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NOTE ON CLASSIFICATION OF SEXTICS

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ABSTRACT. We classify the local and global singularities of sextics which are tame torus curves of type $(2,3)$ and we also show the degenerations among these classes. As an application, two Zariski pairs are found.

INTRODUCTION

The classification of complex algebraic plane curves is a classical problem. For a survey for the classification of curves of degree less than 6, see [N]. The classification of curves of degree 6 is still not completed, though there are some partial answers for instance in a series papers by Urabe on sextics which has only simple singularities (see [U] and its references), and then by Yang extended Urabe's result in [Y].

According to [Li], singular plane curves of interest are given by

(a) a curve which appears as a branch of generic projections of surfaces,
(b) generic plane sections of discriminants of linear systems on projective space $\mathbb{P}^2$ (this includes dual curves),
(c) a curve defined by explicit equation.

Deformation theory is also useful to prove the existence of curves which can be degenerated into curves of type (a), (b), (c).

In this paper, we mainly use the method (c) for a special class of sextics, which is so called torus curves of type (2,3), and actually these curves have also non-simple singularities, by that reason we can not use the methods of Urabe or of Yang.

Let $C = \{(X;Y;Z) \in \mathbb{P}^2; F(X;Y;Z) = 0\}$ be a complex projective plane curve and let (x, y) be the affine coordinates given by $x = X/Z, y = Y/Z$ on $\mathbb{C}^2 := \mathbb{P}^2 - \{Z = 0\}$. For simply we also keep the notation C in the affine chart $\mathbb{C}^2$, where its defining equation $f(x, y)$ is given by $f(x,y) := F(x, y, 1)$. Then C is called *torus curve of type* (p, q) (or (p, q)-torus curve for shortly) if we can write $f = f_p^q + f_q^p$ for some polynomials f_p, f_q of degree p and q respectively in $\mathbb{C}[x, y]$. In this paper, we consider reduced (2,3)-torus curve. A sextic $C = \{f_3^2 + f_2^3 = 0\}$ of type (2,3) is called *tame* if its singularities are sitting only at the intersection of conic C_2 and cubic C_3, where C_2 and C_3 are respectively defined by $f_2 = 0$ and $f_3 = 0$.

In the first section, we assume that conic and cubic passing through the origin of the affine chart $\mathbb{C}^2$, and (C, O) is an isolated singularity. We will classify all the local singularities at the origin (in the sense of topological equivalence) by considering the geometrical relation of conic and cubic. For that purpose the intersection number $\iota = \mathrm{I}(C_2, C_3; O)$ plays an important role. The result is contained in Theorem 1.

In the second section, we study the possible configurations of singularities on tame torus curves of type (2,3) using the local result obtained in the first section. The main result is Theorem 2.

In the third section we study the spaces with fixed configurations and the possible degenerations among them. Actually these degenerations will be used in the investigation of topological

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problem, namely, the computation of the fundamental group of the complement $\mathbb{P}^2 \setminus C$, where C is a tame torus curve of type (2,3). The details will be given in [OP].

Finally in the last section, we give two new examples of weak Zariski pairs. In each pair, both curves are torus curves of type (2,3).

We omit several proofs in this paper, the full version is given in [P].

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1. Local classification problem

1.1. **Some classes of singularities.** We use the following standard notations for "simple" singularities, which have the normal form:

$$\begin{cases} A_n : x^2 + y^{n+1} = 0 \, (n \geq 1) \\ D_n : x^2 y + y^{n-1} = 0 \, (n \geq 4) \\ E_6 : x^3 + y^4 = 0, \; E_7 : x^3 + xy^3 = 0, \; E_8 : x^3 + y^5 = 0 \end{cases}$$

Furthermore we define notations of some other (topological equivalence classes of) singularities which we use later.

$$\begin{cases} B_{p,q} : x^p + y^q = 0 \text{ (Brieskorn-Pham type)} \\ C_{p,q} : x^p + y^q + x^2 y^2 = 0 \\ D_{4,7} : y^4 + x^3 y^2 + x^7 = 0 \\ Sp_1 : (y^2 - x^3)^2 + (xy)^3 = 0 \\ Sp_2 : (y^2 - x^3)^2 - y^6 = 0 \end{cases} \tag{1.1}$$

Note the symmetry $B_{p,q} = B_{q,p}$, $C_{p,q} = C_{q,p}$ and the identities $B_{2,p} = A_{p-1}$, $B_{3,4} = E_6$. Suppose we have a germ (C, O) of a plane curve. Recall three local invariants: $\mu(C, O)$ is the Milnor number at O, $\delta(C, O)$ is the δ-invariant which is the maximal number of nodes in a generic deformation of C, and $r(C, O)$ is the number of the analytic branches. It is well-known that $\delta = (\mu + r - 1)/2$ (see [M]). The invariant triple (μ, r, δ) will be used for the later arguments.

Definition 1.1. Suppose that we have two germs of plane curve singularities (C, p) and (C', p'). We say that (C, p) and (C', p') are *topologically equivalent* if there exists a local homeomorphism ϕ of the respective ambient neighborhoods U, U' such that $\phi(p) = p'$ and $\phi(C \cap U) = C' \cap U'$. We denote this equivalence relation as $(C, p) \sim (C', p')$. When $p = p' = O$, let f and f' are defining polynomials of C and C', then we also write $f \sim f'$.

For a reduced plane curve germ (C, p), the topological equivalence type (or shortly topological type) of the germ (C, p) is determined by the following discrete characteristics (for instance [Z2, Lê1, BK]): the Puiseux pairs of its irreducible components and their linking numbers. Alternatively, it is known that the embedded resolution graph of (C, p) and the multiplicities of the total transforms of (C, p) at infinitely near points (including p) determine the topological type.

1.2. **Setting.** Hereafter, (x, y) is the affine coordinates of $\mathbb{C}^2$ and $f_2(x, y)$, $f_3(x, y)$ are the affine defining polynomial of C_2 and C_3. The sextic C is defined by $f_2(x, y)^3 + f_3(x, y)^2 = 0$. We assume that the origin O is an intersection point of C_2 and C_3 and it is an isolated singularity of C. This implies, in particular, that the conic C_2 and the cubic C_3 have no common component (such a line or the conic itself). We will classify the topological types of the germ (C, O), using the geometry of the respective singularities of the conic C_2 and the cubic C_3 and their mutual position.

