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Regularity of Solutions to Some Variational Inequalities for the Stokes Equations

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1 Introduction

The main purpose of the present paper is to give a regularity result of solutions to the following problem:

PROBLEM (F). Find $u\in K_{\sigma}^{1}(\Omega)$ and $p\in L^{2}(\Omega)$ satisfying

$$a(u, v-u)-(p, \operatorname{div}(v-u))+j(v)-j(u)\geq(f, v-u), \quad (\forall v\in K^{1}(\Omega)). \tag{1.1}$$

Here and hereafter the following notation is employed: Ω is a bounded domain in $\mathbb{R}^{m}$, $m=2$ or 3. The boundary $\partial\Omega$ is composed of two connected components Γ_{0} and Γ which are assumed to be suitably smooth. For the sake of simplicity, we assume that $\overline{\Gamma}_{0}\cap\overline{\Gamma}=\emptyset$. We introduce

$$K^{1}(\Omega)=\{v\in H^{1}(\Omega)^{m}\mid v=0 \text{ on } \Gamma_{0}\},$$

then $K_{\sigma}^{1}(\Omega)$ denotes the solenoidal (div $v=0$) subspace of $K^{1}(\Omega)$. $(\cdot, \cdot)$ denotes the inner product in $L^{2}(\Omega)$ or $L^{2}(\Omega)^{m}$ according as scalar-valued functions or vector-valued functions. We set

$$a(u, v)=\frac{1}{2}\int_{\Omega}\sum_{1\leq i,j\leq m}e_{i,j}(u)e_{i,j}(v)\ dx$$

for $u=(u_{1}, \cdots, u_{m})$ and $v=(v_{1}, \cdots, v_{m})$, where

$$e_{i,j}(v)=\frac{\partial v_{i}}{\partial x_{j}}+\frac{\partial v_{j}}{\partial x_{i}}$$

denotes an element of the defomation tensor $E(v)=[e_{i,j}(v)]$. Finally

$$j(v)=\int_{\Gamma}g|v|\ ds, \quad (ds= \text{ the surface element of } \Gamma), \tag{1.2}$$

where g is a given scalar function defined on Γ.

As was described in Fujita and Kawarada [7], the variational inequiality (1.1) arises in the study of the steady motions of the viscous imcompressible fluid under the *frictinal boundary condition*, where u denotes the flow velocity, p the pressure, f the external forces acting on the fluid, and g is called the modulus function of friction. We now review the boundary condition of this type. Let $\sigma(u,p)$ be the stress vector to Γ. That is, we let $\sigma(u,p) = S(u,p)n$, where $S(u,p) = -pI + E(v)$ stands for the stress tensor and n the unit outer normal to Γ. Then we pose on $\sigma(u,p)$ that

$$|\sigma(u,p)| \leq g \tag{1.3}$$

and

$$\begin{cases} |\sigma(u,p)| < g \implies u = 0, \\ |\sigma(u,p)| = g \implies \begin{cases} u = 0 \text{ or } u \neq 0, \\ u \neq 0 \implies \sigma(u,p) = -gu/|u| \end{cases} \end{cases} \tag{1.4}$$

almost everywhere on Γ. The classical form of the firictional boundary value problem for the Stokes equations dealt with in [7] consists of

$$-\triangle u + \nabla p = f \text{ in } \Omega, \quad \operatorname{div} u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma_0 \tag{1.5}$$

together with (1.3) and (1.4). (F) is a weak form of this problem.

The existence theorem was established in [7]. Assume that

$$f \in L^2(\Omega)^m, \quad g \in L^\infty(\Gamma), \quad g > 0 \text{ a.e. on } \Gamma. \tag{H}$$

Then (F) admits of a solution $\{u,p\}$. The velocity part u is unique. However the uniqueness of the pressure part p depends on cases. That is, in general, p is unique up to an additive constant and the constant is restricted via (1.3).

