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Concentration Behavior of Blow-up Solutions for a Simplified System of Chemotaxis

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1 Introduction

This paper is concerned with a system of partial differential equations proposed by Keller and Segel [19] which is a mathematical model for chemotaxis describing aggregation of organisms sensitive to gradient of a chemical substance. The Keller-Segel model is described as the following system :

$$
\text{(KS)}\qquad \begin{cases} \dfrac{\partial u}{\partial t} = \nabla\cdot(\nabla u - \chi u \nabla v) & \text{in } \Omega, \quad t>0, \\ \tau\dfrac{\partial v}{\partial t} = \Delta v - \gamma v + \alpha u & \text{in } \Omega, \quad t>0, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega, \quad t>0, \\ u(\cdot,0)=u_0, \quad v(\cdot,0)=v_0 & \text{on } \Omega. \end{cases}
$$

Here Ω is a bounded domain in $\mathbf{R}^2$ with smooth boundary $\partial\Omega$, and τ, χ, γ and α are positive constants. $u(x,t)$ and $v(x,t)$ represent the density of the organisms and the concentration of the chemical substance at place x and time t respectively, and u_0 and v_0 are non-negative smooth functions on Ω. Finite-time blow-up of solutions is one of interesting aspects of the Keller-Segel model(see Nanjundiah [25]), and a conjecture in two space dimensions by Childress [10] and Childress and Percus [11] states that there exists a threshold number c such that if $\|u_0\|_{L^1(\Omega)} < c$ then the solution (u,v) exists globally in time, and if $\|u_0\|_{L^1(\Omega)} > c$ then $u(x,t)$ can form a delta function singularity in finite time. Such a blow-up phenomenon is refered to as chemotactic collapse. In the case of radial initial functions (u_0, v_0) on $\Omega = D_L = \{x \in \mathbf{R}^2; |x| < L\}$, the threshold number is conjectured as $c = 8\pi/(\alpha\chi)$, which is supported by [14, 15, 17, 23, 24].

First, Jäger and Luckhaus [17] have dealt with the system :

$$
\text{(JL)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \nabla\cdot(\nabla u - \chi u \nabla v) & \text{in} \quad \Omega, \qquad t > 0, \\ 0 = \Delta v + \alpha(u - \overline{u_0}) & \text{in} \quad \Omega, \qquad t > 0, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on} \quad \partial\Omega, \quad t > 0, \\ u(\cdot, 0) = u_0 & \text{on} \quad \Omega, \end{cases}
$$

which describes the limiting case of $\tau \downarrow 0$ in (KS), where $\overline{w} = (1/|\Omega|)\int_\Omega w dx$, and χ, $\alpha \sim 1$ and $\gamma \sim \tau$. For this system, they showed the global existence of solutions in time when the initial functions have small enough mass, and that there exist radial solutions which blow up at the origin in finite time. Later Herrero and Velázquez [14] succeeded in constructing radial solutions on $\Omega = D_L$ collapsing in finite time by the method of matched asymptotic expansions. Nagai [23] studied another system

$$
\text{(P)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \nabla\cdot(\nabla u - \chi u \nabla v) & \text{in} \quad \Omega, \qquad t > 0, \\ 0 = \Delta v - \gamma v + \alpha u & \text{in} \quad \Omega, \qquad t > 0, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on} \quad \partial\Omega, \quad t > 0, \\ u(\cdot, 0) = u_0 & \text{on} \quad \Omega. \end{cases}
$$

That is, (KS) with $\tau = 0$. He confirmed that blow-up of radial solutions requires the threshold number $8\pi/(\alpha\chi)$ in L^1 norm for radial functions u_0 on $\Omega = D_L$ as follows :

1. If $\|u_0\|_{L^1(\Omega)} < 8\pi/(\alpha\chi)$, then the solution (u, v) exists globally in time and is globally bounded;

2. If $\|u_0\|_{L^1(\Omega)} > 8\pi/(\alpha\chi)$ and $\int_\Omega u_0(x)|x|^2 dx$ is sufficiently small, then the solution (u, v) blows up at the origin in finite time.

Concerning the original system (KS), Herrero and Velázquez [15] also showed the occurence of chemotactic collapse by using the same method as in [14]. Nagai, Senba and Yoshida [24] have proved the time global existence and L^∞ estimate as follows, which also hold in (P):

1. If $\|u_0\|_{L^1(\Omega)} < 4\pi/(\alpha\chi)$, then the solution (u, v) exists globally in time and is globally bounded;

2. If $\Omega = D_L$, (u_0, v_0) be radial in x and $\|u_0\|_{L^1(\Omega)} < 8\pi/(\alpha\chi)$, then the radial solution (u, v) exists globally in time and is globally bounded.

All those works on blow-up have treated the case of radial symmetry. Our aim is to investigate non-radially symmetric case mostly. We deal with the system (P), assuming the following :

(A1) α, γ and χ are positive constants.

(A2) Ω is a bounded domain of $\mathbf{R}^2$ with smooth boundary $\partial\Omega$.

(A3) u_0 is smooth, non-negative and non-trivial on $\overline{\Omega}$.

Two kind of problems are studied here. The first one is related to above results of Nagai, Senba and Yoshida [24]. That is, what happens if $\Omega = D_L$, $4\pi/(\alpha\chi) \leq \|u_0\|_{L^1(D_L)} < 8\pi/(\alpha\chi)$, and u_0 is non-radially symmetric ? We prove the following.

1. There is a criterion for time global existence, which is regarded as an improvement of Nagai [23] mentioned above.

2. If the solution blows up in finite time, then there exists a blow-up point on $\partial\Omega$.

The second problem is on whether chemotactic collapse actually occurs. Around the isolated blow-up point we show the following :

1. If the solution blows up in finite time, then u concentrates and forms a delta function singularity at each isolated blow-up point.

2. If the point is in Ω and on $\partial\Omega$, the concentrated mass of u is no less than $8\pi/(\alpha\chi)$ and $4\pi/(\alpha\chi)$, respectively.

The first result combined with Nagai [23] implies that in radially symmetric case if $\|u_0\|_{L^1(\Omega)} > 8\pi/(\alpha\chi)$ and $\int_{D_L} u_0(x)|x|^2dx$ is sufficiently small, then u forms a delta function singularity at the origin. On the other hand the second result allows us to estimate the number of isolated blow-up points by $\|u_0\|_{L^1(\Omega)}$.

2 Fundamental Properties of Solutions to (P)

In this section, we describe some fundamental properties of solutions to (P). Solutions to (KS) or (JL) satisfy similar properties. Our main results are stated in the following section in details.

¿From now on, we put that

$$\alpha = \gamma = \chi = L = 1 \quad \text{and} \quad D = D_1$$

for simplicity.

Proposition 2.1 *Given a smooth non-negative initial value u_0, we have a unique classical solution (u, v) to (P) defined on a maximal interval of existence $[0, T_{max})$. It is smooth in $\overline{\Omega} \times (0, T_{max})$ and satisfies the following.*

(i) $u(x,t) > 0$, $v(x,t) > 0$ *for any* $(x,t) \in \overline{\Omega} \times (0, T_{max})$.

(ii) $\|u(\cdot,t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)}$ *for any* $t \in [0, T_{max})$.

(iii) $\|v(\cdot,t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)}$ *for any* $t \in [0, T_{max})$.

(iv) *For each* $p \in [1,2)$, *there exists a positive constant* C_p *such that* $\|\nabla v(\cdot,t)\|_{L^p(\Omega)} \leq C_p\|u_0\|_{L^1(\Omega)}$ *for any* $t \in [0, T_{max})$.

(v) *For each* $q \in [1,\infty)$, *there exists a positive constant* C_q *such that* $\|v(\cdot,t)\|_{L^q(\Omega)} \leq C_q\|u_0\|_{L^1(\Omega)}$ *for any* $t \in [0, T_{max})$.

Nagai [23] has shown the existence and uniqueness of solution to (P) and (i). See also Yagi [30]. Identities (ii) and (iii) are shown by a simple calculation. Property (iv) is a consequence of the L^1 estimate of Brezis and Struss [6], and Sobolev's imbedding theorem gives (v).

We here mention what holds for (KS) in short. First, its solution satisfies (i). For each $p \in [1,2)$, $\|v(\cdot,t)\|_{W^{1,p}(\Omega)}$ is estimated by $\|u_0\|_{L^1(\Omega)}$ and $\|v_0\|_{W^{1,p}(\Omega)}$. Finally, $\|v(\cdot,t)\|_{L^q(\Omega)}$ is estimated by $\|u_0\|_{L^1(\Omega)}$ and $\|v_0\|_{L^q(\Omega)}$ for each $q \geq 1$.

Returning to (P), we have the following.

Lemma 2.1 *Let* (u,v) *be a solution to (P). Put*

$$W(t) = \int_\Omega \left\{u\log u - \frac{1}{2}\left(|\nabla v|^2 + v^2\right)\right\} dx.$$

Then, it holds that

$$\frac{d}{dt}W(t) + \int_\Omega u|\nabla\cdot(\log u - v)|^2 dx = 0 \qquad \text{for any } t \in (0, T_{max}). \tag{1}$$

A corresponding identity is known for (KS) by Nagai, Senba and Yoshida [24]. Lemma 2.1 follows similarly in use of

$$\int_\Omega (|\nabla v|^2 + v^2)dx = \int_\Omega uv dx, \tag{2}$$

which is a consequence of the second equation of (P).

Next, we describe the norm behavior of solutions for $T_{max} < \infty$. The following proposition is proven in Appendix.

Proposition 2.2 *If* $T_{max} < \infty$, *then the following relations hold.*

(i) $\lim_{t\to T_{max}} \|u\log u\|_{L^1(\Omega)} = \infty$.

(ii) $\lim_{t\to T_{max}} \|\nabla v\|_{L^2(\Omega)} = \infty$.

(iii) $\lim_{t\to T_{max}} \int_\Omega e^{av(x,t)}dx = \infty$ *for any* $a > 1/2$.

From, by (i) and (iii) it follows that

$$\lim_{t\to T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \lim_{t\to T_{max}} \|v(\cdot,t)\|_{L^\infty(\Omega)} = \infty.$$

Following this fact, we say that the solution blows up in finite time then.

Solutions to (KS) also satisfy Proposition 2.2 just replacing $a > 1/2$ by $a > 1$ in (iii). Concluding the present section, we prepare several notations for later use.

Notation

(i) $B(q,\varepsilon) = \{x \in \mathbf{R}^2 | |x-q| < \varepsilon\}$ and $D = B(0,1)$, where $q \in \mathbf{R}^2$ and $\varepsilon > 0$.

(ii) $A(q,\eta,\varepsilon) = B(q,\eta)\backslash B(q,\varepsilon)$.

(iii) $B_3(Q,\varepsilon) = \{x \in \mathbf{R}^3 | |x-q| < \varepsilon\}$, where $Q \in \mathbf{R}^3$ and $\varepsilon > 0$.

(iv) $\sharp K$ = the number of elements of a set K.

(v) $w^+ = \max\{w,0\}$, $w^- = \max\{-w,0\}$ for a function w.

(vi) $\mathcal{M}(\mathcal{S}) = \{$ Radon measures on $\mathcal{S}\}$, where $\mathcal{S}$ denotes a compace Hausdorff space.

(vii) Weak star limit in $\mathcal{M}(\mathcal{S})$ is denoted by w^*- lim.

(viii) $\delta(\cdot)$ = Dirac's delta function in $\mathbf{R}^2$. $\delta_q(\cdot) = \delta(\cdot - q)$, where $q \in \mathbf{R}^2$.

(ix) $|\Omega|$ = the Lebegue measure of Ω, where Ω is a domain of $\mathbf{R}^2$.

(x) $⨍_{S^2} f d\mu = \frac{1}{4\pi}\int_{S^2} f d\mu$, $⨍_\Omega f dx = \frac{1}{|\Omega|}\int_\Omega f dx$ for a domain Ω of $\mathbf{R}^2$, and $⨍_{\partial D_L} f d\mu = \frac{1}{2\pi L}\int_{\partial D_L} f d\mu$.

Definition

(i) We say that q is a blow-up point of u if there exist $\{t_k\}_{k=1}^\infty \subset [0, T_{max})$ and $\{x_k\}_{k=1}^\infty \subset \overline{\Omega}$ satisfying $u(x_k,t_k) \to \infty$, $t_k \to T_{max} < \infty$ and $x_k \to q \in \overline{\Omega}$ as $k \to \infty$. We denote the set of all blow-up points of u by $\mathcal{B}$.