1.3. **Strategy of classification.** We frequently use the fact that the topological type of a non-degenerate germ depends only on its Newton boundary. See [K, O2] for more details. In our classification, most of the cases can be transformed into non-degenerate forms after several changes of coordinates, and then the Newton principal part (NPP for short) gives us a normal form. However there are two special cases, namely Sp_1 and Sp_2 which can not be transformed into non-degenerate form. For these cases, the topological type can be read off from the explicit resolution of the respective singularity.

From now on, we use (u, v) for local coordinates (i.e. in the sense that we use an analytic change of coordinates), and (x, y) for global coordinates (i.e. affine coordinates). To determine the local type of singularities, it is convenient to use suitable changes of local analytic coordinates. However to see the existence of a sextic of (2,3)-torus type, it is usually better to keep the affine coordinates.

We recall a following lemma [O3, lemma 4.3].

Lemma 1.2. *Assume that $C_f = \{(u,v) \in \mathbb{C}^2; f(u,v) = 0\}$ a germ of a smooth curve at the origin. Let $C_g = \{(u,v) \in \mathbb{C}^2; g(u,v) = 0\}$ be another germ of a curve at the origin. Let d be the multiplicity of g at the origin and let $g_d(u,v)$ be the homogeneous polynomial of degree d, which defines the tangent cone of C_g. Let p, q be positive integers such that $p < dq$. Consider the germ of a plane curve $C = \{(u,v) \in \mathbb{C}^2; f(u,v)^p - g(u,v)^q = 0\}$. Assume that each irreducible component of $g_d(u,v) = 0$ intersects transversely with C_f at the origin. Then topological type of (C, O) is $B_{p,dq}$ and the tangential direction at the origin coincides with that of $f = 0$, where $B_{p,dq}$ is the Brieskorn-Pham singularity introduced in (1.1).*

For $(2,3)$-torus curve, we apply the above lemma with $(p,q) = (2,3)$ or $(p,q) = (3,2)$ and obtain the following result:

Corollary 1.3. *With the transverse intersection condition as above, we have:*

(i) *If (C_3, O) is smooth, then (C, O) is of type $A_{3\iota-1}$ for $\iota = 1, 2, ..., 6$.*

(ii) *If (C_3, O) is singular and (C_2, O) is smooth, then (C, O) is of type $B_{3,2\iota}$, for $\iota = 2, 3, ..., 6$.*

1.4. **Classification steps.** We will divide the situation into 3 cases by the multiplicity m_3 of the cubic C_3 at the origin. The simplest case is the case when (C_3, O) is smooth.
Case A: $m_3 = 1$. In the corollary 1.3 (i), we have to assume that the irreducible components of the tangent cone of C_2 intersects transversely with C_3. The claim is also true without this condition.

Proposition 1.4. *If (C_3, O) is smooth, and it intersects the conic C_2 with multiplicity ι, then (C, O) is of type $A_{3\iota-1}$.*

Proof. Up to an analytic change of coordinates, we may assume that $f_3 = u$ and the conic has the following form $f_2 = cv^\iota + uh$, where $h \in \mathbb{C}\{u,v\}, c \in \mathbb{C}\{v\}$ and $c(0) \neq 0$. Putting weights $w(u) = 3\iota$ and $w(v) = 2$, we can see $f = u^2 + (cv^\iota + uh)^3 = u^3 + c(0)^3 v^{3\iota} +$ higher terms. This implies $(C, O) \sim A_{3\iota-1}$, for $\iota = 1, .., 6$. □

We recall here that the invariant triple (μ, r, δ) is $(k, (k \bmod 2) + 1, [k/2])$ for A_k singularity, where $[\alpha]$ is the greatest integer less than or equal to α.

Proposition 1.5. *For any $1 \le \iota \le 6$ there exists a smooth cubic C_3 and a conic C_2, such that the (C, O) is of type $A_{3\iota-1}$. Furthermore we have a degeneration family: $A_2 \to A_5 \to A_8 \to A_{11} \to A_{14} \to A_{17}$*

Proof. See in [P]. □

Remark 1.6. One can compute the Newton principal part by hand, but usually it is a boring computation. We make a Maple package SCURVE[1], it can be used for compute invariants of singular plane curves, such that intersection number, Milnor number, NPP, toric modification, etc.

The case $m_3 = 1$ is done by Proposition 1.5.

Case B: $m_3 = 2$. We divide Case B into two subcases by m_2.

(B-I) (C_2, O) is smooth $(m_2 = 1)$: there are 8 cases, indicated in the following figures, where the dotted lines denote the cubic C_3, and the straight lines are affine lines. We remark here that C_2 may not irreducible globally, however in the cases I-4 and I-8 it must be irreducible (because C_2 has a tangent line which is a component of C_3).

In the pictures I-3 and I-4, the fat dotted line denotes a line with multiplicity 2.

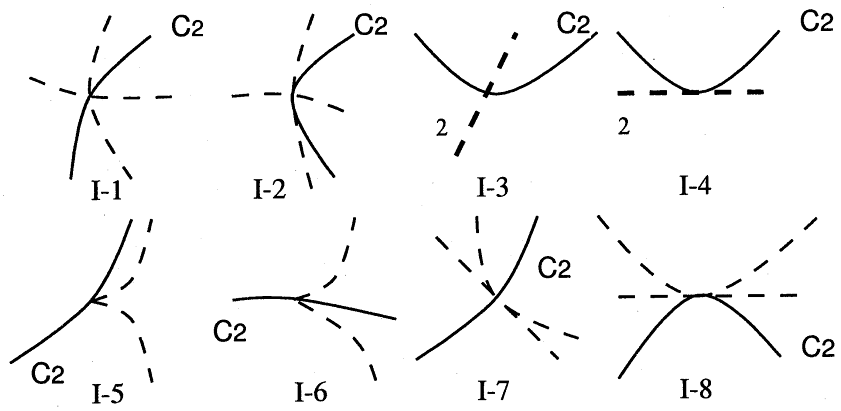


FIGURE 1. Case B-I $(m_3 = 2, m_2 = 1)$.

In the above figure, the cases I-1, I-3, I-5, I-7 satisfy the conditions of the lemma 1.2, and $d = 2$, so that we obtain (C, O) is of $B_{3,4}$ type. In these cases, $\iota = 2$.

The next case I-2 is the most interesting case in this classification, and we consider it at the end of this part.

