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# On Multiple Positive Solutions of Semilinear Elliptic Equations in $\mathbf{R}^n$

Soohyun Bae
Department of Mathematics, Yonsei University
Seoul 120-749, Korea

## 1 Introduction

Diverse physical and geometrical models lead to the elliptic equation

$$\Delta u+K(x)u^{p}+\mu f(x)=0, \tag{1.1}$$

where $n\geq 3$, $\Delta=\sum_{i=1}^{n}\frac{\partial^{2}}{\partial x_{i}^{2}}$ is the Laplace operator, $p>1$, $\mu>0$ is a parameter, and $f$ as well as $K$ is a given locally Hölder continuous function in $\mathbf{R}^{n}$. In particular, the homogeneous equation

$$\Delta u+K(x)u^{p}=0 \tag{1.2}$$

stands for the prescribing scalar curvature problem in Riemannian geometry when $p$ is the critical Sobolev exponent $\frac{n+2}{n-2}$, or Lane-Emden equation in astrophysics when $K(x)=|x|^{l}$. One of many interesting questions is whether these equations possess *multiple* (or *infinitely many*) positive entire solutions in $\mathbf{R}^{n}$.

To illuminate our motivations more clearly, we need the following notations. Set

$$p_{c}=p_{c}(n,l)=\begin{cases}\frac{(n-2)^{2}-2(l+2)(n+l)+2(l+2)\sqrt{(n+l)^{2}-(n-2)^{2}}}{(n-2)(n-10-4l)} & \text{if } n>10+4l,\\ \infty & \text{if } n\leq 10+4l\end{cases} \tag{1.3}$$

for some $l>-2$. Let $m=\frac{2+l}{p-1}$ and

$$\lambda_{1}=\lambda_{1}(n,p,l)=\frac{(n-2-2m)-\sqrt{(n-2-2m)^{2}-4(l+2)(n-2-m)}}{2}, \tag{1.4}$$

$$\lambda_{2}=\lambda_{2}(n,p,l)=\frac{(n-2-2m)+\sqrt{(n-2-2m)^{2}-4(l+2)(n-2-m)}}{2}. \tag{1.5}$$

Observe that $\lambda_{1},\lambda_{2}\in\mathbf{R}$ if and only if $n>10+4l$ and $p\geq p_{c}$. The two numbers, $\lambda_{1}$ and $\lambda_{2}$, play important roles in describing the asymptotic behavior at $\infty$ of positive radial solutions to Lane-Emden equation with $p\geq p_{c}(n,l)$

$$\Delta u+c|x|^{l}u^{p}=0 \tag{1.6}$$

in $\mathbf{R}^n$ for $l > -2$ and $c > 0$. It is known that when $p > \frac{n+2+2l}{n-2}$ and $l > -2$, (1.6) has a positive radial solution $\bar{u}_\alpha$ with $\bar{u}_\alpha(0) = \alpha$ for each $\alpha > 0$ and

$$\lim_{r \to \infty} r^m \bar{u}_\alpha(r) = L, \tag{1.7}$$

where

$$L = L(n,p,l,c) = \left[\frac{l+2}{p-1}\left(n-2-\frac{l+2}{p-1}\right)\frac{1}{c}\right]^{\frac{1}{p-1}} \tag{1.8}$$

(see [7, 14]). Furthermore, $p \geq p_c(n,l)$ if and only if any two positive radial solutions of (1.6) can not intersect each other [14]. By analogy with (1.6), it is natural to expect that under suitable conditions on $K$, (1.2) with $p \geq p_c$ has infinitely many positive solutions which preserve this separation property. This question was studied first by C. Gui [9, 10] and recently by S. Bae, T. K. Chang and D. H. Pahk. In [3], a sufficient condition to guarantee infinite multiplicity for (1.2) is the following:

