give a Fréchet algebra structure on* S_τ.

Proof. We show the statement for the case of 3 generators Z, X, Y for simlicity, which implies the general cases.

Let $U(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^{\alpha n}}$ $(\alpha > 0)$. We set

$$U_\odot(Z,X) = \sum_{n=1}^{\infty} \frac{(Z\odot)^n (X\odot)^n}{n^{\alpha n}}, \quad U_\odot(Z,Y) = \sum_{n=1}^{\infty} \frac{(Z\odot)^n (Y\odot)^n}{n^{\alpha n}}. \tag{4.7}$$

If $\alpha > \tau_0 + \tau_1$, then $U_\odot(Z,X), U_\odot(Z,Y) \in S_\tau$. In fact, the semi-norm of $U_\odot(Z,X)$ is

$$||U_\odot(Z,X)||_{\tau,s} = \sum_{n=1}^{\infty} \frac{(\tau_0 n)^{(\tau_0 n)}(\tau_1 n)^{(\tau_1 n)} s_0^{(\tau_0 n)} s_1^{(\tau_1 n)}}{n^{\alpha n}} = \sum_{k=1}^{\infty} \left(\frac{(\tau_0 s_0)^{\tau_0}(\tau_1 s_1)^{\tau_1}}{n^{\{\alpha-(\tau_0+\tau_1)\}}} \right)^n$$

and it is obviously convergent.

Compute the product

$$\begin{aligned} &U_\odot(Z,X) * U_\odot(Z,Y) \qquad (4.8) \\ &= \sum \frac{1}{n^{\alpha n}} \frac{1}{m^{\alpha m}} \cdot \sum_{p=0}^{min(n,m)} \frac{i^p}{2^p p!} \frac{n!}{(n-p)!} \frac{m!}{(m-p)!} \\ &\qquad (Z\odot)^{n+m+p}(X\odot)^{n-p}(Y\odot)^{m-p} \end{aligned}$$

Then, we get

$$||U_\odot(Z,X) * U_\odot(Z,Y)||_{\tau,s} \geq \sum \frac{1}{l^{2\alpha l}} \cdot \frac{1}{2^l l!} (l!)^2 (3\tau_0 l)^{3\tau_0 l} s_0^{3\tau_0 l} \qquad (4.9)$$

by taking the terms in $||U_\odot(Z,X) * U_\odot(Z,Y)||_{\tau,s}$ for $n = m = l$ and $p = l$. We put the coefficients of $\left(s_0^{3\tau_0}\right)^l$ as a_l in the right hand side of (4.9). Note that the rate a_l/a_{l+1} is equal to

$$\begin{aligned} &\frac{l!(3\tau_0 l)^{3\tau_0 l}(l+1)^{2\alpha(l+1)}2^{l+1}}{(l^{2\alpha l}2^l)(l+1)!(3\tau_0(l+1))^{3\tau_0(l+1)}} \\ = \;& 2\frac{1}{3\tau_0^{3\tau_0}} \cdot (1+1/l)^{(2\alpha-3\tau_0)l} \cdot (l+1)^{2\alpha-3\tau_0-1} \qquad (4.10) \\ \to \;& 0 \quad \text{as } l \to \infty \end{aligned}$$

if we choose $\alpha = \tau_0 + \tau_1 + \varepsilon$, for a sufficiently small $\varepsilon > 0$ because $2\alpha - 3\tau_0 - 1 = -\tau_0 + 2\tau_1 - 1 + 2\epsilon < 0$. Thus, we have (4.8) diverges for any s. As to the case (Hol, τ_*), similar computation gives the following :

Corollary 4.1 *Assume $\tau_1 < \frac{1}{2}$. Then $a * b$ diverges for some elements in S_{Hol,τ_*}.*

