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Bounded Palais-Smale Sequences in Minimax Theorems and Applications to Bifurcation Theory

Louis Jeanjean

Université de Franche-Comtée
Laboratoire de Mathématiques
16, route de Gray
F-25030 Besancon CEDEX, FRANCE
e-mail: jeanjean@math.univ-fcomte.fr

In this talk, we consider a functional I having a mountain pass geometry, a MP geometry for short. We address the central issue of the boundedness of, at least, one of the PS sequences given by the MP lemma and develop a generic approach to handle this question which is closely related to the existence of a critical point for I. This approach is then applied to obtain bifurcation results.

To be more precise let $(X, ||\cdot||)$ be a reflexive Banach space and $I \in C^1(X, \mathbb{R})$ a functional having a MP geometry. By definition there exist two points (v_1, v_2) in X such that setting

$$\Gamma := \{\gamma \in C([0,1], X), \gamma(0) = v_1, \gamma(1) = v_2\}$$

the following holds

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)) > \max\{I(v_1), I(v_2)\}.$$

We recall that for such I, Ekeland's principle implies the existence of a PS (Palais-Smale) sequence for I at the level c, namely a sequence $\{u_n\} \subset X$ satisfying $I(u_n) \to c$ and $I'(u_n) \to 0$ in the dual space of X.

On the problem of finding conditions on I insuring the existence of a bounded PS sequence, a BPS sequence for short, at the MP level, most of the work undertaken deals with specific situations. We mean by this that particular properties of the PDE or Hamiltonian system which corresponds to I are crucially used to prove its existence (as in [ABe] or [J1] where the autonomous feature is essential). A more systematic approach is due to Ghoussoub [G]. His ideas of localizing a PS sequence around dual sets is often a strong help to conclude that it is bounded. We also mention the work of Cerami [Ce] who proved that there exists a (Cerami) sequence $\{u_n\} \subset X$ satisfying $I(u_n) \to c$ and $||I'(u_n)||(1+||u_n||) \to 0$ (see also Th.6, p.140 of [E1]). It is often easier to prove that a Cerami sequence is bounded than an arbitrary PS sequence (see for example [BBF]). However probably the most significant contribution in this area is due to Struwe who introduced a general technique often referred to as the "monotonicity trick" (see [Str1, Str2] and Ch. II, Sec. 9 of [Str3]). The result we now give is contained in [J2]. It pursues the work of Struwe but strongly generalizes and simplifies it.

Theorem 0.1 *Let $(X, ||\cdot||)$ be a Banach space, $J \subset \mathbb{R}^+$ an interval and $(I_\lambda)_{\lambda \in J}$ a family of C^1-functionals on X of the form*

$$I_\lambda(u) = A(u) - \lambda B(u), \ \forall \lambda \in J$$

where $B(u) \geq 0, \forall u \in X$ and $B(u) \to +\infty$ as $||u|| \to \infty$. We assume there are two points v_1, v_2 in X such that setting

$$\Gamma := \{\gamma \in C([0,1], X), \gamma(0) = v_1, \gamma(1) = v_2\}$$

the following holds, $\forall \lambda \in J$

$$c(\lambda) := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(v_1), I_\lambda(v_2)\}.$$

Then, for almost every $\lambda \in J$, there is a sequence $\{v_n\} \subset X$ such that

(i) $\{v_n\}$ is bounded, (ii) $I_\lambda(v_n) \to c(\lambda)$, (iii) $I'_\lambda(v_n) \to 0$ in the dual of X.

We point out that under the assumptions of Theorem 0.1 there needs not exist a BPS sequence at level $c(\lambda)$ for every value of $\lambda \in J$ (see [J2] for an example due to Brezis and Nirenberg). On several occasions Struwe has derived results similar to Theorem 0.1 (see [ST] for a recent example). However these results are always obtained on specific examples where heavy conditions on the family $(I_\lambda)_{\lambda \in J}$, needed in his proofs, are present.