(ii) For $q \in \mathcal{B}$, we say that q is an isolated blow-up point of u if there exists a positive constant η such that

$$\sup_{0\le t<T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega\bigcap A(q,\eta,\varepsilon))} < \infty \qquad \text{for any } \varepsilon \in (0,\eta).$$

We denote the set of all isolated blow-up points of u by $\mathcal{B}_I$.

3 Main Results

In this section, we state main results. First three theorems, devoted to the case where $\Omega = D$, are related to the conjecture by Childress and Percus [11] mentioned in Introduction. Those results include an improvement of Nagai [23], and a description of the concentration behavior of blow-up solutions.

Theorem 1 *Suppose*

$$\Omega = D \quad \text{and} \quad \|u_0\|_{L^1(D)} < 8\pi. \tag{3}$$

Let

$$u_0(x) = u_0(-x) \qquad \text{in } D. \tag{4}$$

Then (P) admits a unique classical solution (u, v) *in* $\overline{D} \times (0, \infty)$ *satisfying*

$$\sup_{0 \le t}\{\|u(\cdot,t)\|_{L^\infty(D)} + \|v(\cdot,t)\|_{L^\infty(D)}\} < \infty. \tag{5}$$

Theorem 2 *Under the circumstances (3), if* $T_{max} < \infty$ *then it holds that*

$$\lim_{t \to T_{max}} \int_{\partial D} e^{av(x,t)} d\mu = \infty \tag{6}$$

for any $a > a*/2$*, where*

$$a_* = \frac{8\pi - \sqrt{8\pi(8\pi - \|u_0\|_{L^1(\Omega)})}}{\|u_0\|_{L^1(\Omega)}}. \tag{7}$$

Theorem 3 *Let (3) hold and* a_* *be the same one as in Theorem 2. If* $T_{max} < \infty$*, then for each* $a > a_*$ *there exists a continuous map* $q(\cdot)$ *from* $[0, T_{max})$ *to* ∂D *satisfying*

$$\liminf_{t \to T_{max}} \int_{D \bigcap B(q(t),\varepsilon)} u(x,t)dx \ge \frac{2\pi}{a} \qquad \text{for any } \varepsilon > 0.$$

In a similar way, we can prove that the solutions to (KS) satisfy the following properties under the assumption (3).

(i′) If (4) holds, then $T_{max} = \infty$ and the solution satisfies (5).

(ii′) If $T_{max} < \infty$, then for $a > 1$ relation (6) holds and it follows that $\mathcal{B} \bigcap \partial D \neq \emptyset$.

(i′) is an improvement of the case 2 of Nagai, Senba and Yoshida [24] mentioned in Introduction.

Now we describe the results on the general domain Ω. The first one is in connection with Theorem 3. Note $a_* > 1/2$ in (7).

Theorem 4 *Suppose that* $4\pi \leq M = \|u_0\|_{L^1(\Omega)} < 8\pi$ *and* $T_{max} < \infty$. *Given a sequence* $\{t_l\}_{l=1}^{\infty}$ *of* $[0, T_{max})$ *with* $\lim_{l\to\infty} t_l = T_{max}$, *we have a subsequence* $\{t'_l\}_{l=1}^{\infty}$ *of* $\{t_l\}_{l=1}^{\infty}$ *and a point* $q \in \mathcal{B} \bigcap \partial\Omega$ *of* u *satisfying*

$$\liminf_{l\to\infty} \int_{\Omega \bigcap B(q,\varepsilon)} u(x, t'_l)dx \geq 4\pi \qquad \text{for any } \varepsilon > 0 \tag{8}$$

and

$$w^*\text{-}\lim_{l\to\infty} \frac{\exp(av(\cdot, t'_l))}{\int_\Omega \exp(av(x, t'_l))dx} = \delta_q \qquad \text{in } \mathcal{M}(\overline{\Omega}) \qquad \text{for any } a \in (1/2, a^*), \tag{9}$$

where

$$a^* = \begin{cases} 2\pi/(M - 4\pi) & \text{if } M > 4\pi \\ \infty & \text{if } M = 4\pi. \end{cases}$$

Furthermore, q *is also a blow-up point of* v.

When $M = 4\pi$, it holds that

$$w^*\text{-}\lim_{l\to\infty} u(\cdot, t'_l) = 4\pi\delta_q \qquad \text{in } \mathcal{M}(\overline{\Omega}). \tag{10}$$

In fact, we have

$$\lim_{l\to\infty} \int_{\Omega \bigcap B(q,\varepsilon)} u(x, t'_l)dx = 4\pi$$

and

$$\lim_{l\to\infty} \int_{\Omega \setminus B(q,\varepsilon)} u(x, t'_l)dx = 0$$

for any $\varepsilon > 0$. Hence we have (10).

Next theorem describes that chemotactic collapse occurs at each isolated blow-up point.

Theorem 5 *Given* $q \in \mathcal{B}_I$, *we have two positive constants* ε, $m \geq m_*$ *and a non-negative function* $f \in L^1(B(q,\varepsilon) \bigcap \Omega) \bigcap C(\overline{B(q,\varepsilon) \bigcap \Omega} \setminus \{q\})$ *such that*

$$w^*\text{-}\lim_{t\to T_{max}} u(\cdot, t) = m\delta_q + f \qquad \text{in } \mathcal{M}(\overline{B(q,\varepsilon) \bigcap \Omega}),$$

where

$$m_* = \begin{cases} 4\pi & \text{if } q \in \partial\Omega, \\ 8\pi & \text{if } q \in \Omega. \end{cases}$$

From property (ii) of Proposition 2.1, this implies the following.

Corollary 1 *If* $T_{max} < \infty$, $\mathcal{B}_I$ *satisfies that*

$$^\sharp\{\mathcal{B}_I \bigcap \Omega\} + \frac{1}{2}{}^\sharp\{\mathcal{B}_I \bigcap \partial\Omega\} \leq \frac{1}{8\pi}\|u_0\|_{L^1(\Omega)}.$$

Combining Nagai [23] with Theorem 5, we get also the following.

Corollary 2 *Suppose that* $\Omega = D$ *and that* u_0 *is radially symmetric in* x. *If* $T_{max} < \infty$, *then there exist a positive constant* $m \geq 8\pi$ *and a non-negative function* $f \in L^1(D) \bigcap C(\overline{D} \backslash \{0\})$ *such that*

$$w^*\text{-} \lim_{t \to T_{max}} u(\cdot, t) = m\delta_0 + f \qquad \text{in } \mathcal{M}(\overline{D}).$$

When $M = 8\pi$, it holds that

$$w^*\text{-} \lim_{t \to T_{max}} u(\cdot, t) = 8\pi\delta_0 \qquad \text{in } \mathcal{M}(\overline{D}). \tag{11}$$

In fact, we have

$$\lim_{t \to T_{max}} \int_{B(0,\varepsilon)} u(x,t)dx = 8\pi$$

and

$$\lim_{t \to T_{max}} \int_{D \backslash B(0,\varepsilon)} u(x,t)dx = 0$$

for any $\varepsilon > 0$. Hence we have (11).

In fact, Nagai [23] has shown that in the radially symmetric case if $T_{max} < \infty$ then $\mathcal{B} = \{0\}$ and also that if $\|u_0\|_{L^1(D)} > 8\pi$ and $\int_D u_0(x)|x|^2 dx \ll 1$ then $T_{max} < \infty$. Corollary 2 describes that in that case all blow-up solutions form a delta function singularity. Method of asymptotic expansion may construct such a solution with $m = 8\pi$, as Herrero and Velázquez [14], [15] have done for (JL) and (KS), respectively.

Plan of this paper is as follows. In Section 4, we prove Theorem 1. In Section 5, we prove Theorem 2. In Section 6, we prove Theorem 4. In Section 7, we prove Theorem 5. In Section 8, we prove Theorem 3. In Appendix, we prove Proposition 2.2, and give a sharp constant in Moser-Onofri type inequality.

4 Time Global Existence via Symmetry

4.1 Proof of Theorem 1

In this section, we show Theorem 1. Most arguments are similar to [24] except for using another kind of Onofri's inequality. The following lemma holds for the general domain similarly to Lemma 3.4 of [24] and proof is omitted. Henceforth, we put $M = \|u_0\|_{L^1(\Omega)}$.

Lemma 4.1 *Suppose that* (u, v) *is a solution of (P). Let* a *be an arbitrary positive constant. Then, the inequality*

$$a \int_\Omega uv dx \leq \int_\Omega u \log u dx + M \log \left(\int_\Omega e^{av(x,t)} dx \right) - M \log M$$

holds for any $t \in [0, T_{max})$.

The following lemma is an immediate consequence of (2), Lemmas 2.1 and 4.1.

Lemma 4.2 *Suppose that (u, v) is a solution of (P). Let $a \in \mathbf{R}$. Then, for any $t \in [0, T_{max})$, the following inequality holds.*

$$\left(a - \frac{1}{2}\right) \int_\Omega \left(|\nabla v|^2 + v^2\right) dx \le W(0) + M \log \left(\int_\Omega e^{av} dx\right) - M \log M.$$

Onofri type inequality is generally referred to as follows:

For a class of functions on S^2 there exist constants $C > 0$ and K such that

$$\log \left(\fint_{S^2} e^f d\mu\right) \le C \int_{S^2} |\text{grad } f|^2 d\mu + \fint_{S^2} f d\mu + K. \tag{12}$$

Moser [21] has proved (12) with $C = 1/(16\pi)$ for H^1 functions on S^2. Onofri [27] and Hong [16] have independently proved it with $C = 1/(16\pi)$ and $K = 0$, which are best possible. Moser [22] and Aubin [2] have proved (12) with $C = 1/(32\pi)$ and $(1+\varepsilon)/(32\pi)$ for C^1 functions satisfying $f(x) = f(-x)$ on S^2 and $\int_{S^2} e^f x d\mu = \vec{0}$, respectively. See [7] and the references therein for their geometric backgrounds.

We make use the following version, of which proof is given in the next subsection.

Proposition 4.1 *If a function w on $\overline{D}$ satisfies that*

$$w \in C^1(\overline{D}), \qquad w(x) = w(-x) \quad \text{on } \partial D \quad \text{and} \quad \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial D, \tag{13}$$

then there exist absolute constants $C > 0$ and K such that

$$\log \left(\fint_D e^w dx\right) \le \frac{1}{16\pi} \int_D |\nabla w|^2 dx + C \fint_D w dx + K. \tag{14}$$

Proof of Theorem 1: Assumption (4) implies

$$v(x, t) = v(-x, t) \qquad \text{for any } (x, t) \in D \times [0, T_{max}),$$

by which together with Proposition 4.1 and Lemma 4.2 it follows that

$$\left(a - \frac{1}{2} - \frac{Ma^2}{16\pi}\right) \int_D \left(|\nabla v|^2 + v^2\right) dx \le W(0) - M \left(\frac{CaM}{|D|} + K - \log M\right).$$

Becauce of $\|u_0\|_{L^1(D)} = M < 8\pi$, we can take a constant a satisfying

$$a - \frac{1}{2} - \frac{Ma^2}{16\pi} > 0.$$

This gives

$$\sup_{0\le t<T_{max}} \int_D \left(|\nabla v|^2 + v^2\right) dx < \infty, \tag{15}$$

and hence $T_{max} = \infty$ by the case (ii) of Proposition 2.2. Lemma 2.1 and (15) imply

$$\sup_{0\le t<T_{max}} \int_\Omega u \log u dx < \infty.$$

Then we have

$$\sup_{0\le t<T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty$$

similary to [24], by which together with the standard arguments of the elliptic equation we have

$$\sup_{0\le t<T_{max}} \|v(\cdot,t)\|_{L^\infty(\Omega)} < \infty. \qquad \square$$

4.2 Moser-Onofri Type Inequality

We prove Proposition 4.1 to complete the proof of Theorem 1.
The following lemma is due to Moser [22]

Lemma 4.3 *If a C^1 function f on S^2 satisfies that $f(x) = f(-x)$ on S^2, then there exists an absolute constant K such that*

$$\log\left(⨍_{S^2} e^f d\mu\right) \le \frac{1}{32\pi}\int_{S^2} |\text{grad } f|^2 d\mu + ⨍_{S^2} f d\mu + K. \tag{16}$$

Proof of Proposition 4.1: Let w be a function on $\overline{D}$ satisfying (13). Given $P \in S^2$, let Π_P be the plane perpendicular to the vector $\overrightarrow{OP}$ and containing the origin $O \in \mathbf{R}^3$. The stereographic projection of S^2 from the north pole P to $\Pi_P \bigcup\{\infty\}$ is denoted by s_P.

