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Vortex State of d-Wave Superconductors in the Ginzburg-Landau Energy

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Abstract

We find a minimizer of a reduced form of the Ginzburg-Landau free energy for d-wave superconductors having distinct degree-one vortices. For a single vortex in the vortex core, we analytically recover the vortex structure with fourfold symmetry.

1 Introduction

In the 1910's, low-temperature superconductivity was observed on metals and alloys (cf. [9]). Recently, high-temperature superconductivity has been found on some copper-oxide superconductors (cf. [12]). The vortex state of high-temperature superconductors is different from the vortex state of low-temperature superconductors. When the applied magnetic field is close to the lower critical field H_{c_1}, the single vortex is expected to be symmetric in low-temperature superconductors but it may be asymmetric (fourfold symmetric) in high-temperature superconductors (cf. [8], [31]). Moreover, as the applied magnetic field is close to the upper critical field H_{c_2}, Abrikosov type vortex lattices are expected to be triangular in low-temperature superconductors but they may be rectangular in high-temperature superconductors (cf. [1], [8], [27], [30], [31] etc).

To distinguish low-temperature and high-temperature superconductivity, an s-wave and a d-wave order parameter were introduced (cf. [13], [21]). Soininen et al. (cf. [3], [28]) introduced the Ginzburg-Landau free energy with an s-wave and a d-wave order parameter. Ren et al. (cf. [24], [25]) present a microscopic derivation of the Ginzburg-Landau equations from the Gor'kov equations by using the finite temperature Green's-function approximation method. From [31], we learned the two fields Ginzburg-Landau free energy is given by:

$$\begin{aligned} G(\Psi_s,\Psi_d,A) = & \int_{\mathbb{R}^2} \kappa^2|\text{ curl } A-H|^2+\alpha_s(T)|\Psi_s|^2 \\ & +\tfrac{1}{2}(1-|\Psi_d|^2)^2+\tfrac{4}{3}|\Psi_s|^4+\tfrac{8}{3}|\Psi_s|^2|\Psi_d|^2+\tfrac{2}{3}(\Psi_s^2\Psi_d^{*^2}+\Psi_d^2\Psi_s^{*^2}) \\ & +2|\Pi\,\Psi_s|^2+|\Pi\,\Psi_d|^2+\{\Pi_x\,\Psi_s\,\Pi_x^*\,\Psi_d^*-\Pi_y\,\Psi_s\,\Pi_y^*\,\Psi_d^*+\text{H.C.}\}\,, \end{aligned} \tag{1.1}$$

where Ψ_s is the s-wave order parameter, Ψ_d is the d-wave order parameter and A is the vector-valued magnetic potential, $\prod = i\nabla - A$, H is a constant applied magnetic field, κ is the Ginzburg-Landau parameter and

$$\alpha_s(T) = C_s/(1 - T/T_c)\,. \tag{1.2}$$

Here C_s is a positive constant, T is the current temperature and T_c is the d-wave transition temperature.

As the current temperature T is close to T_c, Franz et al. [8] observed that in a predominantly d-wave superconductor, the s-wave component is generically very small. They also provided approximation formulas for the order parameters Ψ_d and Ψ_s as follows:

$$|\Psi_s| \ll |\Psi_d|, \quad |\nabla\Psi_s| \ll |\nabla\Psi_d| \quad \text{as } T \to T_c\,. \tag{1.3}$$

Affleck et al. [1] obtained the leading order in $(1 - T/T_c)$ as

$$\Psi_s = \xi\,(\textstyle\prod_x^2 - \prod_y^2)\,\Psi_d\,, \tag{1.4}$$

where ξ is a parameter satisfying that $\xi \to 0$ as $T \to T_c$. In [7], Du derived (1.4) by the formal asymptotic analysis.

We learned from [5] and [6] that it is reasonable to ignore the magnetic field in strongly type II superconductors when the applied magnetic field is close to H_{c1} and $T \to T_c$. Hence it is valuable to study the two fields Ginzburg-Landau model (1.1) without the magnetic field (i.e. $A, H \equiv 0$). Moreover, Rosenstein et al. [6] took (1.3) and (1.4) into (1.1) and modified the free energy (1.1) as follows:

$$G(\Psi_d) = \int_{\mathbb{R}^2} |\nabla\Psi_d|^2 + \frac{1}{2}(1 - |\Psi_d|^2)^2 + \beta\,|\Box\Psi_d|^2\,dx\,dy\,, \tag{1.5}$$

where $\Box = \partial_x^2 - \partial_y^2$ and β is a parameter satisfying that $\beta \to 0$ as $T \to T_c$. Here we have ignored the magnetic field (i.e. $A, H \equiv 0$) for strongly type II superconductors.