I-4. We assume $f_2 = y - x^2$ and $f_3 = y^2 h$, where h is a linear term and $c := h(0,0) \neq 0$. We have $\iota = 4$. Putting $y_1 = y - x^2$, in the new coordinates (x, y_1), we have $\mathrm{NPP}(f) = y_1^3 + c^2 x^8$. Thus $(C, O) \sim B_{3,8}$.

I-6. As C_2 is smooth at O, we can take a local system of coordinates (u, v) so that $f_2 = v$, and $\mathrm{NPP}(f_3) = \alpha v^2 - \beta u^3$, where $\alpha, \beta \neq 0$. Putting weights $w(u) = 1$ and $w(v) = 2$, we obtain $\iota = 3$ and $\mathrm{NPP}(f) = v^3 + \beta^2 u^6$. We can easily see that this polynomial is non-degenerate for any $\beta \neq 0$. Thus $(C, O) \sim B_{3,6}$. This computation applies also for the case of C_2 being two lines.

I-8. The cubic C_3 consists of a line ℓ and a conic C_2'. We may assume that $f_2 = y - x^2$ and $f_3 = y(y + ax^2 + bxy + cy^2)$, where a or b is non-zero. Putting $t_1 = a + 1, t_2 = b, t_3 = c$, and substitute $y = x^2$ we get $f_3(x, x^2) = t_3 x^6 + t_2 x^5 + t_1 x^4$. Hence ι is equal to 4, 5 or 6 (respectively $\mathrm{I}(C_2', C_2; O) = 2, 3$ or 4). By this setting we have:

Proposition 1.7. *Under the above situation the germ (C, O) can be of type $B_{3,2\iota}$, for $\iota = 4, 5, 6$. Furthermore there is a degeneration family: $B_{3,8} \to B_{3,10} \to B_{3,12}$.*

Proof. See in [P]. □

[1]You can get SCURVE by a request e-mail to the author or download from the web-page: http://www.comp.metro-u.ac.jp/~pdtai/scurve/

Now we come back to the case I-2.

We may assume the conic C_2 is defined by $f_2 = b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2$ and the cubic C_3 is defined by $f_3(x,y) = y(a_{11}x + a_{02}y) + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3$, where $b_{01} \neq 0$ and $a_{11} \neq 0$. Because $C_2 \cap \{y = 0\} = C_3 \cap \{y = 0\} = \{O\}$, therefore $\iota = \mathrm{val}_{x=0} Res_y(f)$. We have $Res_y(f) = x^3\varphi(x)$ and

$$\varphi(0) = b_{01}(b_{01}a_{30} - a_{11}b_{20})(b_{02}a_{02} - a_{03}b_{01}) \tag{1.2}$$

Thus $\iota = 3$ iff none of these factors is zero.

First we consider the case $\iota = 3$. Putting weights $w(x) = 1, w(y) = 2$, by computation we have NPP$(f;(x,y))$ is a weighted homogeneous polynomial of degree 6. So that $f(x,y)$ is non-degenerate if and only if

$$(27a_{30}b_{01}^3 - 27a_{11}b_{01}^2b_{20} - 4a_{11}^3)(a_{30}b_{01} - a_{11}b_{20}) \neq 0$$

Thus $f \sim B_{3,6}$ for a generic case. Note that the term $(a_{30}b_{01} - a_{11}b_{20})$ is non-zero by (1.2). Thus to see further degeneration, we solve

$$(27a_{30}b_{01}^3 - 27a_{11}b_{01}^2b_{20} - 4a_{11}^3) = 0$$

in a_{30} and then NPP$(f;(x,y))$ gets a multiply factor. Namely, $(9b_{01}^3y + 9b_{01}^2b_{20}x^2 + 4a_{11}^2x^2)^2$. Thus we need to take the new coordinates (x, y_1) with $y_1 = y - (9b_{01}^2b_{20} + 4a_{11}^2)x^2/(9b_{01}^3)$. In this case we get $C_{3,7}$ as long as the coefficient of x^7 is non-zero. In fact, this coefficient is given by

$$c(4a_{11}^2 + 9b_{20}b_{01}^2)(9a_{02}b_{20}b_{01}^2 - 9a_{21}b_{01}^3 + 9a_{11}b_{11}b_{01}^2 + 4a_{02}a_{11}^2)$$

where c is a non-zero constant.

Thus there are two exceptional cases:

(ι3-1) $4a_{11}^2 + 9b_{20}b_{01}^2 = 0$ or

(ι3-2) $9a_{02}b_{20}b_{01}^2 - 9a_{21}b_{01}^3 + 9a_{11}b_{11}b_{01}^2 + 4a_{02}a_{11}^2 = 0$.

It turns out that the two cases give completely different geometries.

In the first case (ι3-1), putting $b_{20} = -c_{20}^2$, thus the equivalence condition is $a_{11} = \pm 3b_{01}c_{20}/2$. Let assume $a_{11} = 3b_{01}c_{20}/2$, we get NPP$(f;(x,y_1)) = c_1x^5y_1 + c_2x^2y_1^2 + c_3y_1^3$ with c_1, c_2, c_3 are non-zero constants. One can check that this form is topological equivalent to $x^8 + x^2y_1^2 + y_1^3$ (see Case B-II-5 for a detail explanation). Thus $(C,O) \sim C_{3,8}$. We also obtain same result for the case $a_{11} = -3b_{01}c_{20}/2$. An important observation is that *the generic member of this family is not irreducible but it is a union of a line and a quintic.*

Next we consider the case (ι3-2), i.e. $a_{21} = (9b_{01}^2b_{20}a_{02} + 9b_{01}^2a_{11}b_{11} + 4a_{02}a_{11}^2)/(9b_{01}^3)$. We look at the Newton principal part of f with respect to the coordinates (x, y_1). It has two faces, AB and BC where $A = (0,3)$, $B = (2,2)$ and $C = (8,0)$. The face function with respect to AB is non-degenerate, while the discriminant in y_1 of the face function with respect to BC is

$$d = cx^{10}(4a_{11}^2 + 9b_{01}^2b_{20})^2(-9b_{11}^2b_{01}^2 - 16a_{11}a_{12}b_{01} + 16b_{02}a_{11}^2), \ (c \neq 0)$$

Since the factor $4a_{11}^2 + 9b_{01}^2b_{20}$ is already considered in the case (ι3-1), so the generic case take place if the last factor of d, say d_2, is non-zero. In that case we have $(C,O) \sim C_{3,8}$. In comparison with the previous class of $C_{3,8}$ in case (ι3-1) a generic member of this class is irreducible.