**Theorem 1.1** *Let $p \geq p_c(n,l)$ with $l > -2$. Suppose that $K \geq 0$ satisfies*

$$K(x) = c|x|^l + O(|x|^{-d})$$

*near $|x| = \infty$ for some $c > 0$ and*

$$d > n - \lambda_2(n,p,l) - m(p+1).$$

*Then, equation (1.2) possesses infinitely many positive entire solutions satisfying*

$$\lim_{|x| \to \infty} |x|^m u(x) = L(n,p,l,c)$$

*and no two of them can intersect.*

The fact that $p_c(n,l) \to 1$ as $l \to -2$ is a background to study infinite multiplicity in case that $K(x)$ has a similar behavior to $c|x|^{-2}$ at $\infty$. In [10], Gui proved that if $K$ is a positive function satisfying $K(x) = c|x|^{-2} + O(|x|^{-d})$ at $|x| = \infty$ for some $d > 2$, then equation (1.2) with $p > 1$ possesses infinitely many positive entire solutions with the asymptotic behavior

$$\lim_{|x| \to \infty} (\log|x|)^{1/(p-1)} u(x) = L, \tag{1.9}$$

where

$$L = L(n,p,-2,c) = \left[\frac{n-2}{(p-1)c}\right]^{\frac{1}{p-1}} \tag{1.10}$$

and no two of them can intersect. In [2], Bae and Chang established infinite multiplicity without positivity condition on $K$ as follows:

**Theorem 1.2** *Let $p > 1$. If $K \geq 0$ satisfies*

$$K(x) = c|x|^{-2} + O(|x|^{-n}[\log|x|]^q), \tag{1.11}$$

*near $|x| = \infty$ for some constants $c > 0$ and $q > 0$. Then, equation (1.2) possesses infinitely many entire solutions with the asymptotic behavior (1.9) and no two of them can intersect.*

In 1982, W.-M. Ni proved in [13] that if $K(x) = O(|x|)^l$ near $|x| = \infty$ for some $l < -2$, then (1.2) with $p > 1$ possesses infinitely many positive entire solutions which are bounded away from 0.

On the other hand, another natural question is whether (1.1) still could have infinitely many entire solutions. In [4], Bae and Ni confirmed this question positively for (1.1) with $K \equiv 1$, combining the modified version of the barrier method initiated by Gui [9] and asymptotic behaviors near $\infty$ of positive solutions of suitable homogeneous equations. Recently, this equation was studied again by Bae, Chang and Pahk in [3]. Multiplicity results in [3, 4] for the equation

$$\Delta u + u^p + \mu f(x) = 0, \tag{1.12}$$

where $\mu > 0$ is a parameter, can be summarized as follows:

**Theorem 1.3** *Let $p \geq p_c(n, 0)$. Suppose that $f \not\equiv 0$ and*

$$f(x) = O(|x|^{-q})$$

*near $|x| = \infty$, where*

$$q > n - \lambda_2(n, p, 0) - \frac{2}{p-1}.$$

*Then, there exists $\mu_* > 0$ such that for each $\mu \in (0, \mu_*)$, (1.12) possesses infinitely many positive entire solutions with the asymptotic behavior $L(n, p, 0, 1)|x|^{-2/(p-1)}$ at $\infty$.*

The main difference between (1.12) and (1.1) lies in the fact that the part $\Delta u + Ku^p$ of (1.1) does not possess any scaling-invariant structure in general. Hence, the barrier method used in [4] cannot apply to the problem (1.1) directly. A new approach is needed to handle (1.1). In [3], it is observed that a limiting function demonstrating asymptotic behaviors at $\infty$ of positive solutions of equation (1.2) is *continuous* with respect to initial data. This observation makes it possible for the infinitely many pairs of positive solutions of (1.1) constructed by super- and sub-solution arguments to have specific asymptotic behaviors at $\infty$ to discern one another, which is, in fact, the key idea in [4] to get infinite multiplicity for the inhomogeneous problem (1.12). For multiplicity results on the general form (1.1), we refer the readers to [3].

In Theorem 1.1 and Theorem 1.3, the monotonicity of $\bar{u}_\alpha$ with respect to $\alpha$ is essential for the constructions of infinitely many pairs of super- and sub-solutions. It, therefore, seems interesting to ask the multiplicity for (1.1) when $p < p_c$.