It is hard to find the minimizer of (1.5) by the standard direct method. Suppose that $\Psi_d \in H^2(\mathbb{R}^2\,;\mathbb{C})$ is a minimizer of (1.5) over $H^2(\mathbb{R}^2\,;\mathbb{C})$. Then it is easy to check that

$$\begin{aligned} G(\Psi_d + v) = \ & G(\Psi_d) + \int_{\mathbb{R}^2} |\nabla v|^2 - (1 - |\Psi_d|^2)|v|^2 + 2(\Psi_d \cdot v)^2 \\ & + \int_{\mathbb{R}^2} 2|v|^2(\Psi_d \cdot v) + \frac{1}{2}|v|^4 + \beta|\Box v|^2\,, \end{aligned} \tag{1.6}$$

for any test function $v \in C_0^\infty(\mathbb{R}^2)$. Hereafter, $z_1 \cdot z_2 = \frac{1}{2}(\bar{z}_1 z_2 + z_1\bar{z}_2)$ for all $z_1, z_2 \in \mathbb{C}$. Let $v_n(z) = \delta_n v_0(z)\sin[\delta_n^{-2/3}(x+y)]$ for $z = x + iy \in \mathbb{C} \cong \mathbb{R}^2$, where v_0 is a test function with a nonempty compact support and $\{\delta_n\}$ is a sequence of positive numbers such that $\delta_n \to 0$ as $n \to \infty$. Here we use the fact that the complex plane $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$. Now, we replace v in (1.6) by v_n and we obtain that $G(\Psi_d + v_n) \to G(\Psi_d)$ but $\|\Psi_d + v_n\|_{H^2} \to \infty$ as $n \to \infty$. Hence $\Psi_d + v_n$'s form a minimizing sequence but $\Psi_d + v_n$'s have no converging subsequence even weakly converging subsequences in $H^2_{loc}(\mathbb{R}^2\,;\mathbb{C})$. Thus the free energy (1.5) has a defect on minimization.

From [30], we learned a Ginzburg-Landau energy functional (without the magnetic field) as follows:

$$E(\Psi_d)=\int_{\mathbb{R}^2}|\nabla\Psi_d|^2+\frac{1}{2}(1-|\Psi_d|^2)^2+\eta\,(|\partial_x^2\,\Psi_d|^2+|\partial_y^2\,\Psi_d|^2)\,dx\,dy\,, \tag{1.7}$$

where η is a constant depending on the current temperature T. The term $|\partial_x^2\Psi_d|^2+|\partial_y^2\Psi_d|^2$ breaks the circular symmetry and accounts for the square symmetry. Furthermore, Park and Huse [22] introduced a more generalized Ginzburg-Landau free energy (without the magnetic field) for d-wave superconductors as follows:

$$F(\Psi_d)=\int_{\mathbb{R}^2}|\nabla\Psi_d|^2+\frac{1}{2}(1-|\Psi_d|^2)^2+\gamma_1\,|\Delta\Psi_d|^2+\beta_1\,(|\Box\Psi_d|^2-4|\partial_x\partial_y\,\Psi_d|^2)\,dx\,dy\,, \tag{1.8}$$

where $\Delta=\partial_x^2+\partial_y^2$ and β_1,γ_1 are parameters tending to zero as $T\to T_c$.

Hereafter, we assume that $|\Psi_d|\to 1$ and all the derivatives of Ψ_d decay fast as $|(x,y)|\to\infty$. Such an assumption is consistent with the results in [8] and [31]. Using integration by part, we may transform (1.8) into

$$\tilde{G}(\Psi_d)=\int_{\mathbb{R}^2}|\nabla\Psi_d|^2+\frac{1}{2}(1-|\Psi_d|^2)^2+\beta\,|\Box\Psi_d|^2+\gamma\,|\Delta\Psi_d|^2\,dx\,dy\,, \tag{1.9}$$

where β,γ are parameters tending to zero as $T\to T_c$. In this paper, we assume that $\beta,\gamma>0$ and $\beta,\gamma\to 0$ as $T\to T_c$. In particular, such an assumption includes the case that $0<\gamma\ll\beta$ i.e. (1.9) is a small perturbation of (1.5).

In Section 2, we approximate (1.9) by

$$G_\epsilon(\Psi_d)=\int_{\frac{1}{\epsilon}\Omega}|\nabla\Psi_d|^2+\frac{1}{2}(1-|\Psi_d|^2)^2+\beta|\Box\Psi_d|^2+\gamma\,|\Delta\Psi_d|^2\,dx\,dy\,, \tag{1.10}$$

where $0<\epsilon\ll 1$ is a small parameter, Ω is a bounded smooth domain in $\mathbb{R}^2$ having an interior point at the origin and $\frac{1}{\epsilon}\Omega=\{(\frac{x}{\epsilon},\frac{y}{\epsilon}):(x,y)\in\Omega\}$. In the rest of this paper, we prove that the minimizer of (1.10) has distinct degree-one vortices in Section 3. In Section 4, we replace $\frac{1}{\epsilon}\Omega$ in (1.10) by B_{R_0}, where B_{R_0} is a disk with radius R_0 and center at the origin. Here $R_0>0$ is a large constant satisfying $1\ll R_0\le 1/\epsilon$. Then (1.10) becomes