Finally if $d_2 = 0$, it turns out that $(C,O) \sim C_{3,9}$. And it does not degenerate any further (as long as (C_2, O) is smooth and (C_3, O) is A_1).

Summary: In the case $\iota = 3$ there are 4 possibilities $B_{3,6}, C_{3,7}, C_{3,8}$ and $C_{3,9}$. We remark that in this case, the intersection multiplity is not enough to determine the topology.

Next we consider the case $\iota \geq 4$. By Bezout theorem, C_2 is necessarily a smooth conic. Thus we can assume that $f_2(x,y) = y - x^2$. We have $a_{30} = -a_{11}$ and

$$\iota = \begin{cases} 4 & \text{iff} \quad a_{02} + a_{21} \neq 0 \\ 5 & \text{iff} \quad a_{02} + a_{21} = 0, a_{12} \neq 0 \\ 6 & \text{iff} \quad a_{02} + a_{21} = a_{12} = 0, a_{03} \neq 0 \end{cases}$$

Let consider the case $\iota = 4$. We take the coordinate change (x, y_1) with $y_1 = y - x^2$. The Newton principal part is given by

$$\mathrm{NPP}(f;(x,y_1)) = (a_{02}+a_{21})^2 x^8 + 2a_{11}(a_{02}+a_{21})x^5 y_1 + a_{11}^2 x^2 y_1^2 + y_1^3$$

It is degenerate for the weight $P = {}^t(1,3)$, where $f_P = x^2(x^3 a_{21} + x^3 a_{02} + a_{11} y_1)^2$. Taking the coordinate change (x, y_2) with $y_2 = x^3 a_{21} + x^3 a_{02} + a_{11} y_1$. We have

$$\mathrm{NPP}(f;(x,y_2)) = -(a_{02}+a_{21})^3 x^9 / a_{11}^3 + x^2 y_2^2 + y_2^3 / a_{11}^3$$

Hence for generic $a_{ij}'s$ (i.e. $a_{02} + a_{21} \neq 0$) we have $(C,O) \sim C_{3,9}$. Continuing by the same method, we get $C_{3,12}$ and $C_{3,15}$ for $\iota = 5$ and 6 respectively.

Proposition 1.8. *There exists a cubic C_3 and a conic C_2, such that the (C,O) is of type $B_{3,6}$, $C_{3,k}$ (for $k = 7,8,9$) and $C_{3,3k}$ (for $k = 3,4,5$). Furthermore there are the following degeneration families:*
(i) $B_{3,6} \to C_{3,7} \to C_{3,8} \to C_{3,9}$, *with the same* $\iota = 3$.
(ii) $B_{3,6} \to C_{3,9} \to C_{3,12} \to C_{3,15}$, *with* $\iota : 3 \to 4 \to 5 \to 6$.

Proof. See in [P]. □

Remark 1.9. (a) There are two moduli components for the singularity classes $C_{3,8}$ with $\iota = 3$. The generic curve in the component corresponding to the case (ι3-1) decomposes into a line and a quintic, later we will use the notation $C_{3,8}^\sharp$ for the singularity in this moduli.
(b) There are two components for the moduli space of the class $C_{3,9}$, with different intersection numbers. Later we will distinguish by the new notation $C_{3,9}^\natural$ for the case $\iota = 4$.
(c) There are more degenerations for the singularities if we admit ι to increase or if we admit the degenerations for the singularities of C_2 and C_3. For example, we can degenerate $C_{3,7} \to B_{3,8} \to B_{3,10} \to B_{3,12}$, $C_{3,8} \to B_{3,10}$ and $C_{3,9} \to B_{3,12}$ by increasing the intersection number ι. $B_{3,8}$ degenerates also into $B_{4,6}$. For the explicit construction, see §3.

(B-II) (C_2, O) is singular ($m_2 = 2$). There are 9 cases, indicated in Figure 2, where the straight lines are again affine lines.
Attention. In the cases II-1,5,8,9 though in the pictures of C_2 are 2 distinct lines, but 2 lines may also coincide.
II-1. Up to an analytic change of coordinates we may assume $f_3 = uv$, and $f_2 = (a_1 u + b_1 v + h_1)(a_2 u + b_2 v + h_2)$, where a_i, b_i are non-zero constants, and $h_i \in \mathfrak{m}^2$ for $i = 1,2$. Here $\mathfrak{m}$ is the maximal ideal in $\mathbb{C}\{u,v\}$. We have $\iota = 4$. Putting weights $w(u) = w(v) = 1$, we get $f \sim u^6 + v^6 + u^2 v^2$. Thus $(C,O) \sim C_{6,6}$.
II-2. Since conic consists of 2 distinct lines, say ℓ_1 and ℓ_2, where ℓ_1 intersects C_3 transversely and ℓ_2 is a tangent cone direction of C_3, hence $\mathrm{I}(\ell_1, C_3; O) = 2$ and $\mathrm{I}(\ell_2, C_3; O) = 3$. Therefore we may assume $f_3 = uv$, and $f_2 = (a_1 u + b_1 v + h_1)(a_2 u^2 + b_2 v + h_2)$ so that $\iota = 5$. And we obtain $(C,O) \sim C_{6,9}$.
II-3,4. In these case, the intersection number $\iota = 6$. By the same way, we get respectively $C_{9,9}, C_{6,12}$.

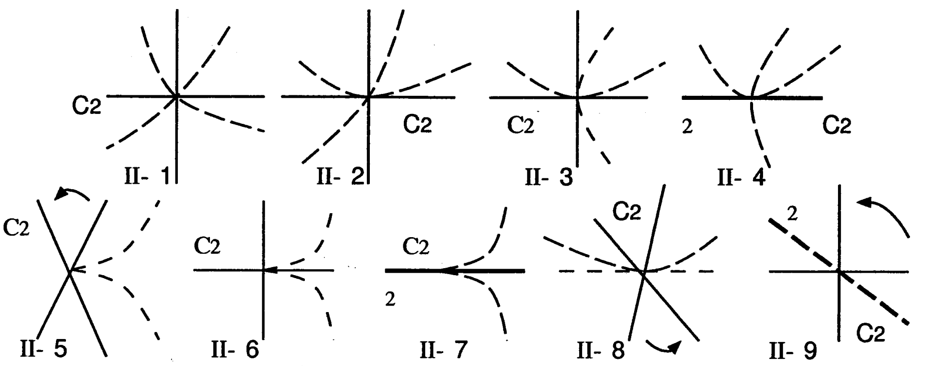


FIGURE 2. Case B-II ($m_3 = m_2 = 2$).