If $\alpha > \tau_1$, then $U_\odot(X), U_\odot(Y) \in S_{\mathrm{Hol},\tau_*}$. In fact, the semi-norm of $U_\odot(X)$ is

$$||U_\odot(X)||_{\tau_*,s_*,t} = \sum_{k=0}^{\infty} \frac{(\tau_1 n)^{(\tau_1 n)} s_1^{(\tau_1 n)}}{n^{\alpha n}} = \sum_{n=0}^{\infty} \left(\frac{(\tau_1 s_1)^{\tau_1}}{n^{\{\alpha-\tau_1\}}} \right)^n$$

and it is obviously convergent. For $U_{\odot}(X), U_{\odot}(Y)$ by taking the terms of $n = m = l$ in $||U_{\odot}(X) * U_{\odot}(Y)||_{\tau_*,s_*,t}$, we see

$$||U_{\odot}(X) * U_{\odot}(Y)||_{\tau_*,s_*,t} \geq \sum \frac{1}{l^{2\alpha l}} \cdot \frac{1}{2^l l!}(l!)^2 t^l \tag{4.11}$$

We put the coefficients of t^l as b_l in the right hand side of (4.11). Note that the rate b_l/b_{l+1} is equal to

$$\frac{l!(l+1)^{2\alpha(l+1)}2^{l+1}}{(l^{2\alpha l}2^l)(l+1)!} = 2 \cdot (1+1/l)^{2\alpha l} \cdot (l+1)^{2\alpha-1} \to 0 \quad \text{as } l \to \infty$$

if we choose $\alpha = \tau_1 + \varepsilon$, for a sufficiently small $\varepsilon > 0$ because $2\alpha - 1 = 2\tau_1 - 1 + 2\epsilon < 0$. Thus, we have $||U_{\odot}(X) * U_{\odot}(Y)||_{\tau_*,s_*,t}$ diverges for any s and $t > 0$.

Note that if $\mathcal{T}_{T_j} = \mathcal{T}_{(\infty,\tau_*)}, \tau_1 > 0$, there is no complementary case. Hence, the argument in this section gives the "only if" part of Theorem 2.1.

4.5 Quotient of $\mathcal{T}_{Tj}$

As defined in §3, let $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ be the free tensor algebras with generators $X_0 = Z, X_1 = X, X_2 = Y$ and with $X_1 = X, X_2 = Y$, respectively. Let $\mathcal{I}$ be the two sided ideal of relations in $\mathcal{T}$ generated by

$$X \otimes Z - Z \otimes X, \;\; Y \otimes Z - Z \otimes Y, \text{ and } X \otimes Y - Y \otimes X - iZ.$$

Denote by $\mathcal{I}_{Tj}$ the closure of $\mathcal{I}$ in $\mathcal{T}_{Tj}$.

In spite of Proposition 4.1 and Corollary 4.1, we see that $\mathcal{A}_{Tj} = \mathcal{T}_{Tj}/\mathcal{I}_{Tj}$ is a Fréchet algebra We denote the induced product from $\otimes$ by $\hat{*}$.

First, we observe the algebra structure of $(\mathcal{A}_{Tj}, \hat{*})$. We remark the following: Let $\widetilde{\mathcal{S}}_{\tau_*}$ be the completion of $\widetilde{\mathcal{S}}$ in $\widetilde{\mathcal{T}}$ with respect to the family of semi-norms $\{||\cdot||_{\tau_*,s_*}\}$.

Theorem 4.4 *For $j = 1, 2, 3$, $(\mathcal{A}_{Tj}, \hat{*})$ is a Z-central, Z-regulated, analytic Fréchet algebra such that*

$$\mathcal{A}_{Tj} = \widetilde{\mathcal{S}}_{\tau_*} \oplus Z \hat{*} \mathcal{A}_{Tj}. \tag{4.12}$$

Proof. Let $T = \sum t_\alpha X_\alpha \in \mathcal{T}_{Tj}$. We remark for every X_α the following:
(i) If X_α does not contain Z, then X_α can be viewed as

$$X_\alpha = S_\alpha + P_\alpha, \text{ where } S_\alpha \in \widetilde{\mathcal{S}}_{\tau_*},\ P_\alpha \in Z \otimes \mathcal{T}_{Tj} + \mathcal{I}_{Tj}, \tag{4.13}$$

and moreover the semi-norms of X_α and S_α are equal.
(ii) If X_α contains Z, then X_α can be viewed as

$$X_\alpha = P_\alpha, \text{ where } P_\alpha \in Z \otimes \mathcal{T}_{Tj} + \mathcal{I}_{Tj}. \tag{4.14}$$

Repeat this computation for X_α. Then, T is written as

$$T = \sum t_\alpha S_\alpha + Z \otimes Q + R \tag{4.15}$$

where $Q \in \mathcal{T}_{Tj}, R \in \mathcal{I}_{Tj}$. Thus we have

$$\mathcal{T}_{Tj}/(Z \otimes \mathcal{T}_{Tj} + \mathcal{I}_{Tj}) \cong \widetilde{\mathcal{S}}_{\tau_*}, \tag{4.16}$$

which yields (iii). The other properties in Theorem 4.4 are obvious.