$$\hat{G}(\Psi_d)=\int_{B_{R_0}}|\nabla\Psi_d|^2+\frac{1}{2}(1-|\Psi_d|^2)^2+\beta|\Box\Psi_d|^2+\gamma\,|\Delta\Psi_d|^2\,dx\,dy\,, \tag{1.11}$$

where $\beta>0$ is a small parameter as $T\to T_c$, $\gamma=C\beta$, and C is a positive constant independent of β. We study then the critical point of (1.11) and find out its single vortex structure with fourfold symmetry. The single vortex structure of d-wave superconductors having fourfold symmetry is well known in physics (cf. [5], [6], [8], [27] and [31]). Here we give a mathematical proof of such a vortex structure.

2 Preliminaries

To investigate vortices in d-wave superconductors, we assume that the order parameter Ψ_d satisfies $|\Psi_d| \to 1$ and all the derivatives of Ψ_d decay fast as $|(x,y)| \to \infty$. Such an assumption is consistent with the results in [8] and [31]. Hence we may approximate (1.9) by

$$G_\epsilon(\Psi_d) = \int_{\frac{1}{\epsilon}\Omega} |\nabla \Psi_d|^2 + \frac{1}{2}(1 - |\Psi_d|^2)^2 + \beta |\Box \Psi_d|^2 + \gamma\, |\Delta \Psi_d|^2 \, dx\, dy\,, \tag{2.1}$$

where $0 < \epsilon \ll 1$ is a small parameter, Ω is a bounded smooth domain in $\mathbb{R}^2$ having an interior point at the origin and $\frac{1}{\epsilon}\Omega = \{(\frac{x}{\epsilon}, \frac{y}{\epsilon}) : (x,y) \in \Omega\}$. Rescaling the spatial variables x, y by ϵ, (2.1) becomes

$$\hat{G}_\epsilon(\Psi_d) = \int_{\Omega} |\nabla \Psi_d|^2 + \frac{1}{2\epsilon^2}(1 - |\Psi_d|^2)^2 + \delta_\epsilon |\Box \Psi_d|^2 + \gamma_\epsilon\, |\Delta \Psi_d|^2 \, dx\, dy\,, \tag{2.2}$$

where

$$\delta_\epsilon = \beta\, \epsilon^2 \quad \text{and} \quad \gamma_\epsilon = \gamma\, \epsilon^2\,. \tag{2.3}$$

Of course, (2.3) implies that $0 < \delta_\epsilon, \gamma_\epsilon = O(\epsilon^2)$ as $\epsilon \to 0+$. In Section 2 and 3, we study (2.2) with an assumption that $0 < \delta_\epsilon, \gamma_\epsilon = O(\epsilon^2)$ as $\epsilon \to 0+$.

This kind of approximation can also be found in s-wave superconductors. The conventional s-wave Ginzburg-Landau free energy (cf. [9]) without the magnetic field is

$$\int_{\mathbb{R}^2} \frac{1}{2}|\nabla u|^2 + \frac{1}{4}(1 - |u|^2)^2\,,$$

where $u \in \mathbb{C}$ is the s-wave order parameter. Under the hypothesis that $|u| \to 1$ and all the derivatives of u decay fast at $|(x,y)| \to \infty$, we may approximate the s-wave Ginzburg-Landau free energy by

$$\int_{\frac{1}{\epsilon}\Omega} \frac{1}{2}|\nabla u|^2 + \frac{1}{4}(1 - |u|^2)^2\,,$$

where $0 < \epsilon \ll 1$ is a small parameter and Ω is a bounded smooth domain in $\mathbb{R}^2$ having an interior point at the origin. Then we rescale the spatial variables by ϵ and obtain the energy functional as follows:

$$E_\epsilon(u) = \int_{\Omega} \frac{1}{2}|\nabla u|^2 + \frac{1}{4\epsilon^2}(1 - |u|^2)^2\,, \tag{2.4}$$

where $u : \Omega \to \mathbb{C}$ is the s-wave order parameter. There are many investigations on the free energy (2.4). For the readers who are interested in these works, please refer to [2], [15], [17], [23] and [29] etc.

