

A proof of the equivalence of two complex structures on the punctured tangent bundle of complex projective space

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1 Introduction

K. Furutani, R. Tanaka, and S. Yoshizawa constructed a complex (Kähler) structure on the punctured (co-)tangent bundle $\mathring{TP}^n(\mathbf{C})$ of the complex projective space $P^n(\mathbf{C})$ by constructing a diffeomorphism of $\mathring{TP}^n(\mathbf{C})$ onto a “complex cone” $\mathcal{A}(\mathbf{C})$ in $M(n+1, \mathbf{C})$, the space of complex matrices ([FT], [FY]).

Motivated by their works, we also constructed a complex (Kähler) structure on $\mathring{TP}^n(\mathbf{C})$ by a purely Riemannian geometric method ([IY], [Ii], [IM]).

Since our complex structure has similar properties as that of Furutani et al., it seems that they coincide, but no proof has been given.

The purpose of the present note is to prove that these structures coincide. (cf. [Na])

2 Tangent bundle of Riemannian manifold

Let M be a Riemannian manifold of dimension n with a Riemannian metric g . Let T_pM denote the tangent space to M at a point p of M , TM the tangent bundle of M , and π the bundle projection of TM onto M . Let ∇ denote the Levi-Civita connection of M , and $\mathcal{K} : TTM \rightarrow TM$ the connection map corresponding to ∇ . Let $T_u^H TM$ resp. $T_u^V TM$ denote the kernel of $\mathcal{K}|_{T_u TM}$ resp. $d\pi|_{T_u TM}$ ($u \in TM$), which is an n -dimensional subspace of $T_u TM$ called the horizontal resp. vertical subspace of $T_u TM$. We have a direct-sum decomposition:

$$T_u TM = T_u^H TM \oplus T_u^V TM.$$

Elements of $T_u^H TM$ resp. $T_u^V TM$ are called horizontal resp. vertical vectors at u . If $u, v \in T_p M$, v_u^H resp. v_u^V will denote the horizontal resp. vertical vector obtained by the horizontal resp. vertical lift of v to $T_u TM$. Note that v_u^H resp. v_u^V are characterized by

$$\mathcal{K}(v_u^H) = 0, \quad d\pi(v_u^H) = v \quad \text{resp.} \quad \mathcal{K}(v_u^V) = v, \quad d\pi(v_u^V) = 0.$$

The standard almost complex structure J_0 on TM is a $(1, 1)$ -tensor field on TM characterized by

$$J_0(v_u^H) = v_u^V, \quad \text{and} \quad J_0(v_u^V) = -v_u^H.$$

(cf. [Do], [GKM], [Sa1], [Sa2])

3 Complex projective space

Let $M(n+1, \mathbf{C})$ denote the space of $(n+1) \times (n+1)$ complex matrices with the bracket operation $[A, B] = AB - BA$, and the norm $\|A\| = \sqrt{\text{tr}(A^*A)}$.

Let $Herm(n+1)$ denote the subspace of $M(n+1, \mathbf{C})$, consisting of Hermitian matrices with the Euclidean inner product $(A, B) \mapsto \text{tr}(AB)$.

A vector tangent to $Herm(n+1)$ at a point $P \in Herm(n+1)$ is denoted by the pair (P, V) with $V \in Herm(n+1)$. The tangent space to $Herm(n+1)$ at P is denoted by

$$T_P Herm(n+1) = \{(P, V) \mid V \in Herm(n+1)\}.$$

The complex projective space $P^n(\mathbf{C})$ of dimension n is represented as a submanifold of $Herm(n+1)$ as follows:

$$P^n(\mathbf{C}) = \{P \in Herm(n+1) \mid P^2 = P, \text{tr}P = 1\}.$$

The tangent space to $P^n(\mathbf{C})$ at a point $P \in P^n(\mathbf{C})$ is given by

$$T_P P^n(\mathbf{C}) = \{(P, Q) \mid Q \in Herm(n+1), PQ + QP = Q\}.$$

Let g denote the induced Riemannian metric on $P^n(\mathbf{C})$, i.e.

$$g((P, Q), (P, R)) = \text{tr}(QR).$$

The tangent bundle to $P^n(\mathbf{C})$ is denoted by $TP^n(\mathbf{C})$.