4.6 Properties for $\mathcal{A}_\tau$

We study algebraic properties on $\mathcal{A}_\tau = \mathcal{T}_\tau/\mathcal{I}_\tau$ where $\tau = (\tau_0, \tau_1, \dots, \tau_1)$. We denote by $\hat{*}$ the induced product from the closure of free tensor algebra $\mathcal{T}_\tau$. We first show the following:

Theorem 4.5 *Assume for* $\tau = (\tau_0, \tau_1, \dots, \tau_1)$ *and* $0 < \tau_0 \leq 2\tau_1 - 1$. *Then, we have*

$$\mathcal{T}_\tau = \mathcal{S}_\tau \oplus \mathcal{I}_\tau \quad (\textit{direct sum}) \tag{4.17}$$

Moreover $(\mathcal{S}_\tau, *)$ *is isomorphic to* $(\mathcal{A}_\tau, \hat{*})$.

Proof. Let ψ be an algebra homomorphism from $(\mathcal{T}, \otimes)$ to $(\mathcal{S}, *)$ defined by

$$\psi(X_{\alpha_1} \otimes \cdots \otimes X_{\alpha_n}) = X_{\alpha_1} * \cdots * X_{\alpha_n} \tag{4.18}$$

where the product $*$ is given by (3.27). We now show that ψ extend continuously to the map from $(\mathcal{T}_\tau, \otimes)$ to $(\mathcal{S}_\tau, *)$. Let Y^k and X^k denote $Y * \cdots * Y$ and $X * \cdots * X$. We first note that

$$Y^m * X^n = \sum_{k=0}^{m} \binom{m}{k} ad(Y)_*^k(X)^n * Y^{m-k}, \tag{4.19}$$

where $ad(Y)_*(X)^n = [Y, X^n]_*$. Using $ad(Y)_*(X)^n = -niZ * X^{n-1}$, we have

$$Y^m * X^n = \sum_{l=0}^{min\{m,n\}} (-i)^l \frac{m!}{l!(m-l)!} \frac{n!}{(n-l)!} Z^l * X^{n-l} * Y^{m-l}. \quad (4.20)$$

Let $\alpha = (\alpha_1, \cdots \alpha_n)$, $\beta = (\beta_1, \cdots, \beta_n)$ be n-tuples of nonnegative integers. By (4.20), we have

$$\begin{aligned} & Y^{\alpha_1} * X^{\beta_1} * Y^{\alpha_2} * X^{\beta_2} * \cdots * Y^{\alpha_n} * X^{\beta_n} \qquad (4.21) \\ = & \sum_{k=(k_1,\cdots,k_n)} (-i)^{k_1+\cdots+k_n} \frac{\alpha_1!}{k_1!(\alpha_1-k_1)!} \frac{\beta_1!}{(\beta_1-k_1)!} \times \cdots \\ & \cdots \times \frac{(\alpha_1+\cdots+\alpha_n-(k_1+\cdots+k_{n-1}))!}{k_n!(\alpha_1+\cdots+\alpha_n-(k_1+\cdots+k_n))!} \frac{\beta_n!}{(\beta_n-k_n)!} \\ & \times Z^{|k|} * X^{|\beta|-|k|} * Y^{|\alpha|-|k|}, \end{aligned}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $|\beta| = \beta_1 + \cdots + \beta_n$ and $|k| = k_1 + \cdots + k_n$.