Let $f_1 = w \circ s_{(0,0,1)}$. Then we observe that

$$\begin{gathered}\int_{S^2_-} |\text{grad } f_1|^2 d\mu = \int_D |\nabla w|^2 dx, \quad \int_{S^2_-} \exp(f_1)\, d\mu = \frac{1}{2}\int_D e^w p_* dx, \\ \int_{S^2_-} f_1 d\mu = \frac{1}{2}\int_D w p_* dx,\end{gathered} \tag{17}$$

where $S^2_- = \{x = (x_1, x_2, x_3) \in S^2 |\ x_3 \le 0\}$ and $p_*(x) = 8/(1+|x|^2)^2$.
Setting $S^2_+ = S^2 \backslash S^2_-$, we can define a C^1 function on S^2 by

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in S^2_- \\ f_1(-x) & \text{if } x \in S^2_+ \end{cases}$$

from (13). Obviously $f \in C^1(S^2)$ satisfies $f(x) = f(-x)$ on S^2, so that (16) is applicable. It follows from (17) that

$$\log\left(\frac{1}{4\pi}\int_D e^w p_* dx\right) \le \frac{1}{16\pi}\int_D |\nabla w|^2 dx + \frac{1}{4\pi}\int_D w p_* dx + K.$$

Proof is complete. □

By the same argument, assumption (13) can be reduced to

$$w \in H^1(D) \quad \text{and} \quad w(x) = w(-x) \quad \text{on } \partial D.$$

In this form Proposition 4.1 is an improvement of Theorem 2.1 of [24] for two dimensional case.

5 Concentration toward Boundaries

5.1 Proof of Theorem 2

Proof of Theorem 2 requires the following proposition. Namely, without assuming $w(-x) = w(x)$, the best constant 16π arises if some terms of the boundary integral are involved in (14).

Proposition 5.1 *The following inequality holds*

$$\log\left(⨍_D e^w dx\right) \le \frac{1}{16\pi}\int_D |\nabla w|^2 dx + \frac{1}{2}⨍_{\partial D} w d\mu + \log\left(⨍_{\partial D} e^{w/2} d\mu\right) + K \quad (18)$$

for any $w \in H^1(D)$, where K is an absolute constant.

Proof of Theorem 2: From Proposition 5.1 and Lemma 4.2 it follows that

$$\left(a - \frac{1}{2} - \frac{Ma^2}{16\pi}\right)\int_D \left(|\nabla v|^2 + v^2\right) dx$$
$$\le M⨍_{\partial D} \frac{a}{2} v d\mu + M\log\left(⨍_{\partial D} e^{av/2} d\mu\right) + W(0) - M(C - \log M).$$

Since $a - 1/2 - Ma^2/(16\pi) > 0$ for $a_* < a < 1$, by the inequality above and Proposition 2.2 and (6) for $a_*/2 < a < 1/2$. Hence, (6) holds for any $a > a_*/2$. □

5.2 Rearrangement Relative to Harmonic Functions

We give the proof of Proposition 5.1. First, the following lemma is due to Moser [21].

Lemma 5.1 *The following inequality holds*

$$\log\left(⨍_\Omega e^w dx\right) \le \frac{1}{16\pi}\int_\Omega |\nabla w|^2 dx + K \quad (19)$$

for any $w \in H_0^1(\Omega)$, where K is an absolute constant.

In Lemma 5.1, we can take $K = 1$, which is best possible. This is proven in Appendix.

The following lemma is due to Alvarez [1].

Lemma 5.2 *We have*

$$\log\left(⨍_{\partial D} e^{w} d\mu\right) \leq \frac{1}{4\pi}\int_D |\nabla w|^2 dx + ⨍_{\partial D} w d\mu$$

for any harmonic function w in D.

We also make use of the following fact proven by Nehari [26].

Lemma 5.3 *Let ρ be a harmonic function on D. Let $\mathcal{U}$ be a subdomain of D with smooth boundary such that $\overline{\mathcal{U}} \subset D$. Then, the following inequality holds.*

$$4\pi \int_{\mathcal{U}} e^{\rho} dx \leq \left(\int_{\partial\mathcal{U}} e^{\rho/2} d\mu\right)^2. \tag{20}$$

If $\rho =$ constant, above (20) is a well-known inequality of isoperimetic.

Based on (20), we can introduce a rearrangement process. A simiar way was followed by Bandle [3] using Bol's inequality instead of (20).

Let ρ be a harmonic function on D satisfying $\rho \in C(\overline{D})$ and let $\rho^* = \log(⨍_D e^{\rho} dx)$. Given a measurable function w on D, let $\mathcal{U}_\xi = \{x \in D \mid w(x) > \xi\}$ and $\mathcal{U}_\xi^*$ be the open ball with center at the origin satisfying

$$a(\xi) = \int_{\mathcal{U}_\xi} e^{\rho} dx = \int_{\mathcal{U}_\xi^*} e^{\rho^*} dx,$$

where $\xi \in \mathbf{R}$. Then we can define the symmetric decreasing rearrangement of w relative to ρ by $w^*(x) = \sup\{\xi \in \mathbf{R} \mid x \in \mathcal{U}_\xi^*\}$. Then the equalities

$$\int_D g(w(x)) e^{\rho} dx = \int_D g(w^*(x)) e^{\rho^*} dx = \int_{-\infty}^{\infty} g(\xi) d(-a(\xi)) \tag{21}$$

hold for any strictly monotone increasing function g on $\mathbf{R}$.

We have the following.

Lemma 5.4 *The property*

$$\int_D |\nabla w|^2 dx \geq \int_D |\nabla w^*|^2 dx$$

holds for a C^1 function w on $\overline{D}$ satisfying

$$w \geq 0 \text{ in } D, \quad w = 0 \text{ on } \partial D.$$

Proof of Lemma 5.4: By co-area formula in differential form (see [12]), we observe that

$$-\frac{d}{d\xi} a(\xi) = \int_{\{x \in D \mid w(x) = \xi\}} \frac{e^{\rho}}{|\nabla w|} d\mu$$

for a.e. $\xi \in \mathbf{R}$. Observing that $\partial\{x \in D | w(x) > \xi\} = \{x \in D | w(x) = \xi\}$, we get by Lemma 5.3 and Sard's lemma that

$$\begin{aligned}
&\int_{\{x\in D|w(x)=\xi\}} |\nabla w| d\mu \qquad (22)\\
&\geq \left(\int_{\{x\in D|w(x)=\xi\}} e^{\rho/2} d\mu\right)^2 \left(\int_{\{x\in D|w(x)=\xi\}} \frac{e^\rho}{|\nabla w|} d\mu\right)^{-1}\\
&\geq -\frac{4\pi}{a'(\xi)} \int_{\{x\in D|w(x)>\xi\}} e^\rho dx\\
&= -\frac{4\pi a(\xi)}{a'(\xi)} \qquad (23)
\end{aligned}$$

for a.e. $\xi \in (0, \max_{x\in\overline{D}} w(x))$. Above relation and co-area formula in the integral form (see [12]) imply

$$\begin{aligned}
\int_D |\nabla w|^2 dx &= \int_0^\infty \int_{\{x\in D|w(x)=\xi\}} |\nabla w| d\mu d\xi\\
&\geq -4\pi \int_0^\infty \frac{a(\xi)}{a'(\xi)} d\xi. \qquad (24)
\end{aligned}$$

Because w^* is radially symmetric and decreasing in $r = |x|$, equalities hold at each step of (23). This fact, together with co-area formula in the integral form, implies

$$\int_D |\nabla w^*|^2 dx = \int_0^\infty \int_{\{x\in D|w^*(x)=\xi\}} |\nabla w^*| d\mu d\xi = -4\pi \int_0^\infty \frac{a(\xi)}{a'(\xi)} d\xi. \qquad (25)$$

The assertion follows from (24) and (25). □

Let $\mathcal{P}$ be the Poisson operator from $C(\partial D)$ to $C^2(D) \bigcap C(\overline{D})$ so that, $\rho = \mathcal{P}g$ solves

$$\Delta\rho = 0 \text{ in } D, \quad \rho = g \text{ on } \partial D.$$

Proof of Proposition 5.1: For $w \in C^1(\overline{D})$, let $\rho = \mathcal{P}(w|_{\partial D})$ and $w_0 = w - \rho$. By (21), we obtain

$$\begin{aligned}
&\log\left(⨍_D e^w dx\right)\\
&\leq \log\left(⨍_D e^{|w_0|} e^\rho dx\right) = \log\left(⨍_D e^{|w_0|^*} e^{\rho^*} dx\right)\\
&= \rho^* + \log\left(⨍_D e^{|w_0|^*} dx\right).
\end{aligned}$$

In use of Lemmas 5.1 and 5.4, the right-hand side is dominated by

$$\begin{aligned}
&\rho^* + \frac{1}{16\pi}\int_D |\nabla |w_0|^*|^2 dx + K\\
&\leq \rho^* + \frac{1}{16\pi}\int_D |\nabla |w_0||^2 dx + K\\
&= \rho^* + \frac{1}{16\pi}\int_D |\nabla w_0|^2 dx + K,
\end{aligned}$$

where K denotes an absolute constant. In use of $\Delta\rho = 0$ and $w_0|_{\partial D} = 0$, we have

$$\int_D \nabla w_0 \cdot \nabla \rho dx = 0.$$

Hence

$$\int_D |\nabla w_0|^2 dx = \int_D |\nabla w|^2 dx - \int_D |\nabla \rho|^2 dx$$

so that

$$\begin{aligned} \log\left(⨍_D e^w dx\right) \frac{1}{16\pi}\int_D |\nabla w|^2 dx \\ + \left\{\log\left(⨍_D e^\rho dx\right) - \frac{1}{16\pi}\int_D |\nabla \rho|^2 dx\right\} + K. \end{aligned} \tag{26}$$

On the other hand Lemmas 5.2 and 5.3 imply

$$\begin{aligned} &\log\left(⨍_D e^\rho dx\right) - \frac{1}{16\pi}\int_D |\nabla\rho|^2 dx \\ &\le 2\log\left(⨍_{\partial D} e^{\rho/2} d\mu\right) - \frac{1}{4\pi}\int_D |\nabla \rho/2|^2 dx \\ &\le \log\left(⨍_{\partial D} e^{\rho/2} d\mu\right) + \frac{1}{2}⨍_{\partial D} \rho d\mu. \end{aligned} \tag{27}$$

Ineqalities (26) and (27) give (18). □

By this proof, we observe that the constant K in Proposition 5.1 is equal to the constant K in Lemma 5.1. In Proposition 5.1, we can take $K = 1$, which is best possible. This is shown in Appendix.

6 Concentration toward Boundaries (continued)

6.1 Proof of Theorem 4

To prove the theorem, we require Brezis-Merle type inequality, of which original form is described as follows ([5]):
Let w be the solution of the boundary value problem

$$-\Delta w = f \text{ in } \Omega, \qquad w = 0 \text{ on } \partial\Omega.$$

Then it follows that

$$\int_\Omega \exp\left(\frac{4\pi - \varepsilon}{\|f\|_{L^1(\Omega)}}|w(x)|\right) dx \le \frac{4\pi^2}{\varepsilon}(\text{diam}\Omega)^2,$$

where $0 < \varepsilon < 4\pi$.
We shall derive a similar inequality relative to the second equation of (P):

$$\text{(E)} \qquad \begin{cases} -\Delta w + w = f & \text{in } \Omega, \\ \dfrac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega.. \end{cases}$$

By Brezis and Struss [6], the weak solution w of (E) with a L^1 function f in Ω belongs to $W^{1,p}(\Omega)\bigcap L^r(\Omega)$ for $1 \le p < 2$ and $1 \le r < \infty$.

Proposition 6.1 *Let $B(q,2\eta) \subset \Omega$ and $0 < \varepsilon < 4\pi$. Then, there exists a positive constant C depending on η, ε and $\|f\|_{L^1(\Omega)}$ such that $\|f^+\|_{L^1(B(q,2\eta))} \le 4\pi - \varepsilon$ implies*

$$\int_{B(q,\eta)} e^{w(x)}dx \le C,$$

where w denotes the weak solution of (E).