To prove Theorem 0.1 we start to notice that the function $\lambda \to c(\lambda)$ is monotone (here nondecreasing, because $B(u) \geq 0, \forall u \in X$). Thus the derivative of $c(\lambda)$, exists almost everywhere and to prove Theorem 0.1 it is enough to show that if $c'(\lambda)$ exists, I_λ has a BPS sequence at the level $c(\lambda)$. We fix a $\lambda_0 \in J$ where $c'(\lambda_0)$ exists. Let $\{\lambda_n\} \subset J$ be a strictly increasing sequence such that $\lambda_n \to \lambda_0$ and $\{\gamma_n\} \subset \Gamma$ be a sequence of paths satisfying

$$\max_{t \in [0,1]} I_{\lambda_n}(\gamma_n(t)) \leq c(\lambda_n) + (\lambda_0 - \lambda_n). \tag{0.1}$$

Such $\{\gamma_n\} \subset \Gamma$ exists because Γ is independent of λ. We prove there is a $K = K(\lambda_0) > 0$ such that

(i) $||\gamma_n(t)|| \leq K$ if $I_{\lambda_0}(\gamma_n(t)) \geq c(\lambda_0) - (\lambda_0 - \lambda_n)$.

(ii) For all $\varepsilon > 0$, $\max_{t\in[0,1]} I_{\lambda_0}(\gamma_n(t)) \leq c(\lambda_0) + \varepsilon$ for $n \in \mathbb{N}$ sufficiently large.

By (ii), $\max_{t\in[0,1]} I_{\lambda_0}(\gamma_n(t)) \to c(\lambda_0)$ and by (i) for all $n \in \mathbb{N}$ sufficiently large, starting from a level strictly below $c(\lambda_0)$, all the "tops" of the paths are contained in the same ball of radius $K > 0$ centred at the origin. Then by a deformation argument we deduce that for all $a > 0$

$$\inf\{\|I'_{\lambda_0}(u)\| : u \in X,\ \|u\| \leq K + 1 \text{ and } |I_{\lambda_0}(u) - c(\lambda_0)| \leq a\} = 0.$$

Namely I_{λ_0} has a PS sequence at level $c(\lambda_0)$ contained in the ball of radius $K+1$ centred at the origin and thus bounded.

Struwe's approach also consists in showing that if $c'(\lambda_0)$ exists, I_{λ_0} has a BPS sequence at level $c(\lambda_0)$ but for this he choose $\{\gamma_n\} \subset \Gamma$ such that

$$\max_{t\in[0,1]} I_{\lambda_0}(\gamma_n(t)) \leq c(\lambda_0) + (\lambda_0 - \lambda_n). \tag{0.2}$$

Then he pursue a contradiction argument supposing there do not exist a BPS sequence for I_{λ_0} at level $c(\lambda_0)$ and searching a contradiction with the variational characterization of $c(\lambda)$ for λ close to λ_0 but distinct. For this he has to deform the paths $\{\gamma_n\}$ with a gradient flow associated to I_{λ_0} on which the contradiction argument is made. Clearly in order to do a suitable deformation there must exist a close connection between the gradients of I_λ and I_{λ_0}. Strong technical requirements are needed there (see [ST]). Our choice of $\{\gamma_n\} \subset \Gamma$ permits to obtain a more general result; Theorem 0.1. (see also 9.5, Ch. II of [Str3]).

The simplicity of the proof of Theorem 0.1 has opened the way to two generalizations. In collaboration with J.F. Toland I removed in [J3] the condition which gives the monotonicity of $c(\lambda)$. Clearly if $\lambda \to c(\lambda)$ is no more monotone the existence of $c'(\lambda)$ almost everywhere is not insured. This generalization is based on the observation that the proof of Theorem 0.1 can be adapted if the existence of $c'(\lambda_0)$ is replaced by the condition

$$\exists\{\lambda_n\} \subset J \text{ such that } \lambda_n \uparrow \lambda_0 \text{ and } \frac{c(\lambda_n) - c(\lambda_0)}{\lambda_0 - \lambda_n} \leq M(\lambda_0) \tag{0.3}$$

for a $M(\lambda_0) < \infty$. It readily follows from a classical result due to Denjoy [Sa, Th. (4.4), p. 270], that the set of points for which (0.3) holds has full measure in J. In [J3] we also extend Theorem 0.1 to a more general family $(I_\lambda)_{\lambda\in J}$.