Note that $\binom{a-n}{b} \le \binom{a}{b}$. Using

$$\begin{aligned} & \binom{\alpha_1+\alpha_2-k_1}{\alpha_2} \cdots \binom{\alpha_1+\cdots+\alpha_n-(k_1+\cdots+k_{n-1})}{\alpha_n} \\ & \le \frac{(\alpha_1+\cdots+\alpha_n)!}{\alpha_1!\cdots\alpha_n!}, \end{aligned}$$

we have

$$\begin{aligned} & \frac{\alpha_1!}{k_1!(\alpha_1-k_1)!} \frac{\beta_1!}{(\beta_1-k_1)!} \times \cdots \\ & \cdots \times \frac{(\alpha_1+\cdots+\alpha_n-(k_1+\cdots+k_{n-1}))!}{k_n!(\alpha_1+\cdots+\alpha_n-(k_1+\cdots+k_n))!} \frac{\beta_n!}{(\beta_n-k_n)!} \\ = & \frac{1}{k_1!\cdots k_n!} \cdot \frac{\alpha_1!\cdots\alpha_n!}{(|\alpha|-|k|)!} \frac{\beta_1!}{(\beta_1-k_1)!} \cdots \frac{\beta_n!}{(\beta_n-k_n)!} \\ & \times \binom{\alpha_1+\alpha_2-k_1}{\alpha_2} \cdots \binom{\alpha_1+\cdots+\alpha_n-(k_1+\cdots+k_{n-1})}{\alpha_n} \\ \le & \frac{1}{k_1!\cdots k_n!} \frac{|\alpha|!}{(|\alpha|-|k|)!} \frac{\beta_1!}{(\beta_1-k_1)!} \cdots \frac{\beta_n!}{(\beta_n-k_n)!}. \end{aligned}$$

Thus, we have

$$\begin{aligned} & ||Y^{\alpha_1} * X^{\beta_1} * Y^{\alpha_2} * X^{\beta_2} * \cdots * Y^{\alpha_n} * X^{\beta_n}||_{\tau,s} \\ \le & \sum_{k=(k_1,\cdots,k_n)} \frac{1}{k_1!\cdots k_n!} \frac{|\alpha|!}{(|\alpha|-|k|)!} \frac{\beta_1!}{(\beta_1-k_1)!} \cdots \frac{\beta_n!}{(\beta_n-k_n)!} \\ & ||Z^{|k|} * X^{|\beta|-|k|} * Y^{|\alpha|-|k|}||_{\tau,s}, \end{aligned}$$

Using the estimate for the semi-norms in Theorem 4.1, we have

$$\begin{aligned}
&||Y^{\alpha_1} * X^{\beta_1} * Y^{\alpha_2} * X^{\beta_2} * \cdots * Y^{\alpha_n} * X^{\beta_n}||_{\tau,s} \qquad (4.22)\\
\leq & \sum_{k=(k_1,\cdots,k_n)} \frac{1}{k_1!\cdots k_n!} \frac{|\alpha|!}{(|\alpha|-|k|)!} \frac{\beta_1!}{(\beta_1-k_1)!} \cdots \frac{\beta_n!}{(\beta_n-k_n)!}\\
&\times(\tau_0|k|)^{\tau_0|k|} \cdot (\tau_1(|\alpha|+|\beta|-2|k|))^{\tau_1(|\alpha|+|\beta|-2|k|)}\\
&(s_0')^{\tau_0|k|}(s_1')^{\tau_1(|\alpha|+|\beta|-2k)}
\end{aligned}$$

for some $s' = s'(\tau_0, \tau_1, s_0, s_1)$. Since $\frac{|\alpha|!}{(|\alpha|-|k|)!} \leq |\alpha|^{|k|}$, we have

$$\begin{aligned}
&||Y^{\alpha_1} * X^{\beta_1} * Y^{\alpha_2} * X^{\beta_2} * \cdots * Y^{\alpha_n} * X^{\beta_n}||_{\tau,s}\\
\leq & \sum_{k=(k_1,\cdots,k_n)} \binom{\beta_1}{k_1} \cdots \binom{\beta_n}{k_n} |\alpha|^{|k|}\\
&\times(\tau_0|k|)^{\tau_0|k|} \cdot (\tau_1(|\alpha|+|\beta|-2|k|))^{\tau_1(|\alpha|+|\beta|-2|k|)}\\
&(s_0')^{\tau_0|k|}(s_1')^{\tau_1(|\alpha|+|\beta|-2k)}
\end{aligned}$$