In [2] and [29], we learn the minimizer of E_ϵ over $H^1_g(\Omega)$ having n degree-one vortices in Ω, where

$$H^1_g(\Omega) = \{u \in H^1(\Omega; \mathbb{C}) \,:\, u = g \quad \text{on } \partial\Omega\}\,,$$

and $g : \partial\Omega \to S^1$ is smooth with degree $n \geq 1$. Furthermore, the minimizer u_ϵ of (2.4) satisfies

(1) $E_\epsilon(u_\epsilon) = n\pi \log \frac{1}{\epsilon} + W_g(a_1, \cdots, a_n) + o_\epsilon(1)$ as $\epsilon \to 0+$,

(2) u_ϵ converges to u_* (up to a subsequence) in $C^2_{loc}(\bar{\Omega}\backslash\{a_1, \cdots, a_n\})$ as $\epsilon \to 0+$,

(3) $(a_1, \cdots, a_n) \in \Omega^n$ is a global minimizer of the renormalized energy W_g defined in [2],

where $o_\epsilon(1)$ is a small quantity which tends to zero as $\epsilon \to 0+$,

$$u_*(z) = \prod_{j=1}^{n} \frac{z - a_j}{|z - a_j|} e^{i\,h(z)}, \quad \forall z \in \Omega, \tag{2.5}$$

and h is a real-valued harmonic function. Since $\mathbb{R}^2$ is isomorphic to $\mathbb{C}$, we may consider $\Omega \subset \mathbb{R}^2 \cong \mathbb{C}$. Note that the domain Ω is assumed star-shaped in [2]. However, Struwe [29] generalized results of [2] for all bounded smooth domains.

For the minimizer of (2.2), we prove:

Theorem I. *Suppose $0 < \delta_\epsilon, \gamma_\epsilon = O(\epsilon^2)$ as $\epsilon \to 0+$. Then there exists a minimizer u_ϵ of (2.2) over $H^1_g(\Omega)$ such that*

(i) $u_\epsilon \in H^2(\Omega)$ *has n degree-one vortices in Ω,*

(ii) $\hat{G}_\epsilon(u_\epsilon) = 2n\pi \log \frac{1}{\epsilon} + O(1)$ *as $\epsilon \to 0+$,*

(iii) u_ϵ *converges to u_* (up to a subsequence) strongly in $L^2(\Omega)$ and weakly in $H^1_{loc}(\Omega\backslash\{a_1, \cdots, a_n\})$,*

(iv) $(a_1, \cdots, a_n) \in \Omega^n$ *is a global minimizer of the renormalized energy W_g in [2].*

Remark. We may consider the energy functional (2.2) with $0 < \delta_\epsilon, \gamma_\epsilon = O(\epsilon^2)$ as a small perturbation of (2.4). However, the perturbation terms are of higher order derivatives. Hence the Euler-Lagrange equation of (2.2) is a singular perturbation problem and the perturbation terms are of the 4-th order derivatives. Until now, there is no general theorem on such a singular perturbation problem.

3 Proof of Theorem I

To prove the existence of a minimizer, we define a comparison map as follows:

$$U_\epsilon(z) = \prod_{j=1}^{n} U_0\left(\frac{z - b_j}{\epsilon}\right) e^{i\,H_\epsilon(z)}, \tag{3.1}$$

for $z \in \Omega \subset \mathbb{C}$, where b_j's are n distinct points in Ω and H_ϵ is a real-valued smooth function in Ω such that

$$U_\epsilon = g \quad \text{on } \partial\Omega, \quad \|H_\epsilon\|_{C^2(\Omega)} = O(1).$$

Hereafter, U_0 is the symmetric vortex solution (cf. [4], [10], [11]) defined by

$$U_0(z) = f(R)\, e^{i\theta} \quad \text{for } z \in \mathbb{C}, \tag{3.2}$$

where $R = |z|$ and (R, θ) is the polar coordinate in $\mathbb{C}$. Moreover, $f(R)$ satisfies

$$\begin{cases} f'' + \frac{1}{R} f' - \frac{1}{R^2} f + (1 - f^2) f = 0 \quad \text{for } R > 0, \\ f(0) = 0, f(\infty) = 1. \end{cases} \tag{3.3}$$

From [4] and [11], the symmetric vortex solution U_0 satisfies

Lemma I.

(i) $f(R) = \alpha_0 R + \alpha_1 R^3 + O(R^5)$ *as* $R \to 0+$, *where* $\alpha_0 > 0, \alpha_1 \in \mathbb{R}$ *are constants,*

(ii) $f(R) = 1 - \frac{1}{2R^2} + O(R^{-4})$ *as* $R \to +\infty$,

(iii) $U_0 = f(R)\, e^{i\theta}$ *is analytic in* $\mathbb{C}$.

Hence it is easy to check that

$$\hat{G}_\epsilon(U_\epsilon) = 2\pi n \log \frac{1}{\epsilon} + O(1) \quad \text{as } \epsilon \to 0+ . \tag{3.4}$$

Now, fix $0 < \epsilon \ll 1$. We claim that $\inf_{u \in H^1_g(\Omega)} \hat{G}_\epsilon(u)$ attains a minimizer $u_\epsilon \in H^2(\Omega)$. Let $\{u_k\}$ be a minimizing sequence such that

$$\hat{G}_\epsilon(u_k) \to \inf_{u \in H^1_g(\Omega)} \hat{G}_\epsilon(u). \tag{3.5}$$