Proposition 6.2 *Let $q \in \partial\Omega$ and $0 < \varepsilon < 2\pi$. Then, there exist constants η_0 with $\eta_0 \in (0,1/4]$ and $C > 0$ depending on ε, $\eta \in (0,\eta_0)$ and $\|f\|_{L^1(\Omega)}$ such that $\eta \in (0,\eta_0)$ and $\|f^+\|_{L^1(\Omega\bigcap B(q,2\eta))} \le 2\pi - \varepsilon$ imply*

$$\int_{\Omega\bigcap B(q,\eta)} e^{w(x)}dx \le C,$$

where w denotes the weak solution of (E).

Above Propositions are proven in the next subsection.

Proof of Theorem 4: Let $\{t_l\}_{l=1}^{\infty} \subset [0, T_{max})$ be a sequence in consideration: $t_l \to T_{max}$. By (iii) of Proposition 2.2, any $a > 1/2$ admits a sequence $\{q_k\}_{k=1}^{\infty}$ of $\overline{\Omega}$ and subsequences $\{t_l^{(k)}\}_{l=1}^{\infty}$ of $\{t_l\}_{l=1}^{\infty}$ such that $\{t_l^{(k+1)}\}_{l=1}^{\infty} \subset \{t_l^{(k)}\}_{l=1}^{\infty}$ for any $k = 1, 2, \cdots$ and that

$$\lim_{l\to\infty} \int_{\Omega\bigcap B(q_k,2^{-k})} \exp\left(\left(\frac{1}{2}+\frac{1}{k}\right)v(x,t_l^{(k)})\right)dx = \infty$$

for any $k = 1, 2, \cdots$. We put $t_k^{(k)} = t'_k$. Let k_0 be an integer satisfying $(\frac{1}{2}+\frac{1}{k_0})M < 4\pi$. We observe that

$$\lim_{l\to\infty} \int_{\Omega\bigcap B(q_k,2^{-k})} \exp\left(\left(\frac{1}{2}+\frac{1}{k}\right)v(x,t'_l)\right)dx = \infty \qquad \text{for any } k \ge k_0.$$

Suppose that $B(q_k, 2^{-k}) \subset \Omega$ for some $k \ge k_0$. Since we have

$$\|\left(\frac{1}{2}+\frac{1}{k}\right)u(\cdot,t)\|_{L^1(\Omega)} = \left(\frac{1}{2}+\frac{1}{k}\right)M < 4\pi,$$

Proposition 6.1 implies that

$$\sup_{l\ge1} \int_{B(q_k,2^{-k})} \exp\left(\left(\frac{1}{2}+\frac{1}{k}\right)v(x,t'_l)\right)dx < \infty.$$

It is a contradiction. Hence, $B(q_k, 2^{-k})\bigcap\partial\Omega \neq \emptyset$ for any $k \ge k_0$. Let q be an accumulating point of $\{q_k\}_{k=1}^{\infty}$. We see that $q \in \partial\Omega$ and that

$$\lim_{l\to\infty} \int_{\Omega\bigcap B(q,\varepsilon)} \exp\left(av(x,t'_l)\right)dx = \infty \qquad \text{for any } a > 1/2 \text{ and } \varepsilon > 0. \quad (28)$$

We then observe that

$$\liminf_{l\to\infty}\int_{\Omega\bigcap B(q,\varepsilon)} u(x,t_l')dx \geq 4\pi \qquad \text{for any } \varepsilon>0. \tag{29}$$

In fact, suppose that

$$\liminf_{l\to\infty}\int_{\Omega\bigcap B(q,\varepsilon_0)} u(x,t_l')dx < 4\pi \tag{30}$$

for some $\varepsilon_0 > 0$. By taking positive constants a and ε such that $a > 1/2$ and that $a - 1/2$ and ε are sufficiently small, Proposition 6.2 and (30) yield that

$$\liminf_{l\to\infty}\int_{\Omega\bigcap B(q,\varepsilon)} \exp\left(av(x,t_l')\right)dx < \infty.$$

It is contrary to (28). Hence, we have (29) and that

$$\limsup_{l\to\infty}\int_{\Omega\setminus B(q,\varepsilon)} u(x,t_l')dx \leq M-4\pi < 4\pi \qquad \text{for any } \varepsilon>0,$$

by which together with Propositions 6.1 and 6.2 it follows that

$$\sup_{l\geq 1}\int_{\Omega\setminus B(q,\varepsilon)} \exp\left(av(x,t_l')\right)dx < \infty \qquad \text{for any } a\in(1/2,a^*) \text{ and } \varepsilon>0. \tag{31}$$

By (28), (31) and (iii) of Proposition 2.2, we have (9). From (31) and (iii) in Proposition 2.2 it follows that q is a blow-up point of v. □

6.2 Brezis-Merle Type Inequalities

To prove previous propositions, we need the following lemma.

Lemma 6.1 *Suppose that w satisfies*

$$-\Delta w + b(x)w = f \quad \text{in } \Omega,$$

where $f \in L^1(\Omega)$ and $b \in L^\infty(\Omega)$. Let $B(q,\eta) \subset \Omega$. Then, for $\varepsilon \in (0,\eta)$ and $p \in (1,2)$ there exists a positive constant C depending on ε, η, p and $\|b\|_{L^\infty}$ such that

$$\int_{B(q,\varepsilon)} e^{w(x)}dx \leq \exp\left(C\|w\|_{W^{1,p}(B(q,\eta))}\right)\int_{|x|<2\eta}\left(\frac{2\eta}{|x|}\right)^{\theta}dx,$$

where

$$\theta = \frac{1}{2\pi}\|f^+\|_{L^1(B(q,\eta))}.$$

Proof of Lemma 6.1: Let φ be a C^∞ function on $\mathbf{R}^2$ such that $0 \le \varphi \le 1$ and

$$\varphi(x) = \begin{cases} 0 & \text{if } x \in B(q,\eta)^c, \\ 1 & \text{if } x \in B(q,\varepsilon). \end{cases}$$

Since the function $\varphi(x)w(x)$ satisfies

$$-\Delta(\varphi w) = -\varphi f + g \quad \text{in } R^2,$$

where $g = (b\varphi - \Delta\varphi)w - 2\nabla\varphi \cdot \nabla w$, we have

$$w(x) = \int_{B(q,\eta)} N(x,y)\varphi(y)f(y)\,dy - \int_{B(q,\eta)} N(x,y)g(y)\,dy \tag{32}$$

for any $x \in B(q,\varepsilon)$. Here $N(x,y) = \frac{1}{2\pi}\log\frac{2\eta}{|x-y|}$. Noting that for $r \ge 1$

$$\int_{B(q,\eta)} |N(x,y)|^r\,dy \le C \quad \text{for any } x \in B(q,\eta),$$

where C is a positive constant depending on η and r, we see that for each $p \in (1,2)$ there exists a positive constant C depending on ε, η and p such that

$$\left|\int_{B(q,\eta)} N(x,y)g(y)\,dy\right| \le C\|w\|_{W^{1,p}(B(q,\eta))} \qquad \text{for any } x \in B(q,\varepsilon). \tag{33}$$

By $N(x,y) \ge 0$ on $B(p,\eta) \times B(p,\eta)$ and $0 \le \varphi(x) \le 1$, we get

$$\int_{B(q,\eta)} N(x,y)\varphi(y)f(y)dy \le \int_{B(q,\eta)} N(x,y)f^+(y)dy. \tag{34}$$

From (32)-(34) it follows that for each $p \in (1,2)$ there exists a positive constant C depending on ε, η and p such that

$$\int_{B(q,\varepsilon)} e^{w(x)}dx \le \int_{B(q,\varepsilon)} \exp\left(C\|w\|_{W^{1,p}(B(q,\eta))} + \int_{B(q,\eta)} N(x,y)f^+(y)dy\right) dx. \tag{35}$$

Jensen's inequality yields that

$$\begin{aligned}
&\int_{B(q,\varepsilon)} \exp\left(\int_{B(q,\eta)} N(x,y)f^+(y)dy\right) dx \\
&\le \int_{B(q,\varepsilon)}\int_{B(q,\eta)} \exp\left(\|f^+\|_{L^1(B(q,\eta))}N(x,y)\right) \frac{f^+(y)}{\|f^+\|_{L^1(B(q,\eta))}}dydx \\
&\le \int_{|x|\le 2\eta} \left(\frac{2\eta}{|x|}\right)^\theta dx,
\end{aligned}$$

by which together with (35) we complete the proof. □

Proof of Proposition 6.1: Let w_p be the weak solution of (E) with f^+.

By using $w \leq w_p$ in Ω, we observe that the estimate is independent of f^-. L^1 estimate (c.f. [6]) is indicated as

$$\|w\|_{W^{1,p}(\Omega)} \leq C_p \|f\|_{L^1(\Omega)} \tag{36}$$

for (E), with $1 \leq p < 2$. Therefore, by Lemma 6.1 we have Proposition 6.1. □

We next give the proof of Proposition 6.2.

Proof of Proposition 6.2: By the similar arguments to those in the proof of Proposition 6.1, the estimate is independent of f^-. By using partition of unity we may assume that Ω is simply connected, by which we observe that there exists a conformal transformation ϕ satisfying $\phi(\Omega) = D$. There exists a continuous function d_* on $\overline{D}$ with $\inf_{x \in D} d_*(x) > 0$ such that

$$-\Delta w_* + d_* w_* = f_* \qquad \text{in } D \tag{37}$$

with

$$\frac{\partial w_*}{\partial n} = 0 \qquad \text{on } D, \tag{38}$$

where $f_* = d_*(f \circ \phi^{-1})$. For each subdomain ω of Ω, we observe that $\|f_*\|_{L^1(\phi(\omega))} = \|f\|_{L^1(\omega)}$. Then we may prove the proposition for the solution to (37) and (38) with a L^1 function f_* on D. Using the Kelvin transform, we extend w_* to the whole space as follows.

$$v(x) = \begin{cases} w_*(x) & \text{if } |x| \leq 1, \\ w_*(x/|x|^2) & \text{if } |x| > 1. \end{cases}$$

Then the function v satisfies

$$-\Delta v + bv = h \quad \text{in } R^2,$$

where

$$b(x) = \begin{cases} d_*(x) & \text{if } |x| \leq 1, \\ |x|^{-4} d_*(x/|x|^2) & \text{if } |x| > 1, \end{cases}$$

$$h(x) = \begin{cases} f_*(x) & \text{if } |x| \leq 1, \\ |x|^{-4} f_*(x/|x|^2) & \text{if } |x| > 1. \end{cases}$$

Let $\eta_0 \in (0, 1/4]$. By Lemma 6.1, for $\eta \in (0, \eta_0)$ and $p \in (1, 2)$ there exists a positive constant C depending on η, η_0 and p such that

$$\int_{B(q,\eta)} e^{v(x)}\, dx \leq \exp\left(C \|v\|_{W^{1,p}(B(q,2\eta))}\right) \int_{|x|<4\eta} \left(\frac{4\eta}{|x|}\right)^{\tilde{\theta}} dx, \tag{39}$$

where

$$\tilde{\theta} = \frac{1}{2\pi} \|h^+\|_{L^1(B(q,2\eta))} \leq \frac{1}{\pi} \|f_*^+\|_{L^1(D \cap B(q,2\eta))}.$$

By L^1 estimate of Brezis and Struss [6], we have

$$\|v\|_{W^{1,p}(B(q,2\eta))} \leq C\|w_*\|_{W^{1,p}(D)} \leq C\|f_*\|_{L^1(D)}.$$

Hence, (39) implies that

$$\int_{D\cap B(q,\eta)} e^{w_*(x)}\,dx \leq \exp\left(C\|f_*\|_{L^1(D)}\right)\int_{|x|<1}\frac{dx}{|x|^\theta},$$

where

$$\theta = \frac{1}{\pi}\|f_*^+\|_{L^1(D\cap B(q,2\eta))}.$$

Thus the proof is complete. □

7 Aggregation at Isolated Blow-up Points

7.1 Proof of Theorem 5

To prove Theorem 5, we begin with the following propositions that will be shown in Section 7.2.