Notice

$$\begin{aligned}
&|k|^{\tau_0|k|}(|\alpha|+|\beta|-2|k|)^{\tau_1(|\alpha|+|\beta|-2|k|)}\\
&\leq (|\alpha|+|\beta|-|k|)^{\tau_0|k|+\tau_1(|\alpha|+|\beta|-2|k|)}\\
&\leq (|\alpha|+|\beta|)^{(\tau_0-2\tau_1)|k|+\tau_1(|\alpha|+|\beta|)}
\end{aligned}$$

Then using $\tau_0 \leq 2\tau_1 - 1$, we have

$$|\alpha|^{|k|}|k|^{\tau_0|k|}(|\alpha|+|\beta|-2|k|)^{\tau_1(|\alpha|+|\beta|-2|k|)} \leq (|\alpha|+|\beta|)^{\tau_1(|\alpha|+|\beta|)}.$$

Thus, we have

$$\begin{aligned}
&||Y^{\alpha_1} * X^{\beta_1} * Y^{\alpha_2} * X^{\beta_2} * \cdots * Y^{\alpha_n} * X^{\beta_n}||_{\tau,s}\\
\leq & \sum_{k=(k_1,\cdots,k_n)} \binom{\beta_1}{k_1} \cdots \binom{\beta_n}{k_n} (\tau_0)^{\tau_0|k|}(s_0')^{\tau_0|k|}(s_1')^{-2\tau_1|k|}\\
&\times(\tau_1(|\alpha|+|\beta|))^{\tau_1(|\alpha|+|\beta|)}(s_1')^{\tau_1(|\alpha|+|\beta|)}\\
&\leq (1+\tau_0^{\tau_0}(s_0')^{\tau_0}(s_1')^{-2\tau_1})^{|\beta|}(\tau_1(|\alpha|+|\beta|))^{\tau_1(|\alpha|+|\beta|)}(s_1')^{\tau_1(|\alpha|+|\beta|)}
\end{aligned}$$

Hence we have

$$\begin{aligned}
&||Y^{\alpha_1} * X^{\beta_1} * Y^{\alpha_2} * X^{\beta_2} * \cdots * Y^{\alpha_n} * X^{\beta_n}||_{\tau,s} \qquad (4.23)\\
&\leq C^{\tau_1(|\alpha|+|\beta|)}(\tau_1(|\alpha|+|\beta|))^{\tau_1(|\alpha|+|\beta|)}
\end{aligned}$$

for some constant C.

Thus, if

$$F = \sum a_{\delta_0\alpha\beta} Z^{\delta_0} \otimes Y^{\alpha_1} \otimes X^{\beta_1} \otimes \cdots \otimes Y^{\alpha_n} \otimes X^{\beta_n}$$

satisfies $||F||_{\tau,s} < \infty$, then for

$$\psi(F) = \sum a_{\delta_0\alpha\beta} Z^{\delta_0} * Y^{\alpha_1} * X^{\beta_1} * \cdots * Y^{\alpha_n} * X^{\beta_n}$$

we have

$$||\psi(F)||_{\tau,s} \leq ||F||_{\tau,\hat{s}} \tag{4.24}$$

for some $\hat{s} = \hat{s}(\tau_0, \tau_1, s_0, s_1)$. Thus, ψ is extended to a continuous map from $\mathcal{T}_\tau$ to $\mathcal{S}_\tau$.

Recall how $\odot$-product is defined in §3.2. We can define $\hat{\odot}$-product in $\mathcal{A}_\tau$ by using $\hat{*}$-product instead of $\otimes$ by a similar manner as $\odot$. Since $a \circ (b \circ c) - (a \circ b) \circ c = \frac{1}{4}[b,[a,c]]$ where $a \circ b = \frac{1}{2}(a \hat{*} b + b \hat{*} a)$, we see that if a, b, c are generators then $X_i \hat{\odot} Y_j = Y_j \hat{\odot} X_i$. Hence $\hat{\odot}$-product is a commutative product (see [4]) without any artificial definition. By this we see, the replacement of $\otimes$ by $*$ gives the identity on $\mathcal{S}_\tau \subset \mathcal{T}_\tau$ and it follows that $\psi|\mathcal{S}_\tau$ is the identity. Since the kernel of ψ contains $\mathcal{I}_\tau$, ψ induces a homomophism $\tilde{\psi} : \mathcal{A}_\tau \to \mathcal{S}_\tau$ which is onto by the above argument. It is easy to see $\tilde{\psi}(Z^k \hat{\odot} X^\alpha \hat{\odot} Y^\beta) = Z^k \odot X^\alpha \odot Y^\beta$. Then we see $\mathcal{A}_\tau$ and $\mathcal{S}_\tau$ are linearly isomorphic which shows Ker $\tilde{\psi} = \mathcal{I}_\tau$. Thus, we obtain Theorem4.5.