Theorem 1.1. *Assume that (H) and moreover that $g \in H^1(\Gamma)$. Let $\{u,p\}$ be a solution of (F). Then $u \in H^2(\Omega)^m$ and $p \in H^1(\Omega)$ with*

$$\|u\|_2 + \|p\|_1 \leq C(\|f\| + \|g\|_{1,\Gamma}),$$

where $C = C(\Omega)$ is a positive constant. Moreover $\{u,p\}$ satisfies (1.5) almost everywhere in Ω. Furthermore we have $\sigma(u,p) \in H^{1/2}(\Gamma)^m$ and

$$-\sigma(u,p) \in g\partial|u| \quad \text{a.e. on } \Gamma.$$

In the above and in what follows, we write $\|\cdot\|$, $\|\cdot\|_s$ and $\|\cdot\|_{s,\Gamma}$ instead of $\|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_{H^s(\Omega)}$ and $\|\cdot\|_{H^s(\Gamma)}$ respectively. For vector-valued functions, as long as there is no possibility of confusion, we use the same symbols. Furthermore, $\partial|\cdot|$ denotes the subdifferential of the function $|z| = (z_1^2 + \cdots + z_2^2)^{1/2}$. Namely,

$$\partial|z| = \begin{cases} z/|z| & (z \neq 0), \\ \{\zeta \in \mathbb{R}^m;\ |\zeta| \leq 1\} & (z = 0). \end{cases}$$

In order to prove Theorem 1.1, we follow the method of Brézis [2]. Namely, we approximate a solution $\{u, p\}$ of the *inequality* (1.1) by solutions $\{u_\varepsilon, p_\varepsilon\}$ of *equations* which are obtained by replacing j by a regular functional j_ε in (1.1). Then the regularity of $\{u_\varepsilon, p_\varepsilon\}$ is studied.

The organization of the present paper is as follows. In §2, we describe a specific definition of the above mentioned regularized problem, which we will refer as (F_ε). The well-posedness and the approximation result are also discussed there. §3 is devoted to a regularity result for (F_ε). In §4, we have the proof of Theorem 1.1. From the view point of physics, some modififcations of (F) are much more interesting. With this connection, in the final section (§5), we state leak or slip boundary value problems of friction type and give regularity results for these problems without the proofs.

Before concluding Introduction, we shall mention a few remarks.

Remark 1.1. To be rigorously, j should be understood as the functional on $H^{1/2}(\Gamma)^m$;

$$j(\eta) = \int_\Gamma g|\eta|\, ds, \qquad (\eta \in H^{1/2}(\Gamma)^m).$$

However, for the sake of simplicity, we will regard j as the functional on $H^1(\Omega)$ through

$$j(v|_\Gamma) = \int_\Gamma g|v|_\Gamma|\, ds,$$

where

$$v|_\Gamma = \text{ the trace of } v \text{ on } \Gamma,$$

and we write as (1.2).

Remark 1.2. It is well-known that (eg., for example, Duvaut and Lions [4]) there are positive constants δ_0 and δ_1 such that

$$a(u,v) \le \delta_0 \|u\|_1 \|v\|_1 \quad (\forall u, v \in H^1(\Omega)), \quad a(v,v) \ge \delta_1 \|v\|_1^2 \quad (\forall v \in K^1(\Omega)).$$

Remark 1.3. Suppose that $\{u, p\}$ is suitably regular and satisfies (1.5) in the classical sense. Multiplying the both sides of $-\Delta u + \nabla p = f$ by $\psi \in K^1(\Omega)$ then integrating over Ω, we have

$$a(u,\psi) - \int_\Omega p \operatorname{div} \psi\, dx = \int_\Gamma \sigma(u,p)\cdot\psi\, ds + \int_\Omega f\cdot\psi\, dx, \quad (\forall \psi \in K^1(\Omega)).$$

According to this identity, we can say that

$$\sigma(u,p) = \omega \text{ on } \Gamma$$

is the Neumann or natural boundary condition corresponding to $a(\cdot,\cdot)$ as the H^1-ellipticity form. Concerning such boundary conditions, we refer to Ladyzhenskaya [10] or Saito [13].