Lemma 3.1 *The standard complex structure j on $P^n(\mathbf{C})$ is given by*

$$j : TP^n(\mathbf{C}) \rightarrow TP^n(\mathbf{C}), \quad j((P, Q)) = (P, \sqrt{-1}[P, Q]).$$

Let \mathbf{C}^{n+1} be the complex $(n+1)$ -space with the norm $\|\cdot\|$;

$$\mathbf{C}^{n+1} = \{p = {}^t(p_1, p_2, \dots, p_{n+1}) \mid p_i \in \mathbf{C}\}.$$

Lemma 3.2 ([FT], [FY]) *For any point P of $P^n(\mathbf{C})$, and for any vector (P, Q) tangent to $P^n(\mathbf{C})$ at P , there exist $p, q \in \mathbf{C}^{n+1}$ that satisfy*

$$(1) \|p\| = 1, \quad (2) p^*q = 0, \quad (3) P = pp^*, \quad (4) Q = pq^* + qp^*.$$

Note that $\|Q\|^2 = 2\|q\|^2$.

Lemma 3.3 ([FT], [FY]) *Let (P, Q) and (P, R) be vectors tangent to $P^n(\mathbf{C})$ at P .*

Then

- (1) $\operatorname{tr} Q = \operatorname{tr}(PQ) = 0$,
- (2) $PQP = 0$,
- (3) $PQ^2 = Q^2P = \frac{1}{2}\|Q\|^2P$,
- (4) $Q^2 = \frac{1}{2}\|Q\|^2P + QPQ$,
- (5) $Q^3 = \frac{1}{2}\|Q\|^2Q$,
- (6) $\|PQ\|^2 = \frac{1}{2}\|Q\|^2$,
- (7) $\operatorname{tr}(PQ^2) = \frac{1}{2}\|Q\|^2$,
- (8) $PQR = QRP$,
- (9) $P(QR + RQ) = \operatorname{tr}(QR)P$,
- (10) $Q(I - 2P) = -(I - 2P)Q$,
- (11) $\operatorname{tr}([P, Q]Q) = 0$,
- (12) $Q[P, Q] = -[P, Q]Q$,
- (13) $\|Q\|^2R + 2(I - 2P)[Q^2, R] = \operatorname{tr}(QR)Q - \operatorname{tr}([P, Q]R)[P, Q]$,

where I denotes the unit matrix.

$T_P P^n(\mathbf{C})$ is a linear subspace of the Euclidean space $T_P \operatorname{Herm}(n+1)$.

Lemma 3.4 *The orthogonal projection $\tau_P : T_P \operatorname{Herm}(n+1) \rightarrow T_P P^n(\mathbf{C})$ is given by*

$$\tau_P(P, V) = (P, PV + VP - 2PVP) = (P, [P, [P, V]]).$$

Proof $(P, PV + VP - 2PVP)$ is tangent to $P^n(\mathbf{C})$ at P , since $(PV + VP - 2PVP)^* = PV + VP - 2PVP$, and $P(PV + VP - 2PVP) + (PV + VP - 2PVP)P = PV + VP - 2PVP$. $(P, V) - (P, PV + VP - 2PVP) = (P, V - (PV + VP - 2PVP))$ is orthogonal to each vector (P, Q) tangent to $P^n(\mathbf{C})$, since $\operatorname{tr}(Q(V - (PV + VP - 2PVP))) = 0$. ■

Let $t \mapsto P(t)$ be a C^∞ curve on $P^n(\mathbf{C})$, and $\xi : t \mapsto \xi(t) = (P(t), Q(t))$ be a vector field along this curve such that $\xi(t)$ is tangent to $P^n(\mathbf{C})$ at $P(t)$. Then the covariant derivative $\nabla_{\frac{d}{dt}} \xi : t \mapsto \nabla_{\frac{d}{dt}} \xi(t)$ of ξ is defined by

$$\nabla_{\frac{d}{dt}} \xi(t) := \tau_{P(t)}(P(t), Q'(t)),$$

which is a vector field along this curve. ξ is called parallel if $\nabla_{\frac{d}{dt}} \xi(t) = (P(t), 0)$ for all t .