Proposition 1.10. *Under the situations* II-1∼II-4, *the type of* (C, O) *can be* $C_{6,6}$, $C_{6,9}$, $C_{9,9}$ *and* $C_{6,12}$. *Furthermore there are the following degeneration families:*

$$C_{6,6} \to C_{6,9} \to C_{9,9} \quad C_{6,9} \to C_{6,12}$$

Proof. See in [P]. □

In the next three cases II-5∼II-7, the cubic C_3 has a cusp, and by a linear change of coordinates we may assume that $\mathrm{NPP}(f_3) = y^2 - x^3$, i.e. $f_3 = y^2 - x^3 + h$ where h is the higher term, and $f_2 = (a_1x + b_1y)(a_2x + b_2y)$.
II-5. This case $a_1, a_2 \neq 0$, the intersection number $\iota = 4$, we may assume $a_1 = 1$. We have $\mathrm{NPP}(f) = (1 + a_2^3)x^6 - 2x^3y^2 + y^4$, and its discriminant in x is $-46656(1 + a_2^3)^2a_2^9y^{20}$. Hence if $1 + a_2^3 \neq 0$, we have $(C, O) \sim B_{4,6}$.
If $1+a_2^3 = 0$, then $\mathrm{NPP}(f) = 3a_1^2(a_1b_2+b_1)x^5y - 2x^3y^2 + y^4$, the isolated singularity condition requires the term $3a_1^2(a_1b_2 + b_1)x^5y$ does not vanish. Thus $f \sim x^5y + x^3y^2 + y^4$. So we get $(C, O) \sim D_{4,7}$ (because $x^5y + x^3y^2 + y^4 \sim x^7 + x^3y^2 + y^4$).

We remark that the intersection multiplity is not enough to determine the topology of (C, O) like in Case B-I-2.
II-6. One branch of the conic is tangent to (C_3, O), we may assume that

$$f_2(x, y) = axy, \quad f_3(x, y) = y^2 - x^3 + b_1yx^2 + b_2xy^2 + b_3y^3$$

In this case $\iota = 5$. This is a degenerate singularity, since $\mathrm{NPP}(f) = (y^2 - x^3)^2$. Taking a canonical toric modification $\pi_1 : X_1 \to \mathbb{C}^2$, we find that $\pi_1^*f = 0$ has again a (2,3)-cusp and it has a simultaneous resolution by one more toric modification for any $b_i \in \mathbb{C}$, $i = 1, 2, 3$. The tower of the weight vectors in the sense of [AO] is given by $\{{}^t(2, 3), {}^t(2, 3)\}$. Thus the topology of (C, O) does not depend on the parameters $a \neq 0$ and b_1, b_2, b_3. Moreover (C, O) is locally irreducible and its Puiseux pairs is $P(C, O) = \{(2, 3), (2, 9)\}$ by [O1]. By taking $a = 1, b_1 = b_2 = b_3 = 0$, we get $f = (y^2 - x^3)^2 + (xy)^3$. Thus $(C, O) \sim Sp_1$. The resolution diagram of Sp_1 is given in Figure 3. The Milnor number is 18 by [A2] or [AO].
II-7. The conic is a double line and it coincides with the tangent direction of (C_3, O). We assume that $f_2(x, y) = -a^2y^2, a \neq 0$ and $f_3(x, y)$ is as above. The intersection number $\iota = 6$. By the same method as Sp_1 case, we obtain this singularity is equivalent to Sp_2 and the resolution graph is given in Figure 4. We see that Sp_2 has 2 irreducible components $f_3 \pm a^3y^3 = 0$, they have same Puiseux pair $\{(2, 3)\}$ and their linking number is 9. Thus $\mu(Sp_2) = 21$ and $\delta(Sp_2) = 11$.

Proposition 1.11. *There are following degeneration families:*

(i) $B_{4,6} \to D_{4,7}$ *in* Case II-5.

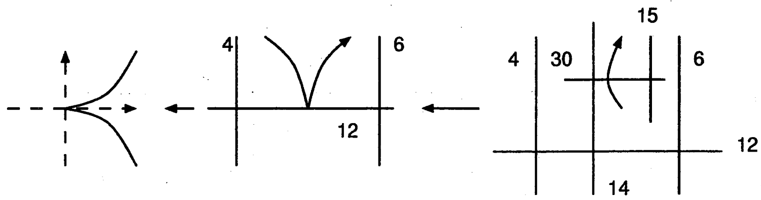


FIGURE 3. Resolution graph of Sp_1.

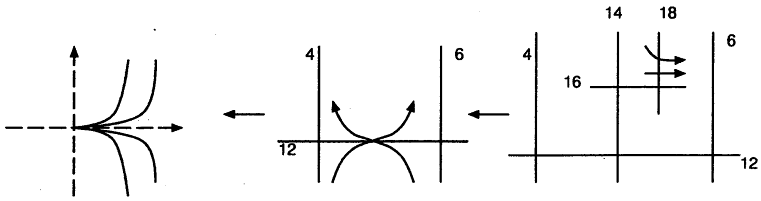


FIGURE 4. Resolution graph of Sp_2.

(ii) $B_{4,6} \to Sp_1 \to Sp_2$ *corresponding to* II-5 $\to$ II-6 $\to$ II-7.

Proof. The proof is obvious from the explanations in Cases II-5,6,7. □

II-8. Assuming $f_3 = y(y - x^2)$ and $f_2 = a(x + by)(x + cy)$, where a is non-zero. We have $\mathrm{NPP}(f) = y^4 + a^3x^6$. Thus $(C, O) \sim B_{4,6}$.
II-9. Assuming $f_3 = x^2(ax + by + k)$ and $f_2 = (cx + y)(dx + y)$, where k is non-zero. We have $\mathrm{NPP}(f) = y^6 + k^2x^4$. Thus $(C, O) \sim B_{4,6}$.

Finally we consider singular cubic with multiplicity 3 ($m_3 = 3$).