Proposition 7.1 *Let (u,v) be the solution for (P). For $q \in \mathcal{B}_I$, there exist $\eta > 0$, $\varepsilon \in (0,\eta)$ and $\theta \in (0,1)$ such that*

$$u, v \in C^{2+\theta,1+\theta/2}\left(\overline{\Omega\bigcap A(q,\eta,\varepsilon)} \times [0,T_{max})\right)$$

and

$$\sup_{0<T<T_{max}}\left\{\|u\|_{C^{2+\theta,1+\theta/2}\left(\overline{\Omega\bigcap A(q,\eta,\varepsilon)}\times[0,T]\right)} + \|v\|_{C^{2+\theta,1+\theta/2}}\left(\overline{\Omega\bigcap A(q,\eta,\varepsilon)} \times [0,T]\right)\right\} < +\infty$$

Proposition 7.2 *Suppose that $T_{max} < \infty$. Let (u,v) be the solution for (P). Then, for $q \in \mathcal{B}_I$ it holds that*

$$\lim_{t\to T_{max}}\int_{\Omega\bigcap B(q,\varepsilon)} u(x,t)dx \geq m_* \qquad \textit{for any } \varepsilon > 0,$$

where m_* is the constant in Theorem 5.

Proof of Theorem 5: Let

$$m(q,\varepsilon) = \lim_{t\to T_{max}}\int_{\Omega\bigcap B(q,\varepsilon)} u(x,t)dx.$$

By Proposition 7.1, we have that $m(q,\cdot)$ is continuous and monotone increasing on $(0,\eta)$. Let $m(q,0) = \lim_{\varepsilon\searrow 0} m(q,\varepsilon)$. Combining those properties together with Proposition 7.2 concludes that $m(q,\cdot) \in C([0,\eta])$ and $m(q,0) \geq m_*$. By Proposition 7.1, there exists a function

$$f \in C\left(\overline{\{\Omega\bigcap B(q,\eta)\}}\backslash\{q\}\right)\bigcap L^1(\Omega\bigcap B(q,\eta))$$

such that

$$\lim_{t\to T_{max}} u(x,t) = f(x) \qquad \text{for any } x \in \overline{\{\Omega\bigcap B(q,\eta)\}}\backslash\{q\}.$$

Then we have that

$$w^*\text{-}\lim_{t\to T_{max}} u(\cdot,t) = m(q,0)\delta_q + f \quad \text{ in } \mathcal{M}\left(\overline{\Omega\bigcap B(q,\eta)}\right).$$

Thus the proof is complete. □

7.2 Localization of Lyapunov Function

We begin with the following lemma. We will proof the lemma in Appendix.

Lemma 7.1 *Let $\eta^* > 0$ and $q \in \partial\Omega$. Suppose that*

$$\sup_{0\leq t<T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega\bigcap B(q,\eta^*))} < \infty$$

and that

$$\sup_{0\leq t<T_{max}} \left(\|v(\cdot,t)\|_{L^\infty(\Omega\bigcap B(q,\eta^*))} + \|\nabla v(\cdot,t)\|_{L^\infty(\Omega\bigcap B(q,\eta^*))}\right) < \infty.$$

Then, there exist $\eta_* \in (0,\eta^*)$ and $\theta \in (0,1)$ such that

$$\|u\|_{C^{\theta,\theta/2}\left(\overline{\Omega\bigcap B(q,\eta_*)}\times[0,T_{max})\right)} < \infty.$$

Proof of Prposition 7.1: By the definition of isolated blow-up points, there exists a positive constant η_0 such that

$$\sup_{0\leq t<T_{max}} \left(\|u(\cdot,t)\|_{L^\infty(\Omega\bigcap A(q,\eta_0,\eta))} + \|v(\cdot,t)\|_{L^\infty(\Omega\bigcap A(q,\eta_0,\eta))}\right) < \infty \tag{40}$$

for any $\eta \in (0,\eta_0)$, from which together with the standard arguments for elliptic equations it follows that

$$\sup_{0\leq t<T_{max}} \|\nabla v(\cdot,t)\|_{L^\infty(\Omega\bigcap A(q,\eta_1,\eta_2))} < \infty \tag{41}$$

for any η_1 and η_2 with $\eta < \eta_2 < \eta_1 < \eta_0$. By the above estimate, Lemma 7.1 and [20, Theorem 10.1 in Section III], for any η_1 and η_2 with $\eta < \eta_2 < \eta_1 < \eta_0$ there exists a positive constant $\theta \in (0,1)$ such that

$$\|u\|_{C^{\theta,\theta/2}\left(\overline{\Omega \bigcap A(q,\eta_1,\eta_2)} \times [0,T_{max})\right)} < \infty. \tag{42}$$

In fact, suppose that $\partial\Omega \bigcap \overline{A(q,\eta_1,\eta_2)} = \emptyset$. By (41) and [20, Theorem 10.1 in Section III], we have (42) in the case of $\partial\Omega \bigcap \overline{A(q,\eta_1,\eta_2)} = \emptyset$.

Suppose that $\partial\Omega \bigcap \overline{A(q,\eta_1,\eta_2)} \neq \emptyset$. For a sufficiently small $\eta^* > 0$, there exist

$$\{q_k\}_{k=1,2,3,\cdots,K} \subset \partial\Omega \bigcap \overline{A(q,\eta_1,\eta_2)}$$

such that

$$\partial\Omega \bigcap \overline{A(q,\eta_1,\eta_2)} \subset \bigcup_{k=1}^{K} B(q_k,\eta_*) \subset \bigcup_{k=1}^{K} B(q_k,\eta^*) \subset A(q,\eta,\eta_1),$$

where η_* is the constant in Lemma 7.1. By Lemma 7.1, we obtain that

$$\|u\|_{C^{\theta,\theta/2}\left(\overline{\Omega \bigcap B(q_k,\eta_*)} \times [0,T_{max})\right)} < \infty \qquad \text{for each } k = 1,2,3,\cdots,K. \tag{43}$$

Since we obtain that

$$\overline{\Omega \bigcap A(q,\eta_1,\eta_2)} \setminus \bigcup_{k=1}^{K} B(q_k,\eta_*) \subset \Omega,$$

by (41) and [20, Theorem 10.1 in Section III] we have that

$$\|u\|_{C^{\theta,\theta/2}\left(\overline{\Omega \bigcap A(q,\eta_1,\eta_2)} \setminus \bigcup_{k=1}^{K} B(q_k,\eta_*) \times [0,T_{max})\right)} < \infty.$$

By which and (43) we get (42) in the case of $\partial\Omega \bigcap \overline{A(q,\eta_1,\eta_2)} \neq \emptyset$. Then, we get (42).

¿From (42) together with [13, Theorem 6.16], for any η_1 and η_2 with $\eta < \eta_2 < \eta_1 < \eta_0$ there exists a positive constant $\theta \in (0,1)$ such that

$$\|v\|_{C^{2+\theta,\theta/2}\left(\overline{\Omega \bigcap A(q,\eta_1,\eta_2)} \times [0,T_{max})\right)} < \infty.$$

Hence, by [20, Theorem 10.1 in Section IV], for any η_1 and η_2 with $\eta < \eta_2 < \eta_1 < \eta_0$ there exists a positive constant $\theta \in (0,1)$ such that

$$\|u\|_{C^{2+\theta,1+\theta/2}\left(\overline{\Omega \bigcap A(q,\eta_1,\eta_2)} \times [0,T_{max})\right)} < \infty.$$

Thus the proof is complete. □

For the proof Proposition 7.2, we note the following.

Lemma 7.2 *There exists a positive constant C such that*

$$W_\varphi(t) \le W_\varphi(0) + C \quad \text{for any } t \in (0, T_{max}), \tag{44}$$

where

$$W_\varphi(t) = \int_\Omega \left(u \log u - \frac{1}{2} uv \right) \varphi dx.$$

Proof: Let η be the constant in Proposition 7.1 and let ε be a positive constant with $\varepsilon \le \eta$. Let φ be a C^∞ function on $\mathbf{R}^2$ such that $0 \le \varphi \le 1$ and

$$\varphi(x) = \begin{cases} 1 & \text{if } x \in \overline{B(q, \varepsilon/2)}, \\ 0 & \text{if } x \in B(q, \varepsilon)^c. \end{cases}$$

Multiplying $(\log u - v)\varphi$ by the first equation of (P) and using Green's formula, we have

$$\begin{aligned} \int_\Omega u_t (\log u - v)\varphi dx &= \int_\Omega \nabla \cdot (\nabla u - u \nabla v)(\log u - v)\varphi dx \\ &= -\int_\Omega u |\nabla(\log u - v)|^2 \varphi dx \\ &\quad - \int_\Omega (\log u - v)(\nabla u - u \nabla v) \cdot \nabla \varphi dx. \end{aligned} \tag{45}$$

We have that

$$\begin{aligned} &\int_\Omega u_t (\log u - v)\varphi dx \\ &= \frac{d}{dt} \int_\Omega (u \log u - uv)\varphi dx - \frac{d}{dt} \int_\Omega u \varphi dx + \int_\Omega u v_t \varphi dx. \end{aligned} \tag{46}$$

Using the second equation of (P), we have that

$$\begin{aligned} \int_\Omega u v_t \varphi dx &= \int_\Omega (-\Delta v + v) v_t \varphi dx \\ &= \frac{1}{2} \frac{d}{dt} \int_\Omega \left(|\nabla v|^2 + v^2 \right) \varphi dx + \int_\Omega (\nabla v \cdot \nabla \varphi) v_t dx. \end{aligned} \tag{47}$$

By Proposition 7.1 and the definitions of φ and isolated blow-up points, there exists a positive constant C such that

$$\begin{aligned} \left| \int_\Omega (\nabla \varphi \cdot \nabla v) v_t dx \right| &= \left| \int_{\Omega \bigcap A(q,\varepsilon,\varepsilon/2)} (\nabla \varphi \cdot \nabla v) v_t dx \right| \\ &\le C \quad \text{for any } t \in (0, T_{max}) \end{aligned} \tag{48}$$

and that

$$\begin{aligned} &\left| \int_\Omega (\nabla u \cdot \nabla \varphi) \log u dx \right| \\ &\le \left| \int_\Omega u \operatorname{div}(\log u \nabla \varphi) dx \right| + \left| \int_{\partial\Omega} u \log u \frac{\partial \varphi}{\partial n} d\mu \right| \\ &= \left| \int_{\Omega \bigcap A(q,\varepsilon,\varepsilon/2)} (\nabla u \cdot \nabla \varphi + u \log u \Delta \varphi) dx \right| \\ &\quad + \left| \int_{\partial\Omega \bigcap A(q,\varepsilon,\varepsilon/2)} u \log u \frac{\partial \varphi}{\partial n} d\mu \right| \le C \quad \text{for any } t \in (0, T_{max}). \end{aligned} \tag{49}$$

By (45)-(49) and Proposition 7.1, we get (44). □

Proof of Propostion 7.2: By (44), there exists a positive constant C such that

$$\int_\Omega (u\log u)\varphi dx \le \frac{1}{2}\int_\Omega uv\varphi dx + W_\varphi(0) + C \qquad \text{for any } t \in [0, T_{max}). \quad (50)$$

By Young's inequality, we get

$$a\int_\Omega uv\varphi dx \le \int_\Omega (u\log u)\varphi dx + \frac{1}{e}\int_\Omega e^{av}\varphi dx$$

for any $t \in [0, T_{max})$ and $a > 0$, from which together with (50) it follows that

$$\left(a - \frac{1}{2}\right)\int_\Omega uv\varphi dx \le \frac{1}{e}\int_\Omega e^{av}\varphi dx + W_\varphi(0) + C \quad (51)$$

for any $t \in [0, T_{max})$ and $a > 0$. We observe that

$$\lim_{t\to T_{max}} \int_\Omega uv\varphi dx = \infty, \quad (52)$$

by which together with (51) it follows that

$$\lim_{t\to T_{max}} \int_\Omega e^{av}\varphi dx = \infty, \qquad \text{for any } a > 1/2. \quad (53)$$

In fact, if we assume that

$$\limsup_{t\to T_{max}} \int_\Omega uv\varphi dx < \infty,$$

then by (50) we have that

$$\limsup_{t\to T_{max}} \int_\Omega (u\log u)\varphi dx < \infty. \quad (54)$$

Then (54) implies

$$\sup_{0\le t<T_{max}} \|u(\cdot,t)\varphi\|_{L^\infty(\Omega)} < \infty$$

similarly to [24]. It is the contradiction. Then, we have (52).