Lemma 3.5 *Let $(P, Q), (P, R)$ be vectors tangent to $P^n(\mathbf{C})$ at P , and $t \mapsto P(t)$ be a C^∞ curve on $P^n(\mathbf{C})$ such that $P(0) = P$ and $\dot{P}(0) := (P(0), P'(0)) = (P, R)$. Let*

$\xi : t \mapsto \xi(t) = (P(t), Q(t))$ be the parallel vector field along this curve with initial condition $\xi(0) = (P(0), Q(0)) = (P, Q)$. Then we have

$$Q'(0) = (I - 2P)(QR + RQ).$$

Proof Since ξ is parallel, we have by Lemma 3.4

$$(P(t), 0) = \nabla_{\frac{d}{dt}} \xi(t) = (P(t), P(t)Q'(t) + Q'(t)P(t) - 2P(t)Q'(t)P(t)).$$

Putting $t = 0$, we have

$$PQ'(0) + Q'(0)P - 2PQ'(0)P = 0. \quad (1)$$

Multiplying (1) by P on the left, resp. on the right, we have (since $P^2 = P$)

$$PQ'(0) + PQ'(0)P - 2PQ'(0)P = 0, \text{ resp. } PQ'(0)P + Q'(0)P - 2PQ'(0)P = 0.$$

Hence we have

$$PQ'(0) = Q'(0)P = PQ'(0)P. \quad (2)$$

On the other hand, since $\xi(t)$ is tangent to $P^n(\mathbf{C})$ at $P(t)$, we have $P(t)Q(t) + Q(t)P(t) = Q(t)$ for all t . Differentiating both sides of this equation with respect to t , and putting $t = 0$, we have $P(0)Q'(0) + P'(0)Q(0) + Q(0)P'(0) + Q'(0)P(0) = Q'(0)$. Since $P(0) = P$, $Q(0) = Q$, and $P'(0) = R$, we have

$$Q'(0) = PQ'(0) + RQ + QR + Q'(0)P. \quad (3)$$

Multiplying (3) by P on the left, we have $PQ'(0) = PQ'(0) + PRQ + PQR + PQ'(0)P$.

Hence

$$PQ'(0)P = -PQR - PRQ. \quad (4)$$

From (2), (3), and (4) we obtain

$$Q'(0) = QR + RQ - 2PQR - 2PRQ = (I - 2P)(QR + RQ). \quad \blacksquare$$

A vector tangent to $TP^n(\mathbf{C})$ at (P, Q) is denoted by $((P, Q), (R, V))$.

The tangent space to $TP^n(\mathbf{C})$ at a point (P, Q) is given by

$$T_{(P,Q)}TP^n(\mathbf{C}) = \{((P, Q), (R, V)) \mid R, V \in \text{Herm}(n+1), PR + RP = R, \\ PV + VP + QR + RQ = V\}.$$

The tangent bundle to $TP^n(\mathbf{C})$ is denoted by $TTP^n(\mathbf{C})$.

Let $(P, Q), (P, R) \in T_P P^n(\mathbf{C})$. Let $t \mapsto P(t)$ be a C^∞ curve on $P^n(\mathbf{C})$ such that $P(0) = P$ and $\dot{P}(0) := (P(0), P'(0)) = (P, R)$, and $\xi : t \mapsto \xi(t) = (P(t), Q(t))$ be the parallel vector field along this curve with initial condition $\xi(0) = (P(0), Q(0)) = (P, Q)$. Then the horizontal lift $(P, R)_{(P,Q)}^H$ of (P, R) to $T_{(P,Q)}TP^n(\mathbf{C})$ is given by

$$(P, R)_{(P,Q)}^H = \dot{\xi}(0) := (\xi(0), \xi'(0)) = ((P(0), Q(0)), (P'(0), Q'(0))).$$

By Lemma 3.5, we have

$$(P, R)_{(P,Q)}^H = ((P, Q), (R, (I - 2P)(QR + RQ))).$$

The vertical lift $(P, R)_{(P,Q)}^V$ of (P, R) to $T_{(P,Q)}TP^n(\mathbf{C})$ is given by

$$(P, R)_{(P,Q)}^V = \dot{\eta}(0) := (\eta(0), \eta'(0)) = ((P, Q), (0, R)),$$

where η is a curve on $TP^n(\mathbf{C})$ given by $\eta(t) = (P, Q + tR)$.