Then by (2.2), (3.4) and (3.5), we have

$$\liminf_{k \to \infty} \int_\Omega |\nabla u_k|^2 + |\Box u_k|^2 + |\Delta u_k|^2 \, dx\, dy < +\infty .$$

Hence there exists a subsequence $\{u_{k_j}\}$ such that

$$\|u_{k_j}\|_{H^2} \le K_\epsilon, \quad \forall j \ge 1, \tag{3.6}$$

where $K_\epsilon > 0$ is a constant independent of j. Thus (3.6) implies

$$u_{k_j} \to u_\epsilon \quad \text{weakly in} \quad H^2(\Omega) \quad \text{as } j \to \infty. \tag{3.7}$$

Therefore by Fatou's lemma, u_ϵ is a minimizer of $\hat{G}_\epsilon$ over $H^1_g(\Omega)$.

From (2.2), (2.4), (3.4) and u_ϵ is a minimizer of $\inf_{u \in H^1_g(\Omega)} \hat{G}_\epsilon(u)$, we obtain

$$E_\epsilon(u_\epsilon) \le \pi\, n \log \frac{1}{\epsilon} + O(1). \tag{3.8}$$

Moreover, by (3.8) and [29], we have

$$E_\epsilon(u_\epsilon) = \pi\, n \log \frac{1}{\epsilon} + O(1)\,. \tag{3.9}$$

Hence (3.4) and (3.9) imply that

$$\delta_\epsilon \int_\Omega |\Box u_\epsilon|^2 \, dx\, dy = O(1)\,, \tag{3.10}$$

and

$$\gamma_\epsilon \int_\Omega |\Delta u_\epsilon|^2 \, dx\, dy = O(1)\,. \tag{3.11}$$

Thus we complete the proof of (ii).

By (3.9), Proposition 1.1 and 1.2 in [16], we complete the proof of (i). Furthermore, we obtain that u_ϵ converges to u_* (up to a subsequence) strongly in $L^2(\Omega)$ and weakly in $H^1_{loc}(\Omega\backslash\{a_1, \cdots, a_n\})$, where $a_1, \cdots, a_n \in \Omega$, $u_*(z) = \prod_{j=1}^{n} \frac{z-a_j}{|z-a_j|} e^{i\,h(z)}, \forall z \in \Omega \subset \mathbb{C}$ and h is a real-valued function. Now we show that h is a harmonic function as follows: Consider the Euler-Lagrange equation of $\hat{G}_\epsilon$ with respect to the minimizer u_ϵ. Then u_ϵ satisfies

$$\Delta u_\epsilon + \frac{1}{\epsilon^2}(1 - |u_\epsilon|^2)u_\epsilon - \delta_\epsilon \Box^2 u_\epsilon - \gamma_\epsilon \Delta^2 u_\epsilon = 0 \quad \text{in } \Omega\,. \tag{3.12}$$

Perform the wedge product with u_ϵ and (3.12). This is a standard trick to erase the cubic nonlinear term in (3.12) (cf. [26] and [29]). Then we have

$$u_\epsilon \wedge \Delta u_\epsilon - \delta_\epsilon\, u_\epsilon \wedge \Box^2 u_\epsilon - \gamma_\epsilon\, u_\epsilon \wedge \Delta^2 u_\epsilon = 0 \quad \text{in } \Omega\,. \tag{3.13}$$

Let $p \in C_0^\infty(\Omega)$ be a test function. Multiply (3.13) by p and integrate it on Ω. Then using integration by parts, we obtain

$$\begin{aligned} &- \int_\Omega (u_\epsilon \wedge \partial_x u_\epsilon)p_x + (u_\epsilon \wedge \partial_y u_\epsilon)p_y \\ &= \delta_\epsilon \int_\Omega (u_\epsilon \wedge \Box u_\epsilon)\Box p + 2(\partial_x u_\epsilon \wedge \Box u_\epsilon)p_x - 2(\partial_y u_\epsilon \wedge \Box u_\epsilon)p_y \\ &\quad + \gamma_\epsilon \int_\Omega (u_\epsilon \wedge \Delta u_\epsilon)\Delta p + 2(\partial_x u_\epsilon \wedge \Delta u_\epsilon)p_x + 2(\partial_y u_\epsilon \wedge \Delta u_\epsilon)p_y \end{aligned} \tag{3.14}$$

Here we have used the following formulas:

$$\begin{aligned} u \wedge \Delta u &= \partial_x (u \wedge \partial_x u) + \partial_y (u \wedge \partial_y u)\,, \\ u \wedge \Box^2 u &= \Box(u \wedge \Box u) - 2(u_x \wedge \Box u_x - u_y \wedge \Box u_y)\,, \\ u \wedge \Delta^2 u &= \Delta(u \wedge \Delta u) - 2(u_x \wedge \Delta u_x + u_y \wedge \Delta u_y)\,. \end{aligned}$$

Hence by $0 < \gamma_\epsilon, \delta_\epsilon = O(\epsilon^2)$, (3.9)-(3.11), (3.14) and Holder inequality, the limit map u_* satisfies

$$u_* \wedge \Delta u_* = 0 \quad \text{in distribution sense}\,. \tag{3.15}$$

Thus u_* is a canonical harmonic map i.e. h is a harmonic function. Therefore we complete the proof of (iii).