Case C: $m_3 = 3$. Similarly, we also divide 2 cases by the multiplicity of the conic.
(C-I) Conic C_2 is smooth at O. Obviously we have $\iota \geq 3$.
C-I-1. $\iota = 3$, applying the corollary 1.3, we get $(C, O) \sim B_{3,6}$.

Now we consider the case $\iota \geq 4$. Then C_2 is irreducible, as (C_2, O) is assumed to be smooth. Therefore we may assume that $f_2 = y - x^2$ and $f_3 = (a_1x + b_1y)(a_2x + b_2y)(a_3x + b_3y)$. We have ι is equal to the lowest degree in x of $f_3(x, x^2) = x^3(a_1 + b_1x)(a_2 + b_2x)(a_3 + b_3x)$.
C-I-2. $\iota = 4$, by symmetry of $a_i's$ we may assume $a_1 = 0$ and a_2, a_3 are none-zero. Putting $y_1 = y - x^2$, then $f(x, y_1) = y_1^3 + b_1^2a_2^2a_3^2x^8 + \text{higher terms} \sim y_1^3 + b_1^2a_2^2a_3^2x^8$. Since $a_1 = 0$ then b_1 must be non-zero. Thus $(C, O) \sim B_{3,8}$.

Similarly, we obtain
C-I-3. $\iota = 5$ if $a_1 = a_2 = 0$ and $a_3 \neq 0$: result is $(C, O) \sim B_{3,10}$.
C-I-4. $\iota = 6$ if $a_1 = a_2 = a_3 = 0$: result is $(C, O) \sim B_{3,12}$.
(C-II) Conic C_2 is singular. We have $\iota = 6$. Since both conic and cubic are products of linear terms, then f is a homogeneous polynomial of degree 6. Since O is an isolated singularity, then f should be non-degenerate and it is a product of linear terms. Thus $(C, O) \sim B_{6,6}$.

1.5. **Statement of the result on the local classification.** Now we can state the result in the local classification:

Theorem 1 (Local Classification). Let $C = \{f = f_2^3 + f_3^2 = 0\}$ is a tame (2,3)-torus curve. Put $C_i = \{f_i = 0\}$ for $i = 2, 3$. The topological type of the germ (C, O) can be read off from C_2 and C_3 as follows:

1. Cubic (C_3, O) is smooth: $A_{3\iota-1}$ ($\iota = 1, \ldots, 6$).
2. Cubic is not smooth at O (i.e., $m_3 \geq 2$).

(a) Conic (C_2, O) is smooth.
  (i) (C_3, O) is A_1 and $\iota = 2 : E_6 = B_{3,4}$.
  (ii) (C_3, O) is A_1 and (C_2, O) is tangent to one of the branch:
  $B_{3,6}$, $C_{3,7}$, $C_{3,8}$, $C_{3,9}$ for $\iota = 3$, and
  $C_{3,3\iota-3}$ for $\iota = 4, 5, 6$.
  (iii) (C_3, O) is not $A_1 : B_{3,2\iota}, \iota = 2, \ldots, 6$.
(b) Conic (C_2, O) is A_1 :
  (i) (C_3, O) is A_1 :
  $C_{6,6}$ for $\iota = 4$, no common tangential cone,
  $C_{6,9}$ for $\iota = 5$, one common tangential cone,
  $C_{9,9}$ for $\iota = 6$, two common tangential cones.
  (ii) (C_3, O) is either A_2 or A_3 :
  $B_{4,6}$ or $D_{4,7}$, for $\iota = 4$, no common tangential cone,
  Sp_1 if $\iota = 5$ and $(C_3, O) = A_2$ and the tangential cones coincide.
(c) C_2 is a line with multiplicity 2:
  (i) (C_3, O) is a A_1 :
  $B_{4,6}$ for $\iota = 4$, no common tangential cone,
  $C_{6,12}$ if $\iota = 6$, a common tangential cone.
  (ii) (C_3, O) is either A_2 or A_3 :
  $B_{4,6}, D_{4,7}$ for $\iota = 4$, no common tangential cone,
  Sp_2 for $\iota = 6$, (C_3, O) is A_2, the same tangential cone.
(d) Cubic C_3 consists of three lines passing through O, and C_2 consists of two lines passing through $O : B_{6,6}$ $(\iota = 6)$.

Remark 1.12. When (C_3, O) is a node (i.e., A_1), C_3 can be reduced (either a line and a conic meeting transversely at O or three lines where two of them are passing through O). (C_3, O) is A_3 if and only if C_3 consists of a line and a conic which are tangent at O. (C_2, O) has a node if it consists of two lines.

Theorem 1-D (Local Degenerations). Under the notation in Theorem 1, we have following degenerations:

1. (C_3, O) is smooth: $A_2 \to A_5 \to A_8 \to A_{11} \to A_{14} \to A_{18}$
2. (C_3, O) is A_1 and (C_2, O) is smooth:
   (a) $B_{3,4} \to B_{3,6} \to C_{3,7} \to C_{3,8} \to C_{3,9}$
   (b) $B_{3,4} \to B_{3,6} \to C_{3,9}^{\natural} \to C_{3,12} \to C_{3,15}$
3. (C_3, O) is A_2 or A_3, (C_2, O) is smooth: $B_{3,4} \to B_{3,8} \to B_{3,10} \to B_{3,12}$
4. (C_3, O) is A_1, (C_2, O) is A_1: $B_{4,6} \to D_{4,7}$, $B_{4,6} \to Sp_1 \to Sp_2$.
5. (C_3, O) is A_2 or A_3, (C_2, O) is A_1: $C_{6,6} \to C_{6,9} \to C_{9,9}$, $C_{6,9} \to C_{6,12}$.

The following is well known.