The horizontal subspace and the vertical subspace of $T_{(P,Q)}TP^n(\mathbf{C})$ are given respectively by

$$T_{(P,Q)}^H TP^n(\mathbf{C}) = \{((P, Q), (R, (I - 2P)(QR + RQ))) \mid (P, R) \in T_P P^n(\mathbf{C})\},$$

and

$$T_{(P,Q)}^V TP^n(\mathbf{C}) = \{((P, Q), (0, R)) \mid (P, R) \in T_P P^n(\mathbf{C})\}.$$

The differential $d\pi$ of the projection $\pi : TP^n(\mathbf{C}) \rightarrow P^n(\mathbf{C})$, $(P, Q) \mapsto P$ is given by

$$d\pi : TTP^n(\mathbf{C}) \rightarrow TP^n(\mathbf{C}), \quad d\pi((P, Q), (R, V)) = (P, R).$$

The connection map \mathcal{K} is given by

$$\mathcal{K} : TTP^n(\mathbf{C}) \rightarrow TP^n(\mathbf{C}), \quad \mathcal{K}((P, Q), (R, V)) = (P, PV + VP - 2PVP).$$

Note that

$$\begin{aligned} (1) \quad d\pi((P, R)_{(P,Q)}^H) &= (P, R), & (2) \quad d\pi((P, R)_{(P,Q)}^V) &= (P, 0), \\ (3) \quad \mathcal{K}((P, R)_{(P,Q)}^H) &= (P, 0), & (4) \quad \mathcal{K}((P, R)_{(P,Q)}^V) &= (P, R). \end{aligned}$$

Lemma 3.6 *Let $((P, Q), (R, V)) \in T_{(P,Q)}TP^n(\mathbf{C})$. Then it is represented as a sum of horizontal and vertical vectors as follows:*

$$\begin{aligned} ((P, Q), (R, V)) &= ((P, Q), (R, (I - 2P)(QR + RQ))) \\ &\quad + ((P, Q), (0, V - (I - 2P)(QR + RQ))) \\ &= (P, R)_{(P,Q)}^H + (P, V - (I - 2P)(QR + RQ))_{(P,Q)}^V. \end{aligned}$$

Proof It suffices to show that $(P, V - (I - 2P)(QR + RQ))$ is tangent to $P^n(\mathbf{C})$ at P , which follows easily from (8) of Lemma 3.3. ■

4 Complex structure on $\overset{\circ}{T}P^n(\mathbf{C})$

Let $\overset{\circ}{T}P^n(\mathbf{C})$ denote the punctured tangent bundle of $P^n(\mathbf{C})$, i.e.

$$\overset{\circ}{T}P^n(\mathbf{C}) = \{(P, Q) \in TP^n(\mathbf{C}) \mid Q \neq 0\},$$

which is an open subset of $TP^n(\mathbf{C})$.

Let define an almost complex structure J on $\overset{\circ}{T}P^n(\mathbf{C})$ by

$$J((j(P, Q))_{(P, Q)}^H) = \sqrt{2}\|Q\|(j(P, Q))_{(P, Q)}^V,$$

$$J((j(P, Q))_{(P, Q)}^V) = -\frac{1}{\sqrt{2}\|Q\|}(j(P, Q))_{(P, Q)}^H,$$

$$J((P, R)_{(P, Q)}^H) = \frac{\|Q\|}{\sqrt{2}}(P, R)_{(P, Q)}^V,$$

$$J((P, R)_{(P, Q)}^V) = -\frac{\sqrt{2}}{\|Q\|}(P, R)_{(P, Q)}^H$$

for (P, R) orthogonal to $j(P, Q)$, $((P, Q) \in \mathring{T}P^n(\mathbf{C}))$. (See [Ii], [IY], [IM].)

It is known that J is integrable, i.e., J is a complex structure on $\mathring{T}P^n(\mathbf{C})$. (See [Ii], [IY], [IM].)