As to $\mathcal{T}_{Hol,\tau_*}$, by the same procedure as above, we have the following:

Theorem 4.6 *Assume $\frac{1}{2} \leq \tau_1$. Then we have*

$$\mathcal{T}_{Hol,\tau_*} = \mathcal{S}_{Hol,\tau_*} \oplus \mathcal{I}_{Hol,\tau_*} \qquad \textit{(direct sum)}. \tag{4.25}$$

Moreover, $(\mathcal{S}_{Hol,\tau}, *)$ is isomorphic to $(\mathcal{A}_{Hol,\tau}, *_\tau)$.

Reminding that $\mathcal{S}_{\infty,\tau_*}$ coincides with $\widetilde{\mathcal{S}}_{\tau_*}[[Z]]$, we have easily

Theorem 4.7 *Let $\tau_1 > 0$. Then, we have*

$$\mathcal{T}_{\infty,\tau_*} = \widetilde{\mathcal{S}}_{\tau_*}[[Z]] \oplus \mathcal{I}_{\infty,\tau_*} \ \textit{(direct sum)}\ . \tag{4.26}$$

Moreover, $(\mathcal{A}_{\infty,\tau_*}, *)$ is isomorphic to $(\widetilde{\mathcal{S}}_{\tau_*}[[Z]], *)$.

5 Degeneration of algebraic structure

In §4, it is shown that, $(\mathcal{S}_{Tj}, *)$ is a Fréchet algebra which is isomorphic to $(\mathcal{A}_{Tj}, \hat{*})$ under certain assumption on the weights τ and τ_*. In this section, we study algebraic structure of $(\mathcal{A}_{Tj}, \hat{*})$ for the cases where τ_0, τ_1 do not necessarily satisfy the condition in Theorems 4.5 and 4.6.

We study the Fréchet algebra $(\mathcal{A}_\tau, \hat{*})$ for $\tau = (\tau_0, \tau_1, \dots, \tau_1)$, satisfying $\tau_0 > 2\tau_1 - 1, \tau_0, \tau_1 > 0$. If further $\tau_1 > \frac{1}{2}$, then we see easily that

$$\mathcal{A}_{\tau_0', \tau_1} \supset \mathcal{A}_{\tau_0, \tau_1} \supset \mathcal{A}_{\tau_0, \tau_1'} \text{ where } \tau_0' = 2\tau_1 - 1, \tau_1' = \tfrac{1}{2}(\tau_0 + 1)$$

where we write $\mathcal{A}_{\tau_0,\tau_1} = \mathcal{A}_{(\tau_0,\tau_1,\dots,\tau_1)}$. By Theorem 4.5 we have $\mathcal{A}_{\tau_0'\tau_1} \cong \mathcal{S}_{\tau_0',\tau_1}$ and $\mathcal{A}_{\tau_0,\tau_1'} \cong \mathcal{S}_{\tau_0,\tau_1'}$, but Proposition 4.1 shows that $\mathcal{A}_{\tau_0,\tau_1} \not\cong \mathcal{S}_{\tau_0,\tau_1}$, since $\mathcal{S}_{\tau_0,\tau_1}$ is not closed in $*$. It follows that $\mathcal{T}_{\tau_0,\tau_1} \neq \mathcal{S}_{\tau_0,\tau_1} \oplus \mathcal{I}^3_{\tau_0,\tau_1}$.

Next, we consider the case $\frac{1}{2} > \tau_1 > 0$. In this case, the algebra $\mathcal{A}_{\tau_0,\tau_1}$ collapses to an almost formal algebra in Z.