Proposition 1.13. *Let C be a reduced curve of degree d in $\mathbb{P}^2$ defined by $F(X, Y, Z)$. Then there is a family C_t for $0 \le t \le 1$ of curves of degree d such that $C_t \cong C$ for $t \ne 0$ and $C_0 \cong B_{d,d}$ where $B_{d,d}$ is the class of d lines meeting at O.*

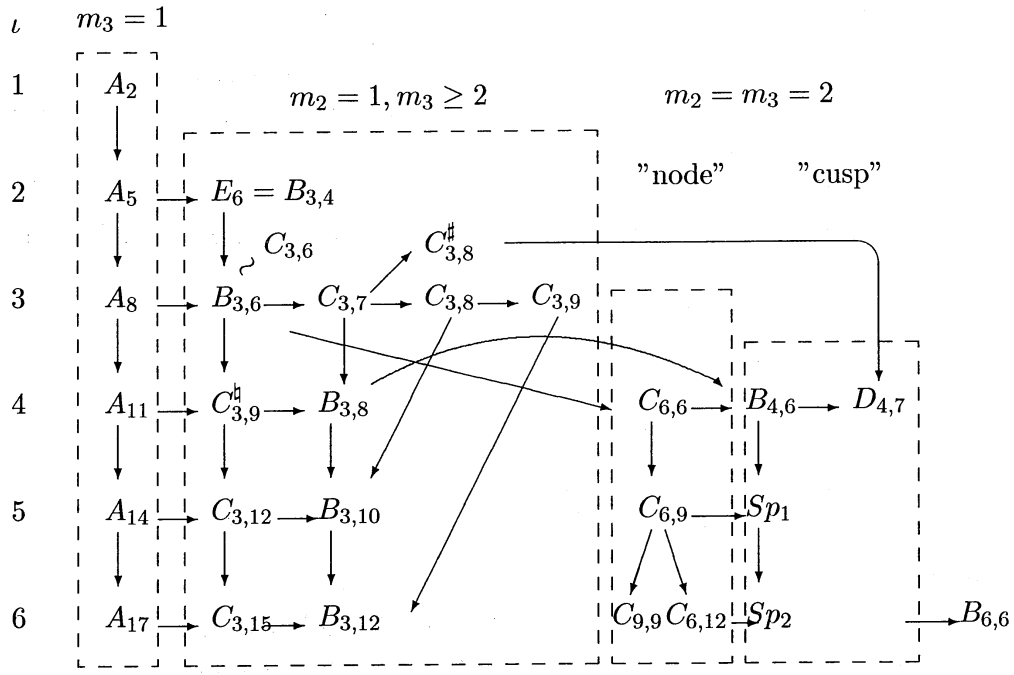
Proof. We follow the method in [OS]. We may assume that the line at infinity $Z = 0$ meets C transversely. Take $F(X, Y, Z, t) := F(X/t, Y/t, Z)t^d$ and define the family $C_t = \{F(X, Y, Z, t) = 0\}$. $\square$

Remark 1.14. For the sake of the global study of sextics, we distinguish $C_{3,8}$ and $C_{3,8}^{\sharp}$, and also $C_{3,9}$ and $C_{3,9}^{\natural}$ though they are locally topologically equivalent.

Table A: Local classification

ι	Singularity type T and the invariant triple (μ, r, δ) of (C,O)
1	A_2(2,1,1)
2	A_5(5,2,3), E_6(6,1,3)
3	A_8(8,1,4), $B_{3,6}$(10,3,6), $C_{3,7}$(11,2,6), $C_{3,8}$(12,3,7), $C^\sharp_{3,8}$(12,3,7), $C_{3,9}$(13,2,7)
4	A_{11}(11,2,6), $C^\natural_{3,9}$(13,2,7), $B_{3,8}$(14,1,7), $C_{6,6}$(13,4,8), $B_{4,6}$(15,2,8), $D_{4,7}$(16,3,9)
5	A_{14}(14,1,7), $C_{3,12}$(16,3,9), $B_{3,10}$(18,1,9), $C_{6,9}$(16,3,9), Sp_1(18,1,9)
6	A_{17}(17,2,9), $C_{3,15}$(19,2,10), $C_{9,9}$(19,2,10), $B_{3,12}$(22,3,12), $C_{6,12}$(19,3,11), Sp_2 (21,2,11), $B_{6,6}$(25,6,15)

Table A': The local degeneration series



2. Global classification

In this section, we consider the possible combination of the local singularities of reduced tame (2,3)-torus curves, using the local classification obtained in Section 1.

Assume that C is an irreducible curve of degree d and denote the set of singularities of C by $\Sigma(C)$. We recall the genus formula

$$g = \tfrac{(d-1)(d-2)}{2} - \textstyle\sum_{P\in\Sigma(C)} \delta(C,P) \geq 0 \quad \text{or} \tag{2.1}$$

$$\chi(C') = 3d - d^2 + \textstyle\sum_{P\in\Sigma(C)} \mu'(C,P) \leq 2 \tag{2.2}$$

where C' is the normalization of C, $\chi(C')$ is the topological Euler characteristic, $\mu'(C,P) := \mu(C,P)+r(C,P)-1$ and $r(C,P)$ is the number of analytic branches at P (see [BK]). We call $\mu'(C,P)$ *the normalized Milnor defect* of C at P. When C is a tame (2,3)-torus curve, then its singularities set $\Sigma(C)$ is given by $C_2 \cap C_3 =: \{P_1, P_2, \dots, P_n\}$. Therefore (2.1) is equivalent to the following

$$\sum_{i=1}^{n} \delta(C, P_i) \leq (6-1)(6-2)/2 = 10. \tag{2.3}$$

Denote $i_k := \mathrm{I}(f_2, f_3; P_k)$, we call $(i_1, \dots, i_n)$ is an *i-vector* and n the *length* of i-vector. By Bezout theorem, we have

$$\sum_{i=1}^{n} i_k = 6. \tag{2.4}$$

When C is reducible curve with $r(C)$ irreducible components, the inequality (2.1) does not hold and (2.2) has to be replaced by the following.

$$\chi(C') = 3d - d^2 + \sum_{P\in\Sigma(C)} \mu'(C,P) \leq 2r(C) \tag{2.5}$$

Our strategy for the global classification is the following steps:
Step 1. Consider every possible i-vector, which satisfies (2.4).
Step 2. List up every possible combination of the local singularities having prescribed i-vector.
Step 3. Prove or disprove the existence of a reduced tame (2,3)-torus curve with the configurations. In this stage, if the inequality (2.2) is not satisfied, we have to look for the reduced curves with the given configuration.

For the later discussion, we recall

Lemma 2.1. *The possible topological types of a singularity on an irreducible quartic curve are $A_1, \dots, A_6$, D_4, D_5 and E_6.*

Proof. See for instance in [W] □

From the above strategy we can enumerate singularities as follows:

Theorem 2 (Global Classification). The configuration of singularities of a tame torus curves C of type $(2,3)$ is given by Table B, where the notation $\{\Sigma(C)\}^*$ is for the cases of reducible C.