Definition 4.1 ([FT], [FY]) *Let define a complex submanifold $\mathcal{A}(\mathbf{C})$ of $M(n+1, \mathbf{C})$, and a mapping Φ by*

$$\mathcal{A}(\mathbf{C}) = \{A \in M(n+1, \mathbf{C}) \mid A^2 = 0, \text{rank} A = 1\},$$

$$\Phi : \mathring{T}P^n(\mathbf{C}) \rightarrow \mathcal{A}(\mathbf{C}), \quad \Phi(P, Q) = \|Q\|^2 P - Q^2 + \frac{\sqrt{-1}}{\sqrt{2}}\|Q\|Q.$$

Remark ([FT], [FY]) *If we choose p, q as in Lemma 3.2, then*

$$\Phi(P, Q) = (\|q\|p + \sqrt{-1}q)(\|q\|p - \sqrt{-1}q)^*.$$

Lemma 4.2 ([FT], [FY]) *Φ is well-defined, and is a diffeomorphism.*

Lemma 4.3 *Let $(P, Q) \in \mathring{T}P^n(\mathbf{C})$. Then we have*

(1) *For a tangent vector $((P, Q), (R, V))$ to $\mathring{T}P^n(\mathbf{C})$ at (P, Q) ,*

$$d\Phi((P, Q), (R, V)) = \left(\Phi(P, Q), 2\text{tr}(QV)P + \|Q\|^2 R - (QV + VQ) \right. \\ \left. + \frac{\sqrt{-1}}{\sqrt{2}\|Q\|}\text{tr}(QV)Q + \frac{\sqrt{-1}}{\sqrt{2}}\|Q\|V \right).$$

(2) *For the horizontal lift to (P, Q) of a tangent vector (P, R) ,*

$$d\Phi((P, R)_{(P, Q)}^H) = \left(\Phi(P, Q), \frac{1}{2}\text{tr}(QR)Q - \frac{1}{2}\text{tr}([P, Q]R)[P, Q] + \frac{1}{2}\|Q\|^2 R \right. \\ \left. - \sqrt{-1}\sqrt{2}\|Q\|\text{tr}(QR)P + \frac{\sqrt{-1}}{\sqrt{2}}\|Q\|(QR + RQ) \right).$$

(3) *For the vertical lift to (P, Q) of a tangent vector (P, R) ,*

$$d\Phi((P, R)_{(P, Q)}^V) = \left(\Phi(P, Q), 2\text{tr}(QR)P - (QR + RQ) \right. \\ \left. + \frac{\sqrt{-1}}{\sqrt{2}\|Q\|}\text{tr}(QR)Q + \frac{\sqrt{-1}}{\sqrt{2}}\|Q\|R \right).$$

(4) For the horizontal lift to (P, Q) of $j(P, Q)$,

$$d\Phi((j(P, Q))_{(P, Q)}^H) = \left(\Phi(P, Q), \sqrt{-1} \|Q\|^2 [P, Q] \right).$$

(5) For the vertical lift to (P, Q) of $j(P, Q)$,

$$d\Phi((j(P, Q))_{(P, Q)}^V) = \left(\Phi(P, Q), -\frac{1}{\sqrt{2}} \|Q\| [P, Q] \right).$$

Lemma 4.4 Let $(P, Q) \in \mathring{T}P^n(\mathbf{C})$, and (P, R) be orthogonal to $j(P, Q)$. Then we have

$$(1) \quad d\Phi(J((j(P, Q))_{(P, Q)}^H)) = \sqrt{-1} d\Phi((j(P, Q))_{(P, Q)}^H),$$

$$(2) \quad d\Phi(J((j(P, Q))_{(P, Q)}^V)) = \sqrt{-1} d\Phi((j(P, Q))_{(P, Q)}^V),$$

$$(3) \quad d\Phi(J((P, R)_{(P, Q)}^H)) = \sqrt{-1} d\Phi((P, R)_{(P, Q)}^H),$$

$$(4) \quad d\Phi(J((P, R)_{(P, Q)}^V)) = \sqrt{-1} d\Phi((P, R)_{(P, Q)}^V).$$

Proposition 4.5 $d\Phi \circ J = \sqrt{-1} d\Phi.$

Theorem 4.6 If we identify $\mathring{T}P^n(\mathbf{C})$ with $\mathcal{A}(\mathbf{C})$ by Φ , then the complex structure J on $\mathring{T}P^n(\mathbf{C})$ coincides with the complex structure $\sqrt{-1}$ on $\mathcal{A}(\mathbf{C})$.

Remark A similar result is obtained for the case of quaternion projective space. (cf. [Ka])

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