When $p$ is the critical Sobolev exponent $\frac{n+2}{n-2}$, Egnell and Kaj studied in [8] the multiplicity for (1.12). By variational methods, they showed that if $f \in L^{2n/(n+2)}(\mathbf{R}^n)$ and $0 \not\equiv f \geq 0$, then (1.12) with $p = \frac{n+2}{n-2}$ has at least two positive weak solutions in $\mathcal{D}^{1,2}$ for $\mu > 0$ small, where the Sobolev space $\mathcal{D}^{1,2}$ is the completion of $\mathbf{C}_0^\infty(\mathbf{R}^n)$ in the $L^2(\mathbf{R}^n)$ norm of $|\nabla u|$. Later, Cao, Li and Zhou verified in [5] that if $0 \not\equiv f \geq 0$ belongs to the dual space $\mathcal{D}_*^{1,2}$ of $\mathcal{D}^{1,2}$ and the dual norm $||f||_*$ of $f$ holds

$$\mu < C_n S^{n/4}/||f||_*,$$

where

$$C_n := \frac{4}{n-2}\left(\frac{n-2}{n+2}\right)^{(n+2)/4}$$

and $S$ is the Sobolev constant for the embedding $\mathcal{D}^{1,2} \hookrightarrow L^{2n/(n-2)}(\mathbf{R}^n)$, then (1.12) has at least two positive weak solutions in $\mathcal{D}^{1,2}$. In fact, there exists $\bar{\mu} > 0$ which is the borderline of existence and nonexistence. We put some results in [1, 5, 8] together as follows:

**Theorem 1.4** *Let $p = \frac{n+2}{n-2}$. Suppose that $0 \not\equiv f \geq 0$ satisfies*

$$f(x) = O(|x|^{-q}) \tag{1.13}$$

*near $|x| = \infty$ for some $q > n$. Then, there exists $\bar{\mu} \geq C_n S^{n/4}/||f||_*$ such that (1.12) has at least two positive solutions $U_\mu > u_\mu$ for each $0 < \mu < \bar{\mu}$ while there is no positive solution of (1.12) for $\mu > \bar{\mu}$, and there exists a unique positive solution $u_{\bar{\mu}}$ of (1.12) when $\mu = \bar{\mu}$. Moreover, as $\mu \to 0+$, $u_\mu \to 0$ in $\mathcal{D}^{1,2}$ and*

$$\lim_{\mu \to 0+} ||U_\mu|| = S^{n/4}.$$

In the next section, we present asymptotic behaviors near $\infty$ which are crucial in establishing Theorem 1.1, 1.2 and 1.3, and interpret multiplicity results to Riemannian geometry. In the final section, related eigenvalue problems are discussed in case $p = \frac{n+2}{n-2}$.

# 2 Asymptotic behavior

**§1.** We first recall the asymptotic behavior at $\infty$ of positive radial solutions $\bar{u}_\alpha$ of equation (1.6) (see [12, 11; Theorem 2.5, Lemma 4.13 and (4.15)] for details).

**Proposition 2.1** *Let $l > -2$ and $c > 0$. For $p \geq p_c(n,l)$, we have that for arbitrarily given $\epsilon > 0$*

$$\bar{u}_\alpha(r) = \frac{L}{r^m} + \frac{a_\alpha}{r^{m+\lambda_1}} + \cdots + O\left(\frac{1}{r^{n-2+\epsilon}}\right) \quad \text{if} \quad p > p_c, \tag{2.1}$$

$$\bar{u}_\alpha(r) = \frac{L}{r^m} + \frac{a_\alpha \log r}{r^{m+\lambda_1}} + \cdots + O\left(\frac{1}{r^{n-2+\epsilon}}\right) \quad \text{if} \quad p = p_c \tag{2.2}$$

*near $\infty$, where $L$ is given by (1.8), $\lambda_1$ is given by (1.4), and*

$$a_\alpha = \alpha^{-\lambda_1/m} a_1 < 0. \tag{2.3}$$

Although Theorem 2.5 in [11] deals only with the case $l = 0$, the arguments in the proof can be proceeded similarly to conclude Proposition 2.1. Another direct consequence of Theorem 2.5 in [11] is the following:

**Proposition 2.2** *Let $v_1$, $v_2$ be two positive radial solutions of the equation*

$$\Delta u + c r^l u^p = 0$$

*near $\infty$, where $c > 0$ and $l > -2$. Suppose that*

$$\lim_{r\to\infty} r^m v_1(r) = L = \lim_{r\to\infty} r^m v_2(r)$$

*and*

$$\lim_{r\to\infty} r^{\lambda_1}(r^m v_1(r) - L) = \lim_{r\to\infty} r^{\lambda_1}(r^m v_2(r) - L) \quad \text{if} \quad p > p_c,$$

$$\lim_{r\to\infty} \frac{r^{\lambda_1}}{\log r}(r^m v_1(r) - L) = \lim_{r\to\infty} \frac{r^{\lambda_1}}{\log r}(r^m v_2(r) - L) \quad \text{if} \quad p = p_c.$$

*Then, $v_1(r) - v_2(r) = O(r^{-m-\lambda_2})$ near $\infty$, where $\lambda_2$ is given by (1.5).*

The existence of a positive radial super-solution of (1.6) having the following asymptotic behavior is verified similarly as in [11] (see [11; Theorems 2.5, 4.1 and Lemmas 4.11, 4.13]).

**Proposition 2.3** *Let $p \geq p_c(n,l), l > -2$ and $c > 0$. Then, for each $\alpha > 0$, there exists a positive radial super-solution $\bar{u}_\alpha^+(r)$ of (1.6) such that $\bar{u}_\alpha^+(r) > \bar{u}_\alpha(r)$ for $r \in [0,\infty)$ and $\bar{u}_\alpha^+(r) - \bar{u}_\alpha(r) = O(r^{-m-\lambda_2})$ as $r \to \infty$.*

Let $K = K(r)$ be a radial function in $\mathbf{R}^n$. The radial version of equation (1.2) is of the form

$$\begin{cases} u'' + \frac{n-1}{r}u' + K(r)u^p &= 0, \\ u(0) = \alpha > 0, \quad u'(0) &= 0. \end{cases} \tag{2.4}$$

For each $\alpha > 0$, the local solution $u_\alpha$ of (2.4) is decreasing and extended locally wherever it exists and remains positive. To obtain a continous family of positive radial solutions of (2.4) for $\alpha > 0$ small, it suffices to construct countable solutions with initial data converging to 0, by the following:

**Lemma 2.4** *Assume that $K \geq 0, \not\equiv 0$. Suppose that there exist three solutions $u_\alpha, u_\beta, u_\gamma$ of (2.4) such that $0 < u_\alpha < u_\beta < u_\gamma$ in $[0, \bar{R})$ for some $\bar{R} \in (0, \infty]$. Then, for each $\alpha < \delta < \beta$, (2.4) possesses a positive radial solution $u_\delta$ in $B_{\bar{R}}$ satisfying*

$$0 < u_\alpha(r) < u_\delta(r) < u_\beta(r)$$

*for $0 \leq r < \bar{R}$.*

Combining Green's identity, Proposition 2.3 and Lemma 2.4, we construct a continuous family of positive radial solutions of (2.4) [3; Proposition 3.1].

**Proposition 2.5** *Let $p \geq p_c(n, l)$ with $l > -2$. Suppose that $K = K(r) \geq 0$ satisfies that*

$$\int_1^\infty (K(r) - cr^l)_- r^{n-1-m(p+1)-\lambda_2} dr < \infty$$

*and either $r^{-l}K(r) \leq cp$ near $\infty$,*

$$\int_1^\infty (K(r) - cr^l)_+ r^{n-1-m(p+1)-\lambda_2} dr < \infty$$

*or*

$$\int_1^\infty (K(r) - cr^l)_+ r^{n-1-mp-\lambda_2} dr < \infty$$

*for some $c > 0$, where $k_\pm = \max(\pm k, 0)$. Then, there exists a positive constant $\alpha^* = \alpha^*(p, K)$ such that for each $\alpha \in (0, \alpha^*]$, equation (2.4) possesses a positive radial solution $u_\alpha$ with $u_\alpha(0) = \alpha$ satisfying*

$$\lim_{r \to \infty} r^m u_\alpha(r) = L(n, p, l, c)$$

*and no two of them can intersect.*

When $K$ satisfies the conditions of Theorem 1.1, infinitely many pairs of super- and sub-solutions of (1.2) are constructed by making use of Proposition 2.5. Then, standard barrier method implies Theorem 1.1. Proposition 2.2 as well as Proposition 2.5 is an important ingredient in establishing Theorem 1.3.