Table B: Global classification

n	i-vector $(i_1,i_2,\dots,i_n)$	Configuration of singularities $\Sigma(C)$
1	(6)	$\{A_{17}\}$, $\{C_{3,15}\}$, $\{C_{9,9}\}$ $\{B_{3,12}\}^\star, \{C_{6,12}\}^\star, \{Sp_2\}^\star, \{B_{6,6}\}^\star$
2	(1,5)	$\{A_2,\sigma\}$, $\sigma = A_{14},\ C_{3,12},\ B_{3,10},\ C_{6,9}, Sp_1$
	(2,4)	$\{A_5,\sigma\}$, $\{E_6,\sigma\}$, $\sigma = A_{11},\ C^{\natural}_{3,9},\ B_{3,8}$ $\{A_5,\sigma'\}^\star$, $\sigma' = C_{6,6},\ B_{4,6},\ D_{4,7}$
	(3,3)	$\{2A_8\}$, $\{A_8, B_{3,6}\}$, $\{A_8, C_{3,7}\}$, $\{A_8, C^{\sharp}_{3,8}\}^\star$, $\{2B_{3,6}\}^\star$
3	$(1,1,4)$	$\{2A_2,\sigma\}$, $\sigma = A_{11},\ C^{\natural}_{3,9},\ B_{3,8},\ C_{6,6},\ B_{4,6}$; $\{2A_2, D_{4,7}\}^\star$
	(1,2,3)	$\{A_2, A_5,\sigma\}$, $\{A_2, E_6,\sigma\}$, $\sigma = A_8,\ B_{3,6},\ C_{3,7}$ $\{A_2, A_5, C_{3,8}\}^\star, \{A_2, A_5, C^{\sharp}_{3,8}\}^\star, \{A_2, E_6, C^{\sharp}_{3,8}\}^\star$
	(2,2,2)	$\{3A_5\}, \{2A_5, E_6\}, \{A_5, 2E_6\}, \{3E_6\}$
4	(1,1,1,3)	$\{3A_2,\sigma\}$, $\sigma = A_8,\ B_{3,6},\ C_{3,7},\ C_{3,8}, C_{3,9}$; $\{3A_2, C^{\sharp}_{3,8}\}^\star$
	(1,1,2,2)	$\{2A_2, 2A_5\}, \{2A_2, A_5, E_6\}, \{2A_2, 2E_6\}$
5	(1,1,1,1,2)	$\{4A_2, A_5\}, \{4A_2, E_6\}$
6	(1,1,1,1,1,1)	$\{6A_2\}$

Proof. We will prove the assertion by three steps (corresponds to 3 steps in the strategy) as follows.

Step 1 + Step 2: The i-vector of length 6 $(1,1,1,1,1,1)$ is obviously given under the generic situation where C_2 and C_3 intersect transversely.

The i-vector with $n = 5$ is given by $(1,1,1,1,2)$ and the possible configurations are $\{A_5, 4A_2\}$ and $\{E_6, 4A_2\}$.

The i-vectors with $n = 4$ is either $(1,1,1,3)$ or $(1,1,2,2)$. The possible configurations are

$(1,1,1,3):\quad \{3A_2,\sigma\}$, where $\sigma = A_8, B_{3,6}, C_{3,7}, C_{3,8}, C^{\sharp}_{3,8}, C_{3,9}$

$(1,1,2,2):\quad \{2A_2, 2A_5\}, \{2A_2, A_5, E_6\}, \{2A_2, 2E_6\}$

The i-vectors with $n = 3$ are (1,1,4), (1,2,3), (2,2,2) and the possible configurations are

$(1,1,4):\quad \{2A_2,\sigma\}$, where $\sigma = A_{11}, C^{\natural}_{3,9}, B_{3,8}, C_{6,6}, B_{4,6}, D_{4,7}$

$(1,2,3):\quad \{A_2, A_5,\sigma\}, \{A_2, E_6,\sigma\}$, where $\sigma = A_8, B_{3,6}, C_{3,7}, C_{3,8}, C^{\sharp}_{3,8}, C_{3,9}$

$(2,2,2):\quad \{3A_5\}, \{3E_6\}, \{2A_5, E_6\}, \{A_5, 2E_6\}$

The i-vectors with $n = 2$ are (1,5),(2,4),(3,3) and the possible configurations are
$(1,5):\quad \{A_2, \sigma\}$, where $\sigma = A_{14}, C_{3,12}, B_{3,10}, C_{6,9}, Sp_1$
$(2,4):\quad \{\tau, \sigma\}$ where $\tau = A_5, E_6\quad \sigma = A_{11}, C^{\natural}_{3,9}, B_{3,8}, C_{6,6}, B_{4,6}, D_{4,7}$
$(3,3):\quad \{\tau, \xi\}$ where $\tau, \xi = A_8, B_{3,6}, C_{3,7}, C_{3,8}, C^{\sharp}_{3,8}, C_{3,9}$
For the case $n = 1$, we have the following obvious possibility.
$(6):\quad \{A_{17}\}, \{C_{3,15}\}, \{C_{9,9}\}, \{B_{3,12}\}, \{C_{6,12}\}, \{Sp_2\}, \{B_{6,6}\}$

Step 3. Existence and Non-existence.

$\underline{n = 6:}$ The configuration $\{6A_2\}$ is given by generic conic and cubic which intersect transversely and this case was first studied by Zariski, [Z1].

$\underline{n = 5:}$ The configuration $\{4A_2, \sigma\}$, $\sigma = A_5$ or E_6 is given by the generic member of the moduli of the singularity A_5 or E_6 discussed in §1.

$\underline{n = 4:}$ Any configurations $\{3A_2, \sigma\}$ where $\sigma = A_8, B_{3,6}, C_{3,7}, C_{3,8}, C_{3,9}$ are obtained by a generic curve in the moduli of singularity σ at the origin. This holds also for $\{3A_2, C^{\sharp}_{3,8}\}^{\star}$, but we notice that a generic member of this class consists of a line and a quintic, though it satisfies the inequality (2.2).

The configuration $\{2A_2, 2A_5\}$ is given by a smooth conic C_2 and a smooth cubic C_3 which are tangent at two points and have two other transverse intersections. The configurations $\{2A_2, A_5, E_6\}$ and $\{2A_2, 2E_6\}$ are obtained by the similar device. The existence of these configurations also shown by explicit equations, in Proposition 3.1 of Section 3.

$\underline{n = 3:}$ For (1,1,4), the existence of the configurations $\{2A_2, \sigma\}$ where $\sigma = A_{11