A control on the size of the BPS sequence obtained in Theorem 0.1 is the second direction of generalization. I show that it is possible to relate the radius of the ball containing the sequence to the quantities $c(\lambda)$ and $c'(\lambda)$. This result is directly applied on bifurcation problems in [J4]. However, first an important consequence of Theorem 0.1 needs to be mentioned:

Special Palais-Smale sequences : Often one is interested in finding a BPS sequence at the MP level for a given functional, namely for a given $\lambda \in J$. Theorem 0.1, just as the improved version given in [J3], is a powerful tool to establish the existence of such sequence. This in particularly so if the problem enjoys some compactness properties:

Corollary 0.1 *Let $(X, ||\cdot||)$ be a Banach space and let $I \in C^1(X, \mathbb{R})$ be of the form $I(u) = A(u) - B(u)$ where B and B' take bounded sets to bounded sets. Suppose there exists $\varepsilon > 0$ such that, for $J = [1-\varepsilon, 1]$, the family $(I_\lambda)_{\lambda \in J}$ defined by*

$$I_\lambda(u) = A(u) - \lambda B(u)$$

satisfies the assumptions of Theorem 0.1 and that for all $\lambda \in J$ any BPS sequences for I_λ at the level $c(\lambda)$ admit a convergent subsequence. Then there exists $\{(\lambda_n, u_n)\} \subset [1-\varepsilon, 1] \times X$ with $\lambda_n \to 1$ and

$$I_{\lambda_n}(u_n) = c(\lambda_n) \text{ and } I'_{\lambda_n}(u_n) = 0$$

such that, if $\{u_n\} \subset X$ is bounded, there hold,

$$I(u_n) = I_{\lambda_n}(u_n) + (\lambda_n - 1)B(u_n) \to \lim_{n\to\infty} c(\lambda_n) = c(1)$$

$$I'(u_n) = I'_{\lambda_n}(u_n) + (\lambda_n - 1)B'(u_n) \to 0 \text{ in the dual of } X.$$

Note that the left continuity of the function $\lambda \to c(\lambda)$ is a consequence of the upper semicontinuity of $c(\lambda)$ (true under general assumptions) and of the monotonicity.

Corollary 0.1 says that, if $\{u_n\}$ is bounded, it is a BPS sequence for I at the MP level. One may wonder about the usefulness of Corollary 0.1 since the existence of a PS sequence for I at the MP level was already known (by Ekeland's principle) and the only remaining problem was, as it is here, to show that it is a bounded sequence. The progress we have made is the following: to show that the sequence is bounded, we use the fact that it is a sequence of real critical points of functionals close to I. The fact that $\{u_n\}$ is a sequence of real critical points (instead of being a sequence of almost critical points of I as in the case of standard PS sequences) often provides additional informations which help to show its boundedness. Imagine that I is defined on a Sobolev space and that its critical points (as those of I_{λ_n}) correspond to solutions of a PDE problem. Then they possess stronger regularity than normally do elements of the ambiant space. Also a use of a maximum principle can often guaranty a sign for u_n, $\forall n \in \mathbb{N}$, and sometimes there exist constraints that u_n must satisfy, for example a Pohozaev's type identity as in [ABe] or [J1]. More globally, for $\lambda \in \mathbb{R}$, let

$$K_\lambda := \{u \in X : I_\lambda(u) = c(\lambda) \text{ and } I'_\lambda(u) = 0\}.$$

If $\cup_{\lambda \in [1-\varepsilon,1]} K_\lambda$ is bounded for a $\varepsilon > 0$ and if for all $\lambda \in [1-\varepsilon, 1]$ any BPS sequence for I_λ at the level $c(\lambda)$ admits a convergent subsequence, I has a critical point.

A bifurcation problem : As we shall see, the approach to boundedness of PS sequences, that we have developed in [J2], is a powerful tool to prove the existence of bifurcation points. In [J4] we consider a family a equations

$$-\Delta u(x) + \lambda u(x) = f(x, u(x)),\ \lambda > 0,\ x \in \mathbb{R}^N, \tag{0.4}$$

where we assume there is $\delta > 0$ such that

(H1) $f : \mathbb{R}^N \times [-\delta, \delta] \to \mathbb{R}$ is Caratheodory.

(H2) $\lim_{|x| \to \infty} f(x, s) = 0$ uniformly for $s \in [-\delta, \delta]$.

(H3) There exists $K > 0$ such that $\limsup_{s \to 0} |f(x, s)s^{-1}| \leq K$ uniformly in $x \in \mathbb{R}^N$.

(H4) $\lim_{s \to 0} F(x, s)s^{-2} = 0$ uniformly in $x \in \mathbb{R}^N$ with $F(x, s) := \int_0^s f(x, t)\, dt$.