Under the assumptions on $K$ as in Proposition 2.5, equation (2.4) with $p \geq p_c(n, l)$ and $l > -2$ has a family $\{u_\alpha\}$ of positive radial solutions indexed by $\alpha \in (0, \alpha^*]$ for some $\alpha^* > 0$ such that $u_\alpha(0) = \alpha$ and $u_\alpha$ is monotonically increasing with respect to $\alpha$. Moreover, it is observed in the proof of Proposition 2.5 that for each $\alpha \in (0, \alpha^*]$, there exist $\gamma < \alpha$ and $\beta > \alpha$ such that $\bar{u}_\gamma \leq u_\alpha \leq \bar{u}_\beta$ in $\mathbf{R}^n$ and thus, $r^m u_\alpha(r) \to L$ as $r \to \infty$. For $\alpha \in (0, \alpha^*]$, set $W(\alpha, t) := r^m u_\alpha(r) - L,\ t = \log r$ and

$$D(\alpha, t) := e^{\lambda_1 t} W(\alpha, t) \quad \text{for } p > p_c,$$

$$D(\alpha, t) := t^{-1} e^{\lambda_1 t} W(\alpha, t) \quad \text{for } p = p_c.$$

Then, it follows from (2.1), (2.2) and (2.3) that for fixed $0 < a < \alpha^*$, $D(\alpha, t)$, $a \le \alpha \le \alpha^*$, are uniformly bounded above and below at $+\infty$, that is, there exists $M = M(a,p)$ such that for all $\alpha \in [a, \alpha^*]$,

$$|W(\alpha,t)| \le Me^{-\lambda_1 t} \quad \text{for } p > p_c$$

and

$$|W(\alpha,t)| \le Mte^{-\lambda_1 t} \quad \text{for } p = p_c.$$

For fixed $-\infty < t < +\infty$, $D(\alpha, t)$ is continuous with respect to $\alpha$. Moreover, $D(\alpha, t)$ converges uniformly on $[a, \alpha^*]$ as $t \to +\infty$, which seems of independent interest.

**Lemma 2.6** *For given $0 < a < \alpha^*$, $D(\alpha, t)$ converges uniformly on $[a, \alpha^*]$ as $t \to +\infty$ provided that*

$$\int_1^\infty |K(r) - cr^l| r^{n-1-m(p+1)-\lambda_2} dr < \infty.$$

An immediate consequence of Lemma 2.6 is that the limit of $D(\alpha, t)$ as $t \to +\infty$ is continuous. This fact is crucial in verifying infinite multiplicity for the general inhomogeneous equation (1.1).

**Proposition 2.7** *Let $p \ge p_c(n,l)$ with $l > -2$. Suppose the assumptions of Proposition 2.5. Then, $D(\alpha) := \lim_{t\to+\infty} D(\alpha, t)$ is continuous for $\alpha > 0$ small.*

§**2.** Now, we consider the asymptotic behavior at $\infty$ of positive radial solutions of the equation

$$\Delta u + c|x|^{-2} u^p = 0 \tag{2.5}$$

near $\infty$ for some $c > 0$. Recall the following asymptotic behavior (see [12; Lemma 5.1]).