Now we prove (iv) as follows: Let $(\tilde{a}_1, \cdots, \tilde{a}_n) \in \Omega^n$ be a global minimizer of the renormalized energy W_g. The definition of W_g can be found in [2]. Then we define another comparison map as follows:

$$V_\epsilon(z) = \begin{cases} u_\epsilon(z - \tilde{a}_j + a_j) & \text{if } z \in B_{\epsilon^\alpha}(\tilde{a}_j)\,, j = 1, \cdots, n\,, \\ \tilde{U}_\epsilon(z) & \text{if } z \in \Omega_{\epsilon^\alpha} \equiv \Omega \setminus \cup_{j=1}^n B_{\epsilon^\alpha}(\tilde{a}_j)\,, \end{cases} \tag{3.16}$$

where $0 < \alpha < 1$ is a constant and $\tilde{U}_\epsilon$ is a minimizer of E_ϵ over $H^1_{\tilde{g}}(\Omega_{\epsilon^\alpha})$. Here the boundary condition $\tilde{g}$ is defined by

$$\tilde{g} = \begin{cases} g & \text{on } \partial\Omega\,, \\ u_\epsilon(\cdot - \tilde{a}_j + a_j) & \text{on } \partial B_{\epsilon^\alpha}(\tilde{a}_j)\,, j = 1, \cdots, n\,. \end{cases} \tag{3.17}$$

Hence by (iii), [2] and [29], $\tilde{U}_\epsilon$ satisfies

$$\tilde{U}_\epsilon \to \prod_{j=1}^n \frac{z - \tilde{a}_j}{|z - \tilde{a}_j|} e^{i\tilde{h}(z)} \quad \text{in } C^2(\Omega_{\epsilon^\alpha}) \quad \text{as } \epsilon \to 0+\,, \tag{3.18}$$

where $\tilde{h}$ is a harmonic function. The convergence of (3.18) may be up to a subsequence. However, this does not effect the following argument. Thus by (3.18) and [2], it is easy to check that

$$\hat{G}_\epsilon(V_\epsilon) = \sum_{j=1}^n \int_{B_{\epsilon^\alpha}(a_j)} \hat{g}_\epsilon(u_\epsilon) + 2\pi n\alpha \log\frac{1}{\epsilon} + 2W_g(\tilde{a}_1, \cdots, \tilde{a}_n) + o_\epsilon(1)\,, \tag{3.19}$$

where $\hat{g}_\epsilon(u) = |\nabla u|^2 + \frac{1}{2\epsilon^2}(1 - |u|^2)^2 + \delta_\epsilon|\Box u|^2 + \gamma_\epsilon|\Delta u|^2$ is the energy density of $\hat{G}_\epsilon$ and $o_\epsilon(1)$ is a small quantity which tends to zero as $\epsilon \to 0+$. On the other hand, by (iii) and [2], we have

$$\int_{\hat{\Omega}_{\epsilon^\alpha}} \frac{1}{2}|\nabla u_\epsilon|^2 + \frac{1}{4\,\epsilon^2}(1 - |u_\epsilon|^2)^2 \geq \pi n\alpha \log\frac{1}{\epsilon} + W_g(a_1, \cdots, a_n) + o_\epsilon(1)\,, \tag{3.20}$$

where $\hat{\Omega}_{\epsilon^\alpha} = \Omega \setminus \cup_{j=1}^n B_{\epsilon^\alpha}(a_j)$. Hence (3.20) implies that

$$\hat{G}_\epsilon(u_\epsilon) \geq \sum_{j=1}^n \int_{B_{\epsilon^\alpha}(a_j)} \hat{g}_\epsilon(u_\epsilon) + 2\pi n\alpha \log\frac{1}{\epsilon} + 2W_g(a_1, \cdots, a_n) + o_\epsilon(1)\,, \tag{3.21}$$

Thus by (3.19) and (3.21), we obtain

$$W_g(a_1, \cdots, a_n) \leq W_g(\tilde{a}_1, \cdots, \tilde{a}_n) + o_\epsilon(1) \tag{3.22}$$

Since $(\tilde{a}_1, \cdots, \tilde{a}_n)$ is a global minimizer of W_g, then we complete the proof of (iv) by (3.22).