In the case of $q \in \Omega$, by (53) and Proposition 6.1, we observe that

$$\limsup_{t\to T_{max}} \int_{\Omega\bigcap B(q,\varepsilon)} u(x,t)dx \ge 8\pi \qquad \text{for any } \varepsilon > 0. \quad (55)$$

By Proposition 7.1, for any $\varepsilon \in (0, \eta]$ there exists a positive constant C depending on ε such that

$$\begin{aligned}\left|\frac{d}{dt}\int_{\Omega\bigcap B(q,\varepsilon)} u(x,t)dx\right| &= \left|\int_{\Omega\bigcap \partial B(q,\varepsilon)} \left(\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) d\mu\right| \\ &\le C \qquad \text{for any } t \in [0, T_{max}). \end{aligned} \quad (56)$$

By (55) and (56), we have that

$$\lim_{t\to T_{max}} \int_{\Omega\bigcap B(q,\varepsilon)} u(x,t)dx \geq 8\pi \qquad \text{for any } \varepsilon > 0.$$

In the case of $p \in \partial\Omega$, by (53), (56) and Proposition 6.2 we observe that

$$\lim_{t\to T_{max}} \int_{\Omega\bigcap B(q,\varepsilon)} u(x,t)dx \geq 4\pi \qquad \text{for any } \varepsilon > 0.$$

Thus the proof is complete. □

8 Concentration toward Boundaries (final)

8.1 Proof of Theorem 3

Theorem 3 is proven through a form of concentration lemma, which is generally referred to as follows.

A family $\mathcal{F}$ of H^1 functions satisfies either one of the following.

1. *All elements $w \in \mathcal{F}$ satisfy an Onofri type's inequality with a sharp constant.*
2. *There exists a sequence $\{w_k\}_{k=1}^{\infty} \subset \mathcal{F}$ such that*

$$\{\exp(w_k)/\|\exp(w_k)\|_{L^1}\}$$

concentrates at a point in L^1 norm.

This type of statement is found in Chang and Yang [9] for $\mathcal{F} \subset H^1(S^2)$. Via similar arguments, we can show the following. Recall that $p_*(x) = 8/(1+|x|^2)^2$.

Proposition 8.1 *Given a one parameter family $\mathcal{F} = \{w(\cdot,t)|0 \leq t < T\}$ with $t \mapsto w(\cdot,t) \in H^1(D)$ continuous, we have the following alternatives.*

(i) *Inequality (60) holds for $w_k = w(\cdot,t_k)$ with some $t_k \nearrow T$.*

(ii) *There exists a continuous map $t \mapsto q(t) \in \partial D$ satisfying*

$$\liminf_{t\to T} \frac{\int_{D\bigcap B(q(t),\varepsilon)} \exp\left(w(x,t)\right) p_*(x)dx}{\int_D \exp\left(w(x,t)\right) p_*(x)dx} \geq \frac{1}{2} \qquad \textit{for any } \varepsilon > 0. \quad (57)$$

Proof of Theorem 3: First, observe that $M < 8\pi$ implies $a_* \in (1/2, 1)$ in (7). We assume that $a \in (a_*, 1)$ is given.

Putting $w(\cdot,t) = av(\cdot,t)$, we suppose the first alternative (i) of Proposition 8.1 so that (60) holds for some $w_k = w(\cdot,t_k)$ with $t_k \nearrow T = T_{max}$.

Then Lemma 4.2 implies

$$\left(a-\frac{1}{2}-\frac{Ma^2(1+\varepsilon)}{16\pi}\right)\int_D\left(|\nabla v(x,t_k)|^2+v(x,t_k)^2\right)dx$$
$$\leq W(0)-M\log M+C_\varepsilon$$

for any $\varepsilon>0$. $(k=1,2,\cdots)$ We take ε satisfying

$$a-\frac{1}{2}-\frac{Ma^2(1+\varepsilon)}{16\pi}>0,$$

to deduce

$$\limsup_{k\to\infty}\int_D\left(|\nabla v(x,t_k)|^2+v(x,t_k)^2\right)dx<\infty.$$

It is contrary to the case (ii) of Proposition 2.2.

The second alternative holds and there exists a continuous map $t\in[0,T_{max})\mapsto q\in\partial D$ satisfying (57) for $w=av$ and $T=T_{max}$.

Here, from the case (iii) of Proposition 2.2 it follows that

$$\lim_{t\to T_{max}}\int_D\exp(av(x,t))dx=\infty$$

by $a_*>1/2$. Therefore,

$$\lim_{t\to T_{max}}\int_{D\bigcap B(q(t),\varepsilon)}\exp(av(x,t))dx=\infty\qquad\text{for any }\varepsilon>0.\tag{58}$$

If we assume

$$\liminf_{t\to T_{max}}\int_{D\bigcap B(q(t),\varepsilon)}u(x,t)dx<\frac{2\pi}{a},\tag{59}$$

then Proposition 6.2 implies

$$\liminf_{t\to T_{max}}\int_{\Omega\bigcap B(q(t),\varepsilon)}e^{av(x,t)}dx<\infty$$

for some $\varepsilon>0$. This is contrary to (58). Thus the proof is complete. □

8.2 Concentration Lemma

For the proof of Proposition 8.1, we make use of the following facts due to Aubin [2], Proposition 3.1 of Chang and Yang [9] and Proposition 2.2 of Chang and Yang [8], respectively.

Lemma 8.1 *Suppose $f\in C^1(S^2)$ with $\int_{S^2}e^f x d\mu=\vec{0}$. Then for each $\varepsilon>0$, there exists a constant K_ε such that*

$$\log\left(⨍_{S^2}e^f d\mu\right)\leq\frac{1+\varepsilon}{32\pi}\int_{S^2}|\text{grad }f|^2d\mu+⨍_{S^2}fd\mu+K_\varepsilon.$$

Lemma 8.2 *Given $f \in H^1(S^2)$ and a conformal transformation ϕ on S^2, let*

$$f_\phi = f \circ \phi + \log(\det(d\phi)).$$

Then, $I(f) = I(f_\phi)$, where

$$I(f) = \frac{1}{16\pi} \int_{S^2} |\text{grad } f|^2 d\mu + \fint_{S^2} f d\mu.$$

Recall that s_P denotes the stereographic projection from the north pole $P \in S^2$ to the plane Π_P containing the equator $E_P \subset S^2$ relative to P. Given $Q \in S^2$ and $\xi \geq 1$, $s_Q^{-1} \circ \xi id \circ s_Q$ becomes a conformal transformation on S^2, where $\xi id(x) = \xi x$ for $x \in \mathbf{R}^2 \bigcup \{\infty\}$. This mapping is denoted by $\phi_{Q,\xi}$. Precisely, $\phi_{Q,\xi}$ is determined by the coset class of $S^2 \times [1,\infty) / S^2 \times \{1\} \cong B_3(0,1)$. Let $\Phi = \{\phi_{Q,\xi} \,|(Q,\xi) \in S^2 \times [1,\infty) / S^2 \times \{1\}\}$,

$$X = \left\{ f \in H^1(S^2) \,\middle|\, \fint_{S^2} e^f d\mu = 1 \right\}$$

and

$$X_0 = \left\{ f \in X \,\middle|\, \int_{S^2} e^f x d\mu = \vec{0} \right\}.$$

The following fact is also well-known (see [8], [9], e.g.). We denote $E_0 = E_{(0,0,1)}$.

Lemma 8.3 *Any function $w \in X$ admits a transformation $\phi \in \Phi$, satisfying $w_\phi \in X_0$. Forthermore,*

(i) $\phi = \phi_{Q,\xi}$ *with $Q \in E_0$ if w is symmetric with respect to x_1x_2 plane.*

(ii) $\{\phi\}$ *changes continuously in Φ, if $\{w\}$ does so in X.*

Admitting those lemmas, we first show the discrete version of the proposition.

Proposition 8.2 *Let $\mathcal{F} = \{w\}$ be a family of C^1 functions on $\overline{D}$ satisfying*

$$\frac{\partial w}{\partial n} = 0 \text{ on } \partial D \qquad \text{and} \qquad \sup_{w \in \mathcal{F}} \|w\|_{L^1(D)} < \infty.$$

Then either one of the following (i) or (ii) holds.

(i) *Any $\varepsilon > 0$ admits a positive constant C_ε such that*

$$\log \left(\fint_D e^w dx \right) \leq \frac{1+\varepsilon}{16\pi} \int_D |\nabla w|^2 dx + C_\varepsilon \quad \text{for any } w \in \mathcal{F}. \tag{60}$$

(ii) *Taking a sequence $\{w_k\}_{k=1}^\infty \subset \mathcal{F}$ such that*

$$w^*\text{-}\lim_{k\to\infty} (e^{w_k} p_*) \Big/ \int_D e^{w_k} p_* dx = d\lambda \qquad \text{in } \mathcal{M}(\overline{D}),$$

there exists some $q \in \partial D$ such that $\lambda(\{q\}) \geq 1/2$.

Proof: Let w be in $\mathcal{F}$. Let $f_1 = w \circ s_{(0,0,1)}$ on S^2_- and define a C^1 function $f(x)$ on S^2 by

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in S^2_-, \\ f_1(x_1, x_2, -x_3) & \text{if } x = (x_1, x_2, x_3) \in S^2_+. \end{cases}$$

Let

$$g = f - \log\left(⨍_{S^2} e^f d\mu\right) \in X. \tag{61}$$

There exists a pair $(Q, \xi) \in E_0 \times [1, \infty)$ such that $(g)_{\phi_{Q,\xi}} \in X_0$ by Lemma 8.3. Therefore, any $\varepsilon > 0$ admits a constant K_ε such that

$$\begin{aligned} &\log\left(⨍_{S^2} \exp\left((g)_{\phi_{Q,\xi}}\right) d\mu\right) \\ &\leq \frac{1+\varepsilon}{32\pi} \int_{S^2} |\text{grad } (g)_{\phi_{Q,\xi}}|^2 d\mu + ⨍_{S^2} (g)_{\phi_{Q,\xi}} d\mu + K_\varepsilon \end{aligned} \tag{62}$$

by Lemma 8.1. The left hand side of (62) is equal to

$$\log\left(⨍_{S^2} e^g d\mu\right) = 0 = \log\left(⨍_{S^2} e^f d\mu\right) - ⨍_{S^2} f d\mu + ⨍_{S^2} g d\mu$$

by (61). In view of Lemma 8.2, we see that the right-hand side of (61) is equal to

$$\begin{aligned} &\frac{1+\varepsilon}{2}\left\{\frac{1}{16\pi}\int_{S^2} |\text{grad } (g)_{\phi_{Q,\xi}}|^2 d\mu + ⨍_{S^2} (g)_{\phi_{Q,\xi}} d\mu\right\} \\ &\quad + \frac{1-\varepsilon}{2} ⨍_{S^2} (g)_{\phi_{Q,\xi}} d\mu + K_\varepsilon \\ &\quad = \frac{1+\varepsilon}{2}\left\{\frac{1}{16\pi}\int_{S^2} |\text{grad } g|^2 d\mu + ⨍_{S^2} g d\mu\right\} \\ &\quad + \frac{1-\varepsilon}{2} ⨍_{S^2} (g)_{\phi_{Q,\xi}} d\mu + K_\varepsilon. \end{aligned}$$

Therefore,

$$\begin{aligned} \log\left(⨍_{S^2} e^f d\mu\right) &\leq \frac{1+\varepsilon}{32\pi}\int_{S^2} |\text{grad } f|^2 d\mu + ⨍_{S^2} f d\mu \\ &\quad + \frac{1-\varepsilon}{2} ⨍_{S^2} ((g)_{\phi_{Q,\xi}} - g) d\mu + K_\varepsilon. \end{aligned}$$

By means of the formulae (17) this means

$$\begin{aligned} \log\left(⨍_D e^w dx\right) &\leq \frac{1+\varepsilon}{16\pi}\int_D |\nabla w|^2 dx + C_\varepsilon \|w\|_{L^1(D)} + K_\varepsilon \\ &\quad + \frac{1-\varepsilon}{2} ⨍_{S^2} ((g)_{\phi_{Q,\xi}} - g) d\mu. \end{aligned} \tag{63}$$

Thus we have proven that any $w \in \mathcal{F}$ admits $(Q, \xi) \in E_0 \times [1, \infty)$ satisfying (63). If $\{(Q, \xi)\}$ is bounded in $E_0 \times [1, \infty)$, then there exists a constant C such that

$$\begin{aligned}\int_{S^2}((g)_\phi - g)d\mu &= \int_{S^2}((f)_\phi - f)d\mu \\ &\leq C\left(\|f\|_{L^1(S^2)} + 1\right) = C\left(\|wp_*\|_{L^1(D)} + 1\right).\end{aligned}$$

In fact, we observe that

$$\begin{aligned}⨍_{S^2} f_\phi d\mu &\leq ⨍_{S^2} f \circ \phi d\mu + \|\log|\det(d\phi)|\|_{L^1(S^2)} \\ &= ⨍_{S^2} f|\det(d\phi)|^{-1}d\mu + \|\log|\det(d\phi)|\|_{L^1(S^2)}.\end{aligned}$$

The family $\mathcal{F}$ has the property (i).