**Lemma 2.8** *Let $p > 1$, $c > 0$ and $u$ be a positive radial solution of (2.5). If*

$$\lim_{r\to\infty} (\log r)^{1/(p-1)} u(r) = L(n, p, -2, c),$$

*then*

$$u(r) = \frac{L}{(\log r)^{1/(p-1)}} - \frac{pL\log(c\log r)}{(p-1)^2(n-2)(\log r)^{p/(p-1)}} + o\left(\frac{1}{(\log r)^{p/(p-1)}}\right),$$

*near $\infty$, where $L$ is given by (1.10).*

It turns out that the asymptotic behavior of the difference of two positive radial solutions of (2.5) is important to establish infinite multiplicity for equation (1.2). In fact, the assumption (1.11) on $K$ at $|x| = \infty$ in Theorem 1.2 comes from the following key observation.

**Proposition 2.9** *Let $p > 1$ and $v_1$, $v_2$ be two positive radial solutions of (2.5). Suppose that*

$$\lim_{r\to\infty}(\log r)^{1/(p-1)}v_1(r) = L = \lim_{r\to\infty}(\log r)^{1/(p-1)}v_2(r).$$

*Then,*

$$\lim_{r\to\infty}(\log r)^m[v_2(r) - v_1(r)] = 0$$

*for any $m > 0$.*

For our convenience, we fix a family $\{\bar{u}_\alpha\}$ of positive radial solutions of (2.4) indexed by $\alpha \in (0, \alpha^*]$ for some $\alpha^* > 0$ such that $\bar{u}_\alpha(0) = \alpha$, $\bar{u}_\alpha$ is monotone with respect to $\alpha$ and

$$\lim_{r\to\infty}(\log r)^{1/(p-1)}\bar{u}_\alpha(r) = L(n, p, -2, c),$$

where $K$ is a smooth positive radial function $\bar{K}$ such that for some $c > 0$,

$$\bar{K}(r) = \frac{1}{1+r^2} \qquad \text{for} \quad 0 \le r \le 1$$

and

$$\bar{K}(r) = \frac{c}{r^2} \qquad \text{for} \quad r \ge 2$$

(see [10; Theorem 5.1 and Lemmas 5.3, 5.6]). Moreover, it follows from Proposition 2.9 that for each $\alpha \in (0, \alpha^*)$,

$$F_\alpha(r) := \bar{u}_{\alpha^*}(r) - \bar{u}_\alpha(r) = o([\log r]^{-m}) \qquad \text{as} \quad r \to \infty$$

for any $m > 0$. This estimation plays a similar role in proving Theorem 1.2 as Proposition 2.3 does in Theorem 1.1. For the radial case, we have the following [2; Proposition 3.1]:

**Proposition 2.10** *Let $p > 1$. Suppose that $K = K(r) \ge 0$ satisfies that*

$$\int_1^\infty |K(r) - cr^{-2}|r^{n-1}(\log r)^{-a}dr < \infty$$

*for some $c > 0, a > 0$. Then, there exists a positive constant $\alpha^* = \alpha^*(p, K)$ such that for each $\alpha \in (0, \alpha^*]$, equation (2.4) possesses a positive radial solution $u_\alpha$ with $u_\alpha(0) = \alpha$ satisfying*

$$\lim_{r\to\infty}(\log r)^{1/(p-1)}u_\alpha(r) = L(n, p, -2, c)$$

*and no two of them can intersect.*

Theorem 1.2 follows from Proposition 2.10 and the particular barrier method initiated by Gui [9, 10] and modified in [3, 4]. An interesting question is whether $[\log|x|]^q$ in (1.11) could be replaced with the form $|x|^q$ with $0 < q < n-2$, which is still left unanswered.

We interpret Theorem 1.1 and Theorem 1.2 in the context of Riemannian geometry. Let $(M, g)$ be an n-dimensional Riemannian manifold and $K$ be a given function. The scalar curvature problem is to find a metric $g_1$ on $M$ conformal to $g$ such that the corresponding scalar curvature to $g_1$ is $K$. The introduction of $u > 0$ by $g_1 = u^{4/(n-2)}g$, $n \geq 3$, brings out the equation

$$\frac{4(n-1)}{n-2}\Delta_g - ku + Ku^{\frac{n+2}{n-2}} = 0, \tag{2.6}$$

where $\Delta_g$ denotes the Laplace-Beltrami operator on $M$ in the $g$ metric and $k$ is the scalar curvature of $(M, g)$. If $M = \mathbf{R}^n$ and $g = \Sigma_{i=1}^n dx_i^2$ is the standard metric, then equation (2.6) reduces to

$$\Delta u + K(x)u^{\frac{n+2}{n-2}} = 0 \qquad \text{in} \quad \mathbf{R}^n.$$

When $p = \frac{n+2}{n-2}$, Theorem 1.1 and Theorem 1.2 are translated as follows:

**Theorem 2.11** *Suppose that $K$ satisfies the assumptions of Theorem 1.1 with $\frac{n+2}{n-2} \geq p_c(n, l)$ or the assumptions of Theorem 1.2. Then, there exist infinitely many Riemannian metrics $g_1$ on $\mathbf{R}^n$ such that* (i) *$K$ is the scalar curvature of $g_1$;* (ii) *$g_1$ is conformal to the standard metric $g$ on $\mathbf{R}^n$;* (iii) *$g_1$ is complete.*

# 3 Positive global solutions

Let $p = \frac{n+2}{n-2}$ and assume that $0 \not\equiv f \geq 0$ and $f \in \mathcal{D}_*^{1,2}$. We call a positive solution in $\mathcal{D}^{1,2}$ of (1.12) in $\mathbf{R}^n$ a positive global solution. Define

$$\bar{\mu} = \sup\{\mu > 0 \,:\, \text{(1.12) has a positive global solution}\}.$$

Denote the minimal solution (the smallest one among all positive solutions) of (1.12) by $u_\mu$ for $0 < \mu \leq \bar{\mu}$ and consider the eigenvalue problem

$$-\Delta\varphi = \lambda p u_\mu^{p-1}\varphi, \qquad \varphi \in \mathcal{D}^{1,2}. \tag{3.1}$$

Let $\lambda_1$ be the least eigenvalue of (3.1), i.e.,

$$\lambda_1 = \lambda_1(\mu) = \inf\left\{\int |\nabla\varphi|^2 \,:\, \varphi \in \mathcal{D}^{1,2}, \int p u_\mu^{p-1}\varphi^2 = 1\right\}.$$

The minimum is achieved by some $\varphi_1 = \varphi_1(\mu) \in \mathcal{D}^{1,2}$ and $\varphi_1 > 0$ in $\mathbf{R}^n$ which is the corresponding eigenfunction of (3.1) for $\lambda = \lambda_1(\mu)$.

**Lemma 3.1** *$\lambda_1(\mu)$ is a continuous function on $(0, \bar{\mu}]$ such that for $0 < \mu < \nu < \bar{\mu}$,*

$$1 = \lambda_1(\bar{\mu}) < \lambda_1(\nu) < \lambda_1(\mu).$$

*Moreover, $\lambda_1(\mu) \to +\infty$ as $\mu \to 0+$.*

In particular, uniqueness of $u_{\bar{\mu}}$ in Theorem 1.4 follows from the fact that $\lambda_1(\bar{\mu}) = 1$. Define $F : \mathbf{R} \times \mathcal{D}^{1,2} \to \mathcal{D}^{1,2}_*$ by

$$F(\mu, u) = \Delta u + (u^+)^p + \mu f(x).$$

It is easy to see that $F(\mu, u)$ is differentiable and for $\mu \in (0, \bar{\mu})$,

$$F_u(\mu, u_\mu)w = \Delta w + p u_\mu^{p-1} w$$

is an isomorphism of $\mathbf{R} \times \mathcal{D}^{1,2}$ onto $\mathcal{D}^{1,2}_*$. Then, the Implicit Function Theorem implies that the solutions of $F(\mu, u) = 0$ near $(\mu, u_\mu)$ are given by a single continuous curve. By making use of a bifurcation result of Crandall and Rabinowitz [6], we conclude that under the condition (1.13) of Theorem 1.4, $(\bar{\mu}, u_{\bar{\mu}})$ is a bifurcation point of $F$.