Theorem 5.1 *Assume* $\tau = (\tau_0, \tau_1, \dots, \tau_1), \tau_0 \geq 0, \frac{1}{2} > \tau_1 > 0$. *(If* $\tau_0 = 0$*, then we read this as Hol or* ∞*.) Then, for any complex number such that* $a \neq 0$*, there exist* $R_a \in \mathcal{I}_\tau$ *and* $H_a \in \mathcal{T}_\tau$ *satisfying*

$$1 = (a - Z) \otimes H_a + R_a.$$

Proof. Set $X^\bullet = \frac{1}{X} \odot (1 - e_\odot^{\frac{2i}{a} X \odot Y})$, $X^\circ = \frac{1}{X} \odot (1 - e_\odot^{\frac{-2i}{a} X \odot Y})$ where $e_\odot^{tX\odot Y} = \sum_{l=0}^\infty \frac{t^l}{l!}(X \odot Y)^l$, and $\frac{1}{X}$ takes the factorization by X for the power series for $(1 - e_\odot^{\frac{2i}{a} X \odot Y})$. Computing the following identity

$$(X^\bullet \otimes X) \otimes X^\circ - X^\bullet \otimes (X \otimes X^\circ) = 0. \tag{5.27}$$

Since the computations modulo $\mathcal{I}_\tau$ is the Moyal product, the product formula gives

$$X^\circ - X^\bullet + (1 - \frac{Z}{a}) \otimes (e_\odot^{\frac{2i}{a} X \odot Y} \otimes X^\circ - X^\bullet \otimes e_\odot^{\frac{-2i}{a} X \odot Y}) \in \mathcal{I}_\tau \tag{5.28}$$

Hence

$$\frac{1}{X} \odot (e_\odot^{\frac{2i}{a} X \odot Y} - e_\odot^{\frac{-2i}{a} X \odot Y}) \in \overline{\langle Z - a \rangle + \mathcal{I}_\tau}$$

where $\overline{\langle Z - a \rangle + \mathcal{I}_\tau}$ is a closure of the two sided ideal generated by $Z - a$ in $\mathcal{T}_\tau$. Thus, we have

$$\begin{aligned} X \otimes (\frac{1}{X} \odot (e_\odot^{\frac{2i}{a} X \odot Y} - e_\odot^{\frac{-2i}{a} X \odot Y})) \\ = (e_\odot^{\frac{2i}{a} X \odot Y} - e_\odot^{\frac{-2i}{a} X \odot Y}) + Z \cdot \frac{2}{a} \otimes (e_\odot^{\frac{2i}{a} X \odot Y} + e_\odot^{\frac{-2i}{a} X \odot Y}) \\ \in \overline{\langle Z - a \rangle + \mathcal{I}_\tau} \end{aligned}$$

Thus we have

$$e_{\odot}^{\frac{2i}{a}X\odot Y} - e_{\odot}^{\frac{-2i}{a}X\odot Y} + 2(e_{\odot}^{\frac{2i}{a}X\odot Y} + e_{\odot}^{\frac{-2i}{a}X\odot Y}) \in \overline{\langle Z-a\rangle + \mathcal{I}_{\mathcal{T}}}.$$

We can write the same equality by replacing a by the complex conjugate $\bar{a}$. Reminding that our system is stable under the complex conjugation. Thus, using that the conjugate mapping is an involutive anti-automorphism, we get

$$e_{\odot}^{\pm\frac{2i}{a}X\odot Y} \in \overline{\langle Z-a\rangle + \mathcal{I}_{\mathcal{T}}}.$$

Since $\partial_X f$, $\partial_Y f$ can be written by using commutator bracket, this shows that

$$(X^m \odot Y^n) \otimes \partial_X^k \partial_Y^l e_{\odot}^{\pm\frac{2i}{a}X\odot Y} \odot (X^{m'} \odot Y^{n'}) \in \overline{\langle Z-a\rangle + \mathcal{I}_{\mathcal{T}}}.$$

Hence, we have $(X^m \odot Y^n) \odot e_{\odot}^{-\frac{2i}{a}X\odot Y} \in \overline{\langle Z-a\rangle + \mathcal{I}_{\mathcal{T}}}$. It follows

$$\Big(\sum_{k=0}^{m} \frac{1}{k!}(\frac{2i}{a}X \odot Y)^k\Big) \odot e_{\odot}^{-\frac{2i}{a}X\odot Y} \in \overline{\langle Z-a\rangle + \mathcal{I}_{\mathcal{T}}}.$$

Taking $m \to \infty$, we have the lemma.