4 Single Vortex Structure in the Vortex Core

In this section, we assume that the single vortex structure is in the vortex core B_{R_0}, where $R_0 > 0$ is a large constant satisfying $1 \ll R_0 \leq \frac{1}{\epsilon}$. Hereafter, we denote B_{R_0} as a disk in $\mathbb{R}^2$ with radius R_0 and center at the origin. To study the vortex structure in the vortex core, we restrict (1.9) in the vortex core B_{R_0} as follows:

$$\hat{G}(\Psi_d) = \int_{B_{R_0}} |\nabla \Psi_d|^2 + \frac{1}{2}(1 - |\Psi_d|^2)^2 + \beta\, |\Box \Psi_d|^2 + \gamma\, |\Delta \Psi_d|^2 \, dx\, dy\,, \tag{4.1}$$

where $\gamma = C\beta, C > 0$ is a constant independent of β, and $\beta > 0$ is a small parameter tending to zero as $T \to T_c$. We investigate (4.1) with $\beta > 0$ a small parameter to see the phase transition of d-wave superconductors.

The Euler-Lagrange equation of (4.1) is

$$\Delta \Psi_d + (1 - |\Psi_d|^2)\Psi_d - \beta\,(\Box^2 + C\,\Delta^2)\Psi_d = 0 \quad \text{in } B_{R_0}. \tag{4.2}$$

Note that $E \equiv \Box^2 + C\,\Delta^2$ is an elliptic operator as $C > 0$. Moreover, by the Lax-Milgram theorem, $E : H_0^2(B_{R_0}\,; \mathbb{C}) \to H^{-2}(B_{R_0}\,; \mathbb{C})$ is invertible and we denote E^{-1} as its inverse. Hence the standard elliptic regularity theorem (cf. [20]) can be applied in (4.2).

We state the main result on (4.2) as follows:

Theorem II. *There exists a solution Ψ_d of (4.2) satisfying*

$$\Psi_d(z, \beta) = f(R)\, e^{i\,\theta} + \beta(a(R)\, e^{-4i\,\theta} + b(R)\, e^{4i\,\theta} + c(R))\, e^{i\,\theta} + O(\beta^2) \quad \text{as } \beta \to 0\,, \tag{4.3}$$

where a, b and c are smooth real-valued functions.

The equation (4.3) implies that the d-wave order parameter Ψ_d is fourfold symmetric in the vortex core. In [27], we learn a well approximated solution of (4.2) with fourfold symmetry. Here we find an exact solution of (4.2) with the fourfold symmetry.

Proof of Theorem II.

To solve (4.2), we set

$$\Psi_d(z, \beta) = U_0(z) + \beta w_1(z) + \beta^2 w_2(z) + \beta^3 w(z, \beta)\,, \tag{4.4}$$

where U_0 is the symmetric vortex solution defined in (3.2) and (3.3). Here w_1 satisfies

$$L\, w_1 - E\, U_0 = 0 \ \text{ in } B_{R_0}\,, \quad w_1 = 0 \ \text{ on } \partial B_{R_0}\,, \tag{4.5}$$

where $Lv = \Delta v + (1 - |U_0|^2)v - 2(U_0 \cdot v)U_0$ is the linearized operator of the equation (4.2) with respect to a trivial solution $(\Psi_d, \beta) = (U_0, 0)$. In addition, w_2 satisfies that

$$\begin{aligned} L\, w_2 \; &= 2(U_0 \cdot w_1)w_1 + |w_1|^2 U_0 + E\, w_1 \quad \text{in } B_{R_0}\,, \\ w_2 \; &= 0 \quad \text{on } \partial B_{R_0}\,. \end{aligned} \tag{4.6}$$

It is easy to check that

$$EU_0 = h_{-3}(R)\,e^{-3i\theta} + h_1(R)\,e^{i\theta} + h_5(R)\,e^{5i\theta}\,, \tag{4.7}$$

where h_{-3}, h_1 and h_5 are real-valued smooth functions. By [14], [18], [19] and [23], L is a bijection from $H_0^1(B_{R_0};\mathbb{C})$ onto $H^{-1}(B_{R_0};\mathbb{C})$. Hence by (4.5)-(4.7), we have

$$w_1 = a(R)\,e^{-3i\theta} + b(R)\,e^{5i\theta} + c(R)\,e^{i\theta}\,, \tag{4.8}$$

$$w_2 = \sum_{k=0}^{2} a_{1-4k}(R)\,e^{i\,(1-4k)\,\theta} + a_{1+4k}(R)\,e^{i\,(1+4k)\,\theta}\,, \tag{4.9}$$

where a, b, c and $a_{1\pm 4k}$'s are smooth real-valued functions.