Let $\mu \in (0, \bar{\mu})$ and $U_\mu$ be a second solution of (1.12). Then, there is another eigenvalue problem

$$-\Delta \varphi = \eta p U_\mu^{p-1} \varphi, \qquad \varphi \in \mathcal{D}^{1,2}. \tag{3.2}$$

Let $\eta_1$ be the least eigenvalue of (3.2), i.e.,

$$\eta_1 = \eta_1(\mu) = \inf\{\int |\nabla \varphi|^2 \,|\, \varphi \in \mathcal{D}^{1,2}, \int p U_\mu^{p-1} \varphi^2 = 1\}.$$

The behavior of $\eta_1$ is the following [1]:

**Lemma 3.2** *For $0 < \mu < \bar{\mu}$,*

$$\frac{1}{p} < \eta_1(\mu) < 1.$$

*Moreover, $\eta_1(\mu) \to 1/p$ as $\mu \to 0+$ while $\eta_1(\mu) \to 1$ as $\mu \to \bar{\mu}$.*

In Theorem 1.4, we suspect that $\bar{\mu} = C_n S^{n/4}/||f||_*$. On the other hand, a fundamental question on (1.12) is the multiplicity of positive entire solutions satisfying $L|x|^{-m}$ at $\infty$ when $\frac{n+2}{n-2} \leq p < p_c$. Furthermore, the existence/nonexistence of singular solutions of (1.12) is also a challenging problem. A singular solution is a positive classical solution in $\mathbf{R}^n \backslash \{0\}$ which converges to zero at $\infty$ and blows up to $\infty$ at the origin.

# References

[1] S. Bae, Positive global solutions of inhomogeneous semilinear elliptic equations with critical Sobolev exponent, in preparation.

[2] S. Bae and T. K. Chang, On a class of semilinear elliptic equations in $\mathbf{R}^n$, preprint.

[3] S. Bae, T. K. Chang and D. H. Pahk, Infinite multiplicity of positive entire solutions for a semilinear elliptic equation, preprint.

[4] S. Bae and W.-M. Ni, Existence and infinite multiplicity for an inhomogeneous semilinear elliptic equation on $\mathbf{R}^n$, preprint.

[5] D. M. Cao, G. B. Li and H. S. Zhou, Multiple solutions for nonhomogeneous elliptic equations involving critical Sobolev exponent, *Proc. Roy. Soc. Edinburgh* **124A** (1994), 1177–1191.

[6] M. G. Crandall and P. H. Rabinowitz, Bifurcation, perturbation of simple eigenvalues and linearized stability, *Arch. Rational Mech. Anal.* **52** (1973), 161–180.

[7] W.-Y. Ding and W.-M. Ni, On the elliptic equation $\Delta u + Ku^{(n+2)/(n-2)} = 0$ and related topics, *Duke Math. J.* **52** (1985) 485–506.

[8] H. Egnell and I. Kaj, Positive global solutions of a nonhomogeneous semilinear elliptic equation, *J. Math. Pures Appl.* **70** (1991), 345–367.

[9] C. Gui, On positive entire solutions of the elliptic equation $\Delta u + K(x)u^p = 0$ and its applications to Riemannian geometry, *Proc. Roy. Soc. Edinburgh* **126A** (1996) 225–237.

[10] C. Gui, Positive entire solutions of the equation $\Delta u + f(x, u) = 0$, *J. Differential Equations* **99** (1992) 245–280.

[11] C. Gui, W.-M. Ni and X. Wang, On the stability and instability of positive steady states of a semilinear heat equation in $\mathbf{R}^n$, *Comm. Pure Appl. Math.* **45** (1992) 1153–1181.

[12] Y. Li, Asymptotic behavior of positive solutions of equation $\Delta u + K(x)u^p = 0$ in $\mathbf{R}^n$, *J. Differential Equations* **95** (1992) 304–330.

[13] W.-M. Ni, On the elliptic equation $\Delta u + K(x)u^{(n+2)/(n-2)} = 0$, its generalizations, and applications in geometry, *Indiana Univ. Math. J.* **31** (1982) 493–529.

[14] X. Wang, On Cauchy problems for reaction-diffusion equations, *Trans. Amer. Math. Soc.* **337** (1993) 549–590.