If $\{(q, \xi)\}$ is not bounded, there exists a sequence $\{(Q_k, \xi_k)\}_{k=1}^{\infty}$ with $\xi_k \to +\infty$. We have

$$⨍_{S^2} \exp\left((g_k)_{\phi_{Q_k,\xi_k}} d\mu\right) = ⨍_{S^2} \exp(g_k)\, d\mu = 1. \tag{64}$$

Passing through the subsequcence if necessary, we have the convergence

$$Q_k \to Q \in E, \qquad w^*\text{-}\lim_{k\to\infty} \exp\left((g_k)_{\phi_{Q_k,\xi_k}}\right) d\mu = d\sigma,$$

and

$$w^*\text{-}\lim_{k\to\infty} e^{g_k} d\mu = d\nu,$$

where $d\sigma$, $d\nu \in \mathcal{M}(S^2)$ with $\sigma(S^2) = \nu(S^2) = 4\pi$.

Let $Q' \in S^2$ be the south pole when $Q \in S^2$ is regarded as a north pole. Under the assumption $\xi_k \to \infty$ we have

$$\phi_{Q_k,\xi_k}(x) \to Q \qquad \text{locally uniformly in } x \in S^2\backslash\{Q'\}.$$

Taking a compact set $K \subset S^2\backslash\{Q'\}$ and a constant $\varepsilon > 0$, we get the inclusion $\phi_{Q_k,\xi_k}(K) \subset B_3(Q, \varepsilon)$ for k: sufficiently large. Hence

$$\int_K \exp\left((g)_{\phi_{Q_k,\xi_k}}\right) d\mu = \int_{\phi_{Q_k,\xi_k}(K)} e^g d\mu \leq \int_{S^2 \bigcap B_3(Q,\varepsilon)} e^g d\mu.$$

This implies

$$\sigma(K) \leq \nu(S^2 \textstyle\bigcap B_3(Q, \varepsilon)),$$

or

$$\sigma(S^2\backslash\{Q'\}) \leq \nu(\{Q\}).$$

Next, we note $(w_k)_{\phi_{Q_k,\xi_k}} \in X_0$. This implies $\int_{S^2} x d\sigma = \vec{0}$. Regard Q as the north pole $(0, 0, 1)$. Then we conclude that

$$\begin{aligned}\sigma(S^2\backslash\{Q'\}) &\geq \int_{S^2_+} x_3 d\sigma = -\int_{S^2_-} x_3 d\sigma \\ &\geq \sigma(\{Q'\}) = 4\pi - \sigma(S^2\backslash\{Q'\}).\end{aligned}$$

This implies

$$2\pi \le \sigma(S^2 \backslash \{Q'\}) \le \nu(\{Q\}).$$

So far we have proven that $g_k = f_k - \log\left(⨍_{S^2} e^{f_k} d\mu\right)$ satisfies

$$w^*\text{-}\lim_{k\to\infty} e^{g_k} d\mu = d\nu \quad \text{in} \quad \mathcal{M}(S^2)$$

with $\nu(\{Q\}) \ge 2\pi$ for some $Q \in E_0$. Putting $q = s_{(0,0,1)}(Q)$ and $\lambda = \nu \circ s^{-1}_{(0,0,1)}$, we observe that q, λ and w_k satisfy the second alternative of the proposition. □

Proof of Propositon 8.1: We follow the argument for the proof of the previous proposition. For each $t \in [0,T)$, $g(\cdot,t) \in X$ is defined subject to $w(\cdot,t)$. This time $g(\cdot,\cdot)$ is a continuous map from $[0,T)$ to $H^1(S^2)$, so that by Lemma 8.3 there exist a continuous map $(Q(\cdot), \xi(\cdot))$ from $[0,T)$ to $(E \times [1,\infty))/(E \times \{1\}) \cong D$ such that $(f)_{\phi_{Q(t),\xi(t)}}(\cdot,t) \in X_0$ for $t \in [0,T)$.

In the case that $\liminf_{t\to T} \xi(t) < +\infty$, there exists a sequence of

$$\{t_k\}_{k=1}^{\infty} \subset [0,T)$$

with $\lim_{k\to\infty} t_k = T$ such that $\sup_{k\ge1} \xi(t_k) < \infty$. Therefore $\{w(\cdot,t_k)\}_{k=1}^{\infty}$ satisfies the property (i) as in the proof of Proposition 8.2.

If $\lim_{t\to T} \xi(t) = \infty$, there exists some $t_0 \in [0,T)$ such that $\xi(t) \ge 2$ for any $t \in [t_0,T)$. This implies that $Q(\cdot)$ is a continuous from $[t_0,T)$ to E_0. Then, similarly we get a continuous map $q(\cdot)$ from $[t_0,T)$ to ∂D such that

$$\liminf_{t\to T} \frac{\int_{D\bigcap B(q(t),\varepsilon)} \exp\left(w(x,t)\right) p_*(x)dx}{\int_D \exp\left(w(x,t)\right) p_*(x)dx} \ge \frac{1}{2} \qquad \text{for any } \varepsilon > 0$$

and the proof is complete. □

A Appendix

A.1 Proof of Proposition 2.2

The following lemma is a modification of inequality ([4])

$$\|w\|_{L^3(\Omega)} \le \varepsilon \|w\|^2_{H^1(\Omega)} \|w \log |w|\|_{L^1(\Omega)} + C_\varepsilon \|w\|_{L^1(\Omega)}$$

for any $w \in H^1(\Omega)$. The proof is done by using a similar way to that in [4] and the following inequality

$$\|w\|^4_{L^4(\Omega)} \le C\|w\|^2_{H^1(\Omega)} \|w\|^2_{L^2(\Omega)} \tag{65}$$

by the Gagliarde-Nerenberg's inequality for two dimensional domain Ω.

Lemma A.1 *For any $N > 1$ it holds that*

$$\|w\|_{L^4(\Omega)}^4 \leq \frac{1}{\log N}\|\nabla w\|_{L^2(\Omega)}^2 \|w^2 \log |w|^2\|_{L^1(\Omega)} + CN^2\|w\|_{L^2(\Omega)}^2$$

for any $w \in H^1(\Omega)$, where C is a positive constant which is independent of $N > 1$.

Proof: Consider a number $N > 1$ and the functin F_N defined on $\mathbf{R}$ by

$$F_N(s) = \begin{cases} 0 & \text{for } |s| \leq N, \\ 2(|s| - N) & \text{for } N < |s| \leq 2N, \\ |s| & \text{for } |s| > 2N. \end{cases}$$

Then it holds that

$$F_N(s) \leq |s| \qquad \text{for any } s \in \mathbf{R}. \tag{66}$$

For each $w \in H^1(\Omega)$, we oberve that $F_N(w) \in H^1(\Omega)$,

$$\||w| - F_N(w)\|_{L^4(\Omega)}^4 \leq (2N)^2 \int_{\{x\in\Omega||w|\leq 2N\}} |w|^2 dx \leq (2N)^2\|w\|_{L^2(\Omega)}^2 \tag{67}$$

and that

$$\begin{aligned} \|F_N(w)\|_{L^2(\Omega)}^2 &\leq \int_{\{x\in\Omega||w|\geq N\}} |w|^2 dx \\ &\leq \frac{1}{\log N^2}\int_{\{x\in\Omega||w|\geq N\}} |w|^2 \log |w|^2 dx \\ &\leq \frac{1}{2\log N}\||w|^2 \log |w|^2\|_{L^1(\Omega)}. \end{aligned} \tag{68}$$

Moreover, the H^1−norm of $F_N(w)$ can be estimated by

$$\|\nabla F_N(w)\|_{L^2(\Omega)} \leq 2\|\nabla w\|_{L^2(\Omega)}. \tag{69}$$

Since we oberve that

$$\|w\|_{L^2(\Omega)} \leq |\Omega|^{1/4}\|w\|_{L^4(\Omega)},$$

we oberve that

$$\|w\|_{H^1(\Omega)}^2 \leq \|\nabla w\|_{L^2(\Omega)}^2 + |\Omega|^{1/2}\|w\|_{L^4(\Omega)}^2. \tag{70}$$

By (65) and (70) we have that

$$\|w\|_{L^4(\Omega)}^4 \leq C\|\nabla w\|_{L^2(\Omega)}^2\|w\|_{L^1(\Omega)}^2 + \frac{1}{2}\|w\|_{L^4(\Omega)}^4 + \frac{1}{2}C^2|\Omega|\|w\|_{L^2(\Omega)}^4,$$

where C is the constant in (65). Then we get that

$$\|w\|_{L^4(\Omega)}^4 \leq 2C\|\nabla w\|_{L^2(\Omega)}^2\|w\|_{L^2(\Omega)}^2 + C^2|\Omega|\|w\|_{L^2(\Omega)}^4,$$

by which together with (66)-(69) we have that

$$\begin{aligned}
\|w\|_{L^4(\Omega)}^4 &\le 8\|F_N(w)\|_{L^4(\Omega)}^4 + 8\||w| - F_N(w)\|_{L^4(\Omega)}^4 \\
&\le 8C\|\nabla F_N(w)\|_{L^2(\Omega)}^2 \|F_N(w)\|_{L^2(\Omega)}^2 \\
&\quad +8C^2|\Omega|\|F_N(w)\|_{L^2(\Omega)}^4 + 8(2N)^2\|w\|_{L^2(\Omega)}^2 \\
&\le \frac{16C}{\log N}\|\nabla w\|_{L^2(\Omega)}^2 \||w|^2 \log |w|^2\|_{L^1(\Omega)} + \left\{32N^2 + 8C^2|\Omega|\right\} \|w\|_{L^2(\Omega)}^2.
\end{aligned}$$

We denote $N^{\max\{16C,1\}}$ and $(32+8|\Omega|C^2)^{1/2}$ by N and C, respectively. Then, the proof is complete. □

We prove Proposition 2.2.

Proof of Proposition 2.2: Multiplying $\log u$ by the first equation of (P) and using the second equation of (P), we have that

$$\frac{d}{dt}\int_\Omega u \log u dx + \int_\Omega u^{-1}|\nabla u|^2 dx + \int_\Omega uv dx = \int_\Omega u^2 dx. \tag{71}$$

Applying Lemma A.1 as $w = u^{1/2}$, we obtain that for any $N > 1$

$$\begin{aligned}
\int_\Omega u^2 dx \le &\frac{1}{4\log N}\int_\Omega u^{-1}|\nabla u|^2 dx \int_\Omega u \log u dx \\
&+ \frac{|\Omega|}{2e\log N}\int_\Omega u^{-1}|\nabla u|^2 dx + CM^2N^2
\end{aligned}$$

and hence

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u \log u dx + \left(1 - \frac{|\Omega|}{2e\log N} - \frac{1}{4\log N}\int_\Omega u\log u dx\right) \\
&\quad \cdot \int_\Omega u^{-1}|\nabla u|^2 dx \le CN^2M^2
\end{aligned}$$

for any $N > 1$. Taking

$$N = \exp\left(\frac{1}{2}\int_\Omega u \log u dx + \frac{|\Omega|}{e}\right) > 1,$$

we obtain

$$\frac{d}{dt}\int_\Omega u\log u dx \le CM^2 \exp\left(\int_\Omega u \log u dx + \frac{2|\Omega|}{e}\right).$$

Then, a standard argument shows that

$$\liminf_{t\to T_{max}} \int_\Omega u \log u dx < \infty.$$

implies

$$\sup_{0\le t<T_{max}} \int_\Omega u \log u dx < \infty. \tag{72}$$

Then Lemma 2.1 implies

$$\sup_{0\le t<T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty$$

similarly to [24]. It is a contradiction. Then we have (i) in this proposition.
By (i) in this proposition and Lemma 2.1, we have that

$$\lim_{t\to T_{max}} \int_\Omega \left(|\nabla v|^2 + v^2\right) dx = \infty,$$

by which together with Lemma 4.2 it follows that

$$\lim_{t\to T_{max}} \int_\Omega e^{av} dx = \infty \qquad \text{for any } a > \frac{1}{2}.$$

Then we have (ii) and (iii) in this proposition. Thus the proof is complete. □

A.2 Best Constants in (17) and (18)

We show to be able to take the constants K in (18) and (19) as 1, which are best possible.