Theorem 5.1 gives the following.

Theorem 5.2 *Under the same assumption as in Theorem 5.1, any element $a-Z$ for $a \neq 0$ in $(\mathcal{A}_{\mathcal{T}}, \hat{*})$ has an inverse, and $\bigcap_{k\geq 0} Z^k \otimes \mathcal{T}_{\mathcal{T}} = \{0\}$. Furthermore, $\mathcal{I}_{\mathcal{T}}$ has no complementary closed subspace in $\mathcal{T}_{\mathcal{T}}$.*

This shows that $\mathcal{A}_{\mathcal{T}}$ is almost formal.

Proof of Theorem 5.2. For a polynomial $p(Z)$ of Z, we define a family of semi-norms:

$$||p(Z)||_{\tau_0,s} = \sum |a_k| k^{\tau_0 k} s^{\tau_0 k}, \quad p(Z) = \sum a_k Z^k \tag{5.29}$$

We denote by $\mathcal{Z}_{\tau_0}$ the completion of $\mathcal{Z} = \{p(Z) :$ polynomial in $Z\}$ by the system of semi-norms (5.29). Then, Z_{τ_0} is a closed algebra of $\mathcal{T}_{\mathcal{T}}$. But in general, $\mathcal{Z}_{\tau_0} + \mathcal{I}_{\mathcal{T}}$ is not a closed subalgebra in $\mathcal{T}_{\mathcal{T}}$. We, however, get $Z_{\tau_0} \cap \mathcal{I}_{\mathcal{T}} = \{0\}$ by Theorem 5.1.

Consider the algebra $(\mathcal{A}_{\mathcal{T}}, \hat{*})$. We denote by $\mathfrak{Z}$ the closure of the algebra generated by Z and 1 in $\mathcal{A}_{\mathcal{T}}$. Then $\mathfrak{Z}$ is a commutative Fréchet algebra, and $a - Z$ $a \neq 0$ is invertible. Now, we get

$$Z_{\tau_0} \cong Z_{\tau_0}/(Z_{\tau_0} \cap \mathcal{I}_{\tau_0}) \subset (Z_{\tau_0} + \mathcal{I}_{\tau_0})\ /\mathcal{I}_{\tau_0} = \mathfrak{Z} \tag{5.30}$$

where $(\overline{Z_{\tau_0} \cap \mathcal{I}_{\tau_0}})$ denotes the closure of $Z_{\tau_0} \cap \mathcal{I}_{\tau_0}$. Thus, Z_{τ_0} is contained in $\mathfrak{Z}$ and $\mathfrak{Z}$ is viewed as a completion of Z_{τ_0} by taking a weaker topology than the previous one.

Proposition 5.1 *$\mathfrak{Z}$ is contained densely in the space of formal power series ring $\mathbb{C}[[Z]]$ and also contains the space of $\mathbb{C}((Z))$ of convergence series in Z.*

Proof. Since Z_τ does not clash, $\mathfrak{Z}$ is contained densely in $\mathbb{C}[[Z]]$. Let D_n be the disk with the radius $\frac{1}{n}$ with the boundary C_n with the center at the origin. Let $f(\theta)$ be a continuous function on C_n. By the completeness of $\mathfrak{Z}$, we have

$$\hat{f} = \frac{1}{2\pi i}\int_{C_n} f(\theta)(\theta - z)^{-1} d\theta \in \mathfrak{Z}. \tag{5.31}$$

$\hat{f}$ is holomorphic on D_n and extends continuously to C_n. Conversely, such function is written as the form. By moving n, we see that $\mathfrak{Z}$ contains every function which converges on an appropriate disk.

Remark Thus, if $\tau_0 > 2\tau_1 - 1$, and $\tau_1 < \frac{1}{2}$, the algebra $(\mathcal{A}_\tau, \hat{*})$ collapses to the trivial one, if we insert to Z a non-zero number a. This fact may assert that $\mathfrak{Z}$ is a local ring.

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