(H5) There exist $A > 0$, $d \in]0, 2[$ and $\alpha \in]0, \frac{2(2-d)}{N}[$ such that

$$F(x, s) \geq A(1 + |x|)^{-d}|s|^{2+\alpha} \text{ for all } s \in [-\delta, \delta].$$

We say that $\gamma = 0$ is a bifurcation point if there exists a sequence $\{(\lambda_n, u_n)\} \subset \mathbb{R}^+ \times H^1(\mathbb{R}^N)\backslash\{0\}$ of solutions of (0.4) with $\gamma_n \to 0$ and $||u_n||_{H^1(\mathbb{R}^N)} \to 0$. Our main result is:

Theorem 0.2 *Assume that (H1)-(H5) hold. Then $\lambda = 0$ is a bifurcation point for (0.4).*

On equations of type (0.4) all the works of bifurcation we know require f to be defined on all $\mathbb{R}^N \times \mathbb{R}$ and to satisfy some structures conditions as (SQC) (see for example [Stu1, Stu2, Stu3] and the references within).

A first originality of our result is that only conditions on $f(x, .)$ around zero are required. Our main achievement however is that our conditions around zero are mild. To prove Theorem 0.2 we first modify f by $\tilde{f}$ outside $[-\delta, \delta]$. The function $\tilde{f}$ is choosen so that for all $u \in H = H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \tilde{F}(x, u)\, dx \leq \frac{K}{2}||u||_2^2 \text{ where } \tilde{F}(x, s) := \int_0^s \tilde{f}(x, t)\, dt. \tag{0.5}$$

Then we introduce, for $\lambda > 0$, the family of functionals $I_\lambda : H \to \mathbb{R}$ with

$$I_\lambda(u) = ||\nabla u||_2^2 + \lambda\ ||u||_2^2 - 2\int_{\mathbb{R}^N} \tilde{F}(x, u)\, dx$$

associated to the modify version of (0.4). We show, using (H5), that there exists $\lambda_0 > 0$, such that for all $\lambda \in]0, \lambda_0]$, the following sets are non empty

$$\Gamma_\lambda := \{\gamma \in C([0, 1], H^1(\mathbb{R}^N)), \gamma(0) = 0 \text{ and } I_\lambda(\gamma(1)) < 0\}$$

and

$$c(\lambda) := \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > I_\lambda(0) = 0.$$

Namely I_λ has, for $\lambda \in]0, \lambda_0]$, a MP geometry. Moreover the function $\lambda \to c(\lambda)$ is non decreasing and thus we are in the situation of Theorem 0.1. We deduce that, when $c'(\lambda)$ exists, I_λ has a BPS sequence $\{u_m\} \subset H^1(\mathbb{R}^N)$ at the level $c(\lambda)$. Because of the compacity condition (H2) this leads to a nontrivial critical point of I_λ. This is not enough to get our bifurcation result, we also need precise informations on the size of these critical points. This is achieved refining the proof of existence of $\{u_m\} \subset H^1(\mathbb{R}^N)$. Broadly speaking the argument goes as follows. First we show, that $||u_m||_2$ is controlled from above by $c(\lambda)$ and $c'(\lambda)$. We mean by this that $||u_m||_2$ goes to zero, uniformly in $m \in \mathbb{N}$, if $c(\lambda) \to 0$ and $c'(\lambda) \to 0$. Thus, since

$$||\nabla u_m||_2^2 + \lambda ||u_m||_2^2 - 2 \int_{\mathbb{R}^N} \tilde{F}(x, u_m)\, dx \to c(\lambda)$$

we see from (0.5) that $||\nabla u_m||_2$ is also controlled by $c(\lambda)$ and $c'(\lambda)$. Hence $\{u_m\}$ is contained in a ball centred at the origin whose radius goes to zero when $c(\lambda) \to 0$ and $c'(\lambda) \to 0$. Now test functions show that $c(\lambda)\lambda^{-1} \to 0$ as $\lambda \to 0$ and this implies the existence of a strictly decreasing sequence $\lambda_n \to 0$ such that $c(\lambda_n) \to 0$ and $c'(\lambda_n) \to 0$. Calling $\{u_n\}$ the sequence of solutions corresponding to $\{\lambda_n\}$ we immediately deduce that $||u_n||_{H^1(\mathbb{R}^N)} \to 0$ and this proves the bifurcation for the modified problem. We then show that bifurcation occurs in the L^∞- norm and thus for (0.4).

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