Taking (4.4) into (4.2), we obtain that

$$\begin{aligned} L\,w = \;& 2[(U_0\cdot(w_2+\beta w))w_1 + (U_0\cdot w_1)(w_2+\beta w)] + \beta|w_2+\beta w|^2 U_0 \\ &+\beta(U_0\cdot(w_2+\beta w))(w_2+\beta w) + 2(w_1\cdot(w_2+\beta w))U_0 \\ &+|w_1+\beta(w_2+\beta w)|^2(w_1+\beta(w_2+\beta w)) + Ew_2 + \beta Ew \quad \text{in } B_{R_0}\,. \end{aligned} \tag{4.10}$$

Hence (4.10) is equivalent to

$$\begin{aligned} E^{-1}L\,w = \;& E^{-1}\{2[(U_0\cdot(w_2+\beta w))w_1 + (U_0\cdot w_1)(w_2+\beta w)] + \beta|w_2+\beta w|^2 U_0 \\ &+\beta(U_0\cdot(w_2+\beta w))(w_2+\beta w) + 2(w_1\cdot(w_2+\beta w))U_0 \\ &+|w_1+\beta(w_2+\beta w)|^2(w_1+\beta(w_2+\beta w))\} + w_2 + \beta w \quad \text{in } B_{R_0}\,. \end{aligned} \tag{4.11}$$

Note that (4.11) has a trivial solution $(w,\beta) = (w_3, 0)$, where w_3 satisfies that

$$\begin{aligned} L\,w_3 &= 2[(U_0\cdot w_2)w_1 + (U_0\cdot w_1)w_2] + 2(w_1\cdot w_2)U_0 + |w_1|^2 w_1 + E\,w_2 \ \text{ in } B_{R_0}\,, \\ w_3 &= 0 \ \text{ on } \partial B_{R_0}\,. \end{aligned} \tag{4.12}$$

Since U_0, w_1, w_2 are smooth functions and L is bijective from $H_0^1(B_{R_0};\mathbb{C})$ onto $H^{-1}(B_{R_0};\mathbb{C})$, then the standard elliptic regularity theorem implies that w_3 is also a smooth function. Furthermore, since E is bijective from $H_0^2(B_{R_0};\mathbb{C})$ onto $H^{-2}(B_{R_0};\mathbb{C})$ and $H^{-1}(B_{R_0};\mathbb{C})$ is embedded in $H^{-2}(B_{R_0};\mathbb{C})$, then E is a bijection from $H_0^2(B_{R_0};\mathbb{C}) \cap H^3(B_{R_0};\mathbb{C})$ onto $H^{-1}(B_{R_0};\mathbb{C})$. We denote E^{-1} as the inverse of E. Hence $E^{-1}L$ is a bijection from $H_0^1(B_{R_0};\mathbb{C})$ onto $H_0^2(B_{R_0};\mathbb{C}) \cap H^3(B_{R_0};\mathbb{C})$. Thus by the implicit function theorem, (4.11) has a unique solution $w \in H_0^1(B_{R_0};\mathbb{C})$ as $|\beta|$ is sufficiently small. Moreover, the standard elliptic regularity theorem may imply the smoothness of w. Therefore (4.2) has a solution Ψ_d satisfying (4.4) as $|\beta|$ is sufficiently small. By (4.4), (4.8) and (4.9), we obtain (4.3) and complete the proof of Theorem II.

Final Remark: By (1.4) with $A \equiv 0$ and (4.3), we have

$$\Psi_s(z) = \xi\Box[U_0 + \beta(a(R)\,e^{-4i\theta} + b(R)\,e^{4i\theta} + c(R))\,e^{i\theta} + O(\beta^2)] \quad \text{as } \beta \to 0\,. \tag{4.13}$$

Since $U_0(z) = f(R)\,e^{i\theta}$, then

$$\Box U_0(z) = \frac{1}{2}(f' + \frac{1}{R}f)'\,e^{-i\theta} + \frac{1}{2}[(f' - \frac{1}{R}f)' - \frac{2}{R}(f' - \frac{1}{R}f)]\,e^{3i\theta}\,. \tag{4.14}$$

Hence by (i), (ii) of Lemma I and (4.14), $\Box U_0$ satisfies

$$\Box U_0(z) = 4\alpha_1 R\, e^{-i\theta} + O(R^3) \quad \text{as } R \to 0+\,, \tag{4.15}$$

and

$$\Box U_0(z) = -\frac{1}{2R^2}\, e^{-i\theta} + \frac{3}{2R^2}\, e^{3i\theta} + O(R^{-4}) \quad \text{as } R \to +\infty\,. \tag{4.16}$$

By (4.15) and (4.16), the degree of $\Box U_0$ is minus one in B_{r_1} and three in B_{R_1} as $0 < r_1 \ll 1$ and $R_1 \gg 1$. Moreover, by [4] and [11], it is easy to check that

$$\frac{d}{dz}\,\Box\, U_0(z) \neq 0 \quad \text{if } \Box U_0(z) = 0\,. \tag{4.17}$$

Hence (iii) of Lemma I and (4.17) imply that $\Box U_0$ has only simple zeros in $\mathbb{C}$. Thus $\Box U_0$ has a single zero with degree minus one at the origin and another four zeros with degree one away from the origin. Therefore as $|\beta|$ is sufficiently small, Ψ_s has a single zero with degree minus one near the origin and another four zeros with degree one away from the origin. This indicates the four-lobe structure of Ψ_s in the vortex core. The numerical simulation can be found in [7], [8] and [31].

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