Remark 1.4. In the subsequent sections, C denotes various generic constant. If it depends on parameters $q_1, \cdots, q_M$ which may not be numbers, we shall indicate it by $C = C(q_1, \cdots, q_M)$.

2 Regularized Problem (F_ε)

Let $\varepsilon > 0$. We introduce

$$j_\varepsilon(v) = \int_\Gamma g\rho_\varepsilon(v)\, ds, \qquad (v \in H^1(\Omega)^m), \tag{2.1}$$

where

$$\rho_\varepsilon(v) = \begin{cases} |v| - \varepsilon/2 & (|v| > \varepsilon), \\ |v|^2/(2\varepsilon) & (|v| \le \varepsilon). \end{cases} \tag{2.2}$$

Then we consider

PROBLEM (F_ε). Find $u_\varepsilon \in K^1_\sigma(\Omega)$ and $p_\varepsilon \in L^2(\Omega)$ satisfying

$$\begin{aligned} a(u_\varepsilon, v - u_\varepsilon) - (p_\varepsilon, \operatorname{div}(v - u_\varepsilon)) + j_\varepsilon(v) - j_\varepsilon(u_\varepsilon) \\ \ge (f, v - u_\varepsilon), \quad (\forall v \in K^1(\Omega)). \end{aligned} \tag{2.3}$$

Theorem 2.1. *Assume that (H) and let $\varepsilon > 0$. Then (F_ε) admits a unique solution $\{u_\varepsilon, p_\varepsilon\}$ with*

$$\|u_\varepsilon\|_1 + \|p_\varepsilon\| \le C(\Omega)(\|f\| + \|g\|_{L^2(\Gamma)}).$$

Furthermore, $\{u_\varepsilon, p_\varepsilon\}$ is a weak solution of (1.5) together with

$$-\sigma(u_\varepsilon, p_\varepsilon) = g\alpha_\varepsilon(u_\varepsilon) \quad \text{a.e.} \quad \textit{on } \Gamma, \ (\textit{In particular } \sigma(u_\varepsilon, p_\varepsilon) \in L^2(\Gamma)^m).$$

Namely, $\{u_\varepsilon, p_\varepsilon\}$ satisfies

$$a(u, \psi) - (p, \operatorname{div}\psi) + \int_\Gamma g\alpha_\varepsilon(u) \cdot \psi\, ds = (f, \psi) \quad (\forall \psi \in K^1(\Omega)), \tag{2.4}$$

where we have put

$$\alpha_\varepsilon(v) = \begin{cases} v/|v| & (|v| > \varepsilon) \\ v/\varepsilon & (|v| \le \varepsilon). \end{cases} \tag{2.5}$$

Remark 2.1. In Theorem 2.1, $\sigma(u, p)$ is understood as a functional on $H^{1/2}(\Gamma)^m$ defined by

$$\langle \sigma, \eta \rangle = a(u_\varepsilon, \psi_\eta) - (p_\varepsilon, \operatorname{div}\psi_\eta) - (f, \psi_\eta), \quad (\forall \eta \in H^{1/2}(\Gamma)^m),$$

where $\psi_\eta \in K^1(\Omega)$ is any extension of η.

Proof of Theorem 2.1. From standard theory of convex analysis (e.g., Ekeland and Temam [5], or Glowinski [8]), the minimization problem: Find $u \in K^1_\sigma(\Omega)$ satisfying

$$\mathcal{J}_\varepsilon(u) = \inf_{v \in K^1_\sigma(\Omega)} \mathcal{J}_\varepsilon(v), \quad \mathcal{J}_\varepsilon(v) = \frac{1}{2}a(v, v) - (f, v) + j_\varepsilon(v)$$

has a unique solution u which is characteraized by

$$a(u, v-u) + j_\varepsilon(v) - j_\varepsilon(u) \geq (f, v-u), \quad (\forall v \in K^1_\sigma(\Omega)). \tag{2.6}$$

We are going to show that a scalar function p can be taken as $\{u, p\}$ sloves (2.3). Le $\phi \in K^1_\sigma(\Omega)$ and $t > 0$. Substituting into (2.6) $v = u + t\phi$ and letting $t \to 0$, we have

$$a(u, \phi) + \int_\Gamma g\alpha_\varepsilon(u) \cdot \phi \, ds = (f, \phi), \quad (\forall \phi \in K^1_\sigma(\Omega)).$$

By using this, in the same line as Solonnikov and Ščadilov [16], we can take a unique $p \in L^2(\Omega)$ satisfying (2.4).

Thanks to the convexity of j_ε,

$$\int_\Gamma g\alpha_\varepsilon(v) \cdot (w - v) \, ds \leq j_\varepsilon(w) - j_\varepsilon(v), \quad (\forall v, w \in H^1(\Omega)^m). \tag{2.7}$$

In view of (2.4) and (2.7), we can easily verify that $\{u, p\}$ solves (F_ε). On the other hand, (2.4) yields

$$\langle \sigma, \psi_\eta \rangle + \int_\Gamma g\alpha_\varepsilon(u) \cdot \psi_\eta \, ds = 0 \quad (\forall \eta \in H^{1/2}(\Gamma)^m),$$

where $\psi_\eta \in K^1(\Omega)$ is any extension of η. Consequently, it follows from $g\alpha_\varepsilon(u) \in L^2(\Gamma)^m$ that $\sigma(u, p) \in L^2(\Gamma)^m$ and

$$-\sigma(u, p) = g\alpha_\varepsilon(u) \quad \text{a.e. on } \Gamma,$$

which completes the proof. □

Remark 2.2. As mentioned above, in order to derive (2.4), we follow the method of [16], in which the following facts are applied. Through Riesz's representation theorem, we define an operator B from $L^2(\Omega)$ to $K^1(\Omega)$ by

$$(Bq, v)_{H^1(\Omega)^m} = (p, \operatorname{div} v), \quad (\forall q \in L^2(\Omega); \ \forall v \in K^1(\Omega)).$$

The range $R(B)$ of B forms a closed subspace of $K^1(\Omega)$. Moreover, the orthogonal decomposition

$$K^1(\Omega) = R(B) \oplus K^1_\sigma(\Omega)$$

holds. For the proof, we refer to Saito et al. [14].

Theorem 2.2. *Assume that (H) holds. Let $\{u, p\}$ and $\{u_\varepsilon, p_\varepsilon\}$ be solutions of (F) and (F_ε), respectively. Then we have:*

$$\|u_\varepsilon - u\|_1 + \|\tilde{p}_\varepsilon - \tilde{p}\| \leq C(\Omega, g)\sqrt{\varepsilon}, \tag{2.8}$$

where $\tilde{p}$ stands for the normalization of p subject to

$$\tilde{p} = p - \frac{1}{|\Omega|} \int_\Omega p \, dx, \quad (|\Omega|\text{: the measure of } \Omega),$$

and where the meaning of $\tilde{p}_\varepsilon$ is same.

Proof. Since the derivation of $\|u - u_\varepsilon\|_1 \le C(\Omega, g)\sqrt{\varepsilon}$ is essentially same as Kikuchi and Oden [9], we omit to mention it and proceed to the estimate of the pressure part. Putting $q_\varepsilon = \tilde{p}_\varepsilon - \tilde{p}$, we have

$$(2.9) \qquad a(u - u_\varepsilon, \phi) = (q_\varepsilon, \operatorname{div} \phi) \quad (\forall \phi \in H_0^1(\Omega)^m).$$

In view of Babušca-Aziz's lemma ([2]), we can take $w_\varepsilon \in H_0^1(\Omega)^m$ subject to div $w_\varepsilon = q_\varepsilon$ in Ω with $\|w_\varepsilon\|_1 \le C(\Omega)\|q_\varepsilon\|$. Now substituting $\phi = w_\varepsilon$ into (2.9), we deduce

$$\|q_\varepsilon\|^2 = a(u - u_\varepsilon, w_\varepsilon) \le \delta_0 \|u - u_\varepsilon\|_1 \|w_\varepsilon\|_1 \le \delta_0 C(\Omega) \|u - u_\varepsilon\|_1 \|q_\varepsilon\|.$$

Combining this with the estimate of the velocity part, we arrive at (2.8). □

3 Regularity Results for ($\mathbf{F}_\varepsilon$)

Concering a regularity of a solution $\{u_\varepsilon, p_\varepsilon\}$ of (F_ε), we have

Theorem 3.1. *Assume that (H) and $g \in H^1(\Gamma)$ hold. For any $\varepsilon > 0$, let $\{u_\varepsilon, p_\varepsilon\}$ be a solution of (F_ε). Then $u_\varepsilon \in H^2(\Omega)^m$ and $p_\varepsilon \in H^1(\Omega)$ with*

$$(3.1) \qquad \|u_\varepsilon\|_2 + \|p_\varepsilon\|_1 \le C(\Omega)(\|f\| + \|g\|_{1,\Gamma}).$$

We firstly review a regularity result for the Stokes equations under the Neumann boundary condition.

Lemma 3.1. *Let $f \in L^2(\Omega)^m$ and $\omega \in H^{1/2}(\Gamma)^m$. Suppose that $\{u, p\} \in H^1(\Omega)^m \times L^2(\Omega)$ is a weak solution of (1.5) with*

$$\sigma(u, p) = \omega \ \ on\ \Gamma.$$

Namely, $\{u, p\}$ satisfies

$$a(u, \psi) - (p, \operatorname{div} \psi) = \int_\Gamma \omega \cdot \psi \, ds - (f, \psi), \quad (\forall \psi \in K^1(\Omega)).$$

Then $u \in H^2(\Omega)^m$ and $p \in H^1(\Omega)$ with

$$\|u\|_2 + \|p\|_1 \le C(\Omega)(\|f\| + \|\omega\|_{1/2,\Gamma}).$$

Lemma 3.1 in the case of $\omega \equiv 0$ was described in Solonnikov [15] with a mention on Solonnikov and Ščadilov [16] concerning the method of the proof. However it seems that the complete proof for the case of $\omega \not\equiv 0$ is not explicitly stated in these papers; In this connection, we refer to a forthcoming paper Saito [13].

Lemma 3.2. *Let $\{u_\varepsilon, p_\varepsilon\}$ be a solution of (F_ε), and put $\omega_\varepsilon = g\alpha_\varepsilon(u_\varepsilon)|_\Gamma$. Then we have $\omega_\varepsilon \in H^{1/2}(\Gamma)^m$. Thus, from Lemma 3.1, $u_\varepsilon \in H^2(\Omega)^m$ and $p_\varepsilon \in H^1(\Omega)$.*

Proof. Firstly we have $\alpha_\varepsilon(u_\varepsilon) \in H^{1/2}(\Gamma)^m$ with

$$\|\alpha_\varepsilon(u_\varepsilon)\|_{1/2,\Gamma} \leq C(\Omega,\varepsilon)\|u_\varepsilon|_\Gamma\|_{1/2,\Gamma}. \tag{3.2}$$

This is essentially due to Brézis [2], where he dealt with the scalar case. It is possible to extend his result into our vector-values case; See [14] or [12]. Let us denote by $\tilde{g} \in H^1(\Omega)$ the weak harmonic extension of $g \in H^{1/2}(\Gamma)$:

$$\Delta\tilde{g} = 0 \text{ in } \Omega, \quad \tilde{g} = 0 \text{ on } \Gamma_0, \quad \tilde{g} = g \text{ on } \Gamma.$$

It follows from the maximum principle that $\|\tilde{g}\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\Gamma)}$. On the other hand, we take the weak harmonic extension $\tilde{\alpha}_\varepsilon \in H^1(\Omega)$ of $\alpha_\varepsilon(u_\varepsilon)$. That is, we extend each component of $\alpha_\varepsilon(u_\varepsilon)$ into Ω by the harmonic function. By the definition of α_ε and again using the maximum principle, we have $\|\tilde{\alpha}_\varepsilon\|_{L^\infty(\Omega)} \leq \|\alpha_\varepsilon(v)\|_{L^\infty(\Gamma)} \leq C(m)$. Therefore, since $\tilde{g}\tilde{\alpha}_\varepsilon \in H^1(\Omega)^m$, the trace $\omega_\varepsilon \in H^{1/2}(\Gamma)^m$. □

Remark 3.1. Our chooice of a regularized functional is based on the Yosida regularization. Namely, putting $\rho(z) = |z|$ for $z \in \mathbb{R}^m$, then we have

$$\text{“the Yosida regularization of } \partial\rho\text{”} = \nabla\rho_\varepsilon = \alpha_\varepsilon. \tag{3.3}$$

A property of the Yosida regularization (or a direct calculation) gives

$$|\alpha_\varepsilon(z) - \alpha_\varepsilon(w)| \leq \frac{1}{\varepsilon}|z-w|, \qquad (z, w \in \mathbb{R}^m)$$

which is needed to prove (3.2). On the other hand, in view of (3.3) and Proposition 3 (Appendice I, Brézis [2]), we also have

$$\text{“the Yosida regularization of } \partial j\text{”} = \text{“ the Gâteaux defivative of } j_\varepsilon\text{”}.$$

We proceed to the derivation of (3.1); We need another device.

Lemma 3.3. *Let $\beta_\varepsilon = u_\varepsilon|_\Gamma$. Under the same assumptions of Theorem 3.1, we have*

$$\|\beta_\varepsilon\|_{3/2,\Gamma} \leq C(\Omega)(\|f\| + \|g\|_{1,\Gamma}). \tag{3.4}$$

Because of the limitation of the page number, we cannnot state the complete proof of Lemma 3.3; Below we shall describe a sketch of the proof under a simple situation. Namely, we assume that

$$\Omega = \mathbb{R}^2_+ \equiv \{x = (x_1, x_2); x_2 > 0\}, \quad \Gamma = \{x = (x_1, x_2); x_2 = 0\}$$

and, for $R > 0$, put

$$\mathcal{O}_R = \{x = (x_1, x_2); |x| > R\} \cap \Omega.$$

Moreover we assume that $u_\varepsilon \equiv 0$ in $\mathcal{O}_{R/2}$. We simply write as $u = u_\varepsilon$ and $p = p_\varepsilon$. Put

$$\varphi = -\frac{\partial v}{\partial x_1}, \quad v = \frac{\partial u}{\partial x_1}.$$

Multiplying $-\triangle u + \nabla p = f$ by φ then integrating over Ω, we have

$$a(u,\varphi) - \int_\Omega p \operatorname{div} \varphi \, dx = \int_{-R}^{R} \sigma(u,p)\cdot\varphi \, dx_1 + \int_\Omega f\cdot\varphi\, dx. \tag{3.5}$$

We obtain

$$a(u,\varphi) = a(v,v) \geq \delta_1 \|v\|_1^2,$$

since

$$\int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial \varphi_l}{\partial x_k}\, dx = \int_\Omega \frac{\partial v_i}{\partial x_j}\frac{\partial v_l}{\partial x_k}\, dx, \quad (i,j,k,l = 1,2).$$

By virtue of $\nabla_z \alpha_\varepsilon(z) w\cdot w \geq 0$ for $z, w \in \mathbb{R}^m$, we get

$$\begin{aligned}
\int_{-R}^{R} \sigma(u,p)\cdot\varphi\, dx_1 &= -\int_{-R}^{R} \frac{\partial}{\partial x_1}(g\alpha_\varepsilon(u))\cdot v\, dx_1 \\
&= -\int_{-R}^{R} g'\alpha_\varepsilon(u)\cdot v\, dx_1 - \int_{-R}^{R} g(\nabla_u \alpha_\varepsilon(u)v\cdot v)\, dx_1 \\
&\leq \int_{-R}^{R} |g'|\cdot|\alpha_\varepsilon(u)|\cdot|v|\, dx_1 \\
&\leq C\|g'\|_{L^2(\Gamma)}\|v\|_1.
\end{aligned}$$

Moreover we can easily check that

$$\int_\Omega p \operatorname{div}\varphi\, dx = 0, \qquad \int_\Omega f\cdot\varphi\, dx \leq C\|f\|\|v\|_1.$$

Substituing these results of calculations into (3.5), we have

$$\|v\|_1 \leq C(\|f\| + \|g\|_{1,\Gamma})$$

which implies that $\beta_\varepsilon \in H^{3/2}(\Gamma)$ and

$$\|\beta_\varepsilon\|_{3/2,\Gamma} \leq C(\|f\| + \|g\|_{1,\Gamma}).$$

4 Proof of Theorem 1.1

Let $\varepsilon > 0$, and let $\{u_\varepsilon, p_\varepsilon\}$ be a solution of (F_ε). By virtue of Theorem 3.1, sequences $\|u_\varepsilon\|_2$ and $\|p_\varepsilon\|_1$ are bounded as $\varepsilon \downarrow 0$, respectively. Hence, there are subsequences $\{u_{\varepsilon'}\}$ and $\{p_{\varepsilon'}\}$ such that

$$u_{\varepsilon'} \to u^* \text{ weakly in } H^2(\Omega)^m, \quad p_{\varepsilon'} \to p^* \text{ weakly in } H^1(\Omega)$$

and

$$\|u^*\|_2 + \|p^*\|_1 \leq C(\Omega)(\|f\| + \|g\|_{1,\Gamma}).$$

According to Theorem 2.2, $\{u^*, p^*\}$ is a solution of (F). Next let $\{u, p\}$ be any solution of (F). By the uniqueness of the velocity part, we have $u = u^*$. On the other hand, $p-p^* = k$ and a constant k is restricted via (1.3). Therefore $p \in H^1(\Omega)$, and we deduce

$$\sigma(u,p) - \sigma(u,p^*) = kn \quad \text{a.e. on } \Gamma.$$

This, together with (1.3), implies that $|k| \le 2g$ holds almost everywhere on Γ. Hence $|k| \le 2|\Gamma|^{-1/2}\|g\|_{L^2(\Gamma)}$, where $|\Gamma|$ denotes the measure of Γ. By making use of this estimate, we have

$$\begin{aligned}\|u\|_2 + \|p\|_1 &\le \|u\|_2 + \|p^*\|_1 + |k|\sqrt{|\Omega|} \\ &\le C(\Omega)(\|f\| + \|g\|_{1,\Gamma}),\end{aligned}$$

which completes the proof.

5 Other Problems of Friction Type

In gerenal, for a vector-valued function v, let v_N and v_T denote the normal component and the tangential components of v, respectively;

$$v_N = v \cdot n, \quad v_T = v - v_N n.$$

5.1 Leak Problem of Friction Type

We consider the Stokes flow $\{u,p\}$ satisfying (1.5) together with

$$(5.1) \qquad |\sigma_N(u,p)| \le g_N$$

and

$$(5.2) \qquad \begin{cases} |\sigma_N(u,p)| < g_N \implies u_N = 0, \\ |\sigma_N(u,p)| = g_N \implies \begin{cases} u_N = 0 \text{ or } u_N \ne 0, \\ u_N \ne 0 \implies \sigma_N(u,p)u_N \le 0 \end{cases} \end{cases}$$

almost everywhere on Γ, and

$$(5.3) \qquad u_T = 0 \text{ on } \Gamma.$$

The above problem was introduced H. Fujita ([6]) and is called the *leak boundary value problem of friction type.* As was described in [6], this can be reduced to

PROBLEM (LF). Find $u \in K^1_{L,\sigma}(\Omega)$ and $p \in L^2(\Omega)$ satisfying

$$(5.4) \quad a(u, v-u) - (p, \text{div}\,(v-u)) + j_N(v) - j_N(u) \ge (f, v-u), \quad (\forall v \in K^1_L(\Omega)),$$

where

$$K^1_L(\Omega) = \{v \in K^1(\Omega);\ v_T = 0 \text{ on } \Gamma\}, \quad K^1_{L,\sigma}(\Omega) = K^1_L(\Omega) \cap K^1_\sigma(\Omega),$$

and

$$j_N(v) = \int_\Gamma g_N |v_N|\, ds.$$

Concerning the existence and the uniqueness/non-uniqueness, we know ([6]): Assume that $f \in L^2(\Omega)^m$, $g_N \in L^\infty(\Gamma)$ and $g_N > 0$ a.e. Then there exists a solution $\{u, p\}$ of (LF). The velocity part u is unique and p is unique up to an additive constant and the constant is restricted via (5.1).

The following theorem is proved in Saito [12].

Theorem 5.1. *In additon to the assumptions mentioned above, we assume that* $g_N \in H^1(\Gamma)$. *Let* $\{u, p\}$ *be a solution of (LF). Then* $u \in H^2(\Omega)^m$ *and* $p \in H^1(\Omega)$ *with*

$$\|u\|_2 + \|p\|_1 \leq C(\|f\| + \|g_N\|_{1,\Gamma}).$$

Moreover we have $\sigma_N(u, p) \in H^{1/2}(\Gamma)$ *and*

$$-\sigma_N(u, p) \in g\partial|u_N| \quad \text{a.e. on } \Gamma.$$

5.2 Slip Problem of Friction Type

The *slip boundary value problem of friction type* consists of (1.5) together with

$$|\sigma_T(u, p)| \leq g_T \tag{5.5}$$

and

$$\begin{cases} |\sigma_T(u, p)| < g_T & \Longrightarrow \quad u_T = 0, \\ |\sigma_T(u, p)| = g_T & \Longrightarrow \quad \begin{cases} u_T = 0 \text{ or } u_T \neq 0, \\ u_T \neq 0 \Longrightarrow \sigma_T(u, p) u_T \leq 0 \end{cases} \end{cases} \tag{5.6}$$

almost everywhere on Γ, and

$$u_N = 0 \text{ on } \Gamma \tag{5.7}$$

The weak formulation using the variational inequality is as follows.

PROBLEM (SF). Find $u \in K^1_{S,\sigma}(\Omega)$ and $p \in L^2(\Omega)$ satisfying

$$a(u, v - u) - (p, \text{div } (v - u)) + j_T(v) - j_T(u) \geq (f, v - u), \quad (\forall v \in K^1_S(\Omega)), \tag{5.8}$$

where

$$K^1_S(\Omega) = \{v \in K^1(\Omega);\ v_N = 0 \text{ on } \Gamma\}, \quad K^1_{S,\sigma}(\Omega) = K^1_S(\Omega) \cap K^1_\sigma(\Omega),$$

and

$$j_T(v) = \int_\Gamma g_T |v_T| \, ds.$$

As was mentioned in [6], (SF) admits a solution $\{u, p\}$ if $f \in L^2(\Omega)^m$, $g_T \in L^\infty(\Gamma)$ and $g_N > 0$ a.e. The velocity part u is unique and p is unique except for an additive constant. In this case, the restriction for an additive constant is absent.

Theorem 5.2. *In additon to the assumptions mentioned above, we assume that $g_T \in H^1(\Gamma)$. Let $\{u, p\}$ be a solution of (SF). Then $u \in H^2(\Omega)^m$ and $p \in H^1(\Omega)$ with*

$$\|u\|_2 + \|p\|_1 \leq C(\|f\| + \|g_T\|_{1,\Gamma}).$$

Moreover we have $\sigma_T(u,p) \in H^{1/2}(\Gamma)^m$ and

$$-\sigma_T(u,p) \in g\partial|u_T| \quad \text{a.e. on } \Gamma.$$

For the proof, we refer to [12].

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