Lemma A.2 *Inequality (19) holds for any $w \in H_0^1(\Omega)$, where $K \ge 1$. When Ω is a ball, if $K < 1$, there exists a function in $H_0^1(\Omega)$ which does not satisfy the inequality (19).*

For the proof of Lemma A.2, the structure of the following problem is nessecarry.

$$\text{(EF)} \qquad \begin{cases} -\Delta w = \dfrac{\sigma}{\int_\Omega e^w dx} e^w & \text{in } D, \\ w = 0 & \text{on } \partial D, \end{cases}$$

where σ is a constant. The following fact is due to [28, Theorem 3.1].

Lemma A.3 *For each $\sigma \in (0, 8\pi)$, there exists a unique solution $w(x) = 2\log\left(\frac{1+\mu}{|x|^2+\mu}\right)$ to (EF), where $\mu = (8\pi/\sigma) - 1$.*

For each $\sigma > 0$ and $w \in H_0^1(\Omega)$, we put

$$J_\sigma^\Omega(w) = \frac{1}{2}\int_\Omega |\nabla w|^2 dx - \sigma \log\left(\int_\Omega e^w dx\right).$$

For a measurable function h on a domain $\omega \subset \mathbf{R}^2$, let us put

$$\mu(t) = |\{x \in \omega |\ |h(x)| > t\}| \qquad \text{for each } t \ge 0$$

and

$$h^*(s) = \sup\{t \geq 0 | \mu(t) \geq s\} \qquad \text{for each } s \in [0, |\omega|].$$

Then h^* is a decreasing function from $[0, |\omega|]$ to $[0, \infty]$ called the decreasing rearrangement, and the symmetrized rearrangement $h^\sharp$ of h is defined by

$$h^\sharp(x) = h^*(\pi |x|^2) \qquad \text{for any } x \in \omega^\sharp,$$

where $\omega^\sharp = D_l$ with $|\omega^\sharp| = |D_l|$.

Proof of Lemma A.2: We may assume $|\Omega| = \pi$ without loss of generality, that is, $\Omega^\sharp = D$.

For each $w \in H_0^1(\Omega)$, the symmetrized rearrangement $w^\sharp$ of w satisfies that

$$\int_\Omega e^w dx = \int_D e^{w^\sharp} dx, \qquad \int_\Omega |\nabla w|^2 dx \geq \int_D |\nabla w^\sharp|^2 dx. \tag{73}$$

For each $\sigma \in (0, 8\pi)$, by Lemma 5.1 we observe that there exists a minimizer of J_σ^Ω in $H_0^1(\Omega)$, by which together with (73) it follows that

$$J_\sigma^\Omega(w) \geq J_\sigma^D(w^\sharp) \geq J_\sigma^D(v_\sigma) \qquad \text{for any } w \in H_0^1(\Omega), \tag{74}$$

where v_σ is the minimizer of J_σ^D in $H_0^1(D)$. By using the standard arguments, we have that the minimizer v_σ is a solution to (EF), by which together with Lemma A.3 it follows that

$$v_\sigma(x) = 2 \log \left(\frac{1+\mu}{|x|^2 + \mu} \right), \tag{75}$$

where $\mu = (8\pi/\sigma) - 1$. By a simple calculation, we have

$$\begin{aligned} \min_{w \in H_0^1(\Omega)} J_\sigma^D(w) &= J_\sigma^D(v_\sigma) \\ &= -\frac{8\pi}{1+\mu} \left(1 + \log \pi + \mu \log \frac{1+\mu}{\mu} \right) \end{aligned} \tag{76}$$

for each $\sigma \in (0, 8\pi)$, where $\mu = (8\pi/\sigma) - 1$.

Noting

$$\begin{aligned} \lim_{\sigma \nearrow 8\pi} J_\sigma^D(v_\sigma) &= \lim_{\mu \searrow 0} \left\{ -\frac{8\pi}{1+\mu} \left(1 + \log \pi + \mu \log \frac{1+\mu}{\mu} \right) \right\} \\ &= -8\pi(1 + \log \pi), \end{aligned}$$

we observe that for each $\varepsilon > 0$ and $w \in H_0^1(D)$ there exists a positive constant δ such that

$$\delta \log \left(\int_D e^w dx \right) < \frac{\varepsilon}{2}$$

and

$$\left| J_\sigma^D(v_\sigma) + 8\pi(1 + \log \pi) \right| < \frac{\varepsilon}{2} \qquad \text{for any } \sigma \in (8\pi - \delta, 8\pi).$$

Then we have that

$$\begin{aligned} J^D_{8\pi}(w) &> J^D_\sigma(w) - \frac{\varepsilon}{2} \\ &\geq J^D_\sigma(v_\sigma) - \frac{\varepsilon}{2} > -8\pi(1+\log\pi) - \varepsilon \end{aligned}$$

for any $\sigma \in (8\pi - \delta, 8\pi)$, by which it follows that

$$J^D_{8\pi}(w) \geq -8\pi(1+\log\pi) \qquad \text{for any } w \in H^1_0(D). \tag{77}$$

By (74) and (77), we have

$$J^\Omega_{8\pi}(w) \geq J^D_{8\pi}(w^\sharp) \geq -8\pi(1+\log\pi)$$

for any $w \in H^1_0(\Omega)$. This means the first part of the lemma.

By a simple calculation, we have that

$$J^D_{8\pi}(v_\sigma) = -\frac{8\pi}{1+\mu}(1+\log\pi) \qquad \text{for any } \sigma \in (0, 8\pi).$$

This means the second part of the lemma. □

Proposition A.1 *Inequality (18) holds for any $w \in H^1(D)$, where $K \geq 1$. If $K < 1$, there exists a function in $H^1(D)$ which does not satisfy the inequality (18).*

Proof of Proposition A.1: By the proof of Proposition 5.1, we observe that the constant K in Proposition 5.1 is equal to the constant K in Lemma 5.1, by which together with Lemma A.2 it follows the first half of the proposition. By using the similar calculation to one in the proof of Lemma A.2, if $K < 1$ we observe that the function v_σ in (75) does not satisfies (18) for some $\sigma \in (0, 8\pi)$. This means the second half of the proposition. □

A.3 Proof of Lemma 7.1

In this subsection, we proof Lemma 7.1.

Proof of Lemma 7.1 In order to prove this lemma, we begin by introducing a diffeomorophism which straightens the boundary portion near a point $q \in \partial\Omega$. Throught translation and rotation of the coordinate system, we may assume that q is the origin and the inner normal to $\partial\Omega$ at q is pointing in the direction of the positive x_2 axis. Then, there exists a smooth function $\phi(x_1)$ defined for $|x_1|$ sufficiently small such that (i) $\phi(0) = 0$ and $\phi'(0) = 0$; and (ii) $\partial\Omega \bigcap \mathcal{O} = \{(x_1, x_2) | x_2 = \phi(x_1)\}$ and $\Omega \bigcap \mathcal{O} = \{(x_1, x_2) | x_2 > \phi(x_1)\}$, where $\mathcal{O}$ is a neighborhood of q. For $y \in \mathbf{R}^2$ near 0, we define a mapping $x = \Phi(y) = (\Phi_1(y), \Phi_2(y))$ by

$$\Phi_1(y) = y_1 - y_2\phi'(y_1), \qquad \Phi_2(y) = y_2 + \phi(y_1).$$

Since $\phi'(0) = 0$, the defferential map $d\Phi$ of Φ satisfies $d\Phi(0) = I$, where I is the identity map. Then, Φ has the inverse mapping $y = \Phi^{-1}(x)$ on $\{x||x| < r\}$ for some $r \in (0,1)$. We denote $\Phi^{-1} = \Psi = (\Psi_1, \Psi_2)$. We can take a sufficiently small $\eta_* \in (0, (1/2)\min(r, \eta^*))$ such that

$$B(0,\eta_*) \subset \Phi(B(0, 5\eta_*/4)), \qquad \Phi(B(0, 3\eta_*/2)^+) \subset \Omega \bigcap B(0, \min(r, \eta^*)),$$

where $B(0,\eta_*)^+ = B(0,\eta_*) \bigcap \{y \in \mathbf{R}^2 | y_2 > 0\}$. With this transformation the solution $w(y,t) = u(\Phi(y), t)$ satisfies

$$\frac{\partial w}{\partial t} - \sum_{i,j=1,2} a_{ij} \frac{\partial^2 w}{\partial y_i \partial y_j} + \sum_{j=1}^{2} b_j \frac{\partial w}{\partial y_j} + cw = 0 \quad \text{in } B(0, 3\eta_*/2)^+ \times (0, T_{max}),$$

where

$$\begin{aligned} a_{ij}(y) &= \nabla_x \Psi_i(x) \cdot \nabla_x \Psi_j(x), \\ b_j(y) &= -\Delta_x \Psi_j(x) + \nabla_x v(x,t) \cdot \nabla_x \Psi_j(x), \\ c(y,t) &= v(x,t) - u(x,t), \qquad x = \Phi(y). \end{aligned}$$

Define the function $\tilde{w}$ on $B(0, 3\eta/2) \times [0, T_{max})$ by

$$\tilde{w} = \begin{cases} w(y,t) & \text{if } y_2 \geq 0, \\ w(y_1, -y_2, t) & \text{if } y_2 < 0. \end{cases}$$

For $i, j = 1, 2$, we put

$$\begin{aligned} \tilde{a}_{ij} &= \begin{cases} a(y) & \text{if } y_2 \geq 0, \\ (-1)^{\delta_{i2}+\delta_{j2}} a(y_1, -y_2, t) & \text{if } y_2 < 0, \end{cases} \\ \tilde{b}_j(y,t) &= \begin{cases} b_j(y,t) & \text{if } y_2 \geq 0, \\ (-1)^{\delta_{j2}} b_j(y_1, -y_2, t) & \text{if } y_2 < 0, \end{cases} \\ \tilde{c} &= \begin{cases} c(y,t) & \text{if } y_2 \geq 0, \\ c(y_1, -y_2, t) & \text{if } y_2 < 0, \end{cases} \end{aligned}$$

where δ_{ij} is the Kronecker's delta. We then observe that

$$\frac{\partial \tilde{w}}{\partial t} + \tilde{\mathcal{L}} \tilde{w} = 0 \qquad \text{in } B(0, 3\eta_*/2) \times (0, T_{max}),$$

where

$$\tilde{\mathcal{L}} \tilde{w} = -\sum_{i,j=1,2} \frac{\partial}{\partial y_i} \left(\tilde{a}_{ij} \frac{\partial \tilde{w}}{\partial y_j} \right) + \sum_{j=1}^{2} \left(\sum_{i=1}^{2} \frac{\partial \tilde{a}_{ij}}{\partial y_i} + \tilde{b}_j \right) \frac{\partial \tilde{w}}{\partial y_j} + \tilde{c} \tilde{w}.$$

The coefficients satisfy

$$\tilde{a}_{ij} \in W^{1,\infty}(B(0, 3\eta_*/2)), \qquad \tilde{b}_j, \tilde{c} \in L^\infty(B(0, 3\eta_*/2) \times [0, T_{max}))$$

for $i, j = 1, 2$. By which together with [20, Theorem 10.1 in Section III], it holds that

$$\|w\|_{C^{\theta,\theta/2}(\overline{B(0,5\eta_*/4)} \times [0,T_{max}))} < \infty.$$

By which and $\overline{B(0,\eta_*)} \subset \Phi(\overline{B(0, 5\eta_*/4)})$, we oberve that

$$\|u\|_{C^{\theta,\theta/2}(\overline{\Omega \bigcap B(0,\eta_*)} \times [0,T_{max}))} < \infty.$$

Thus, the proof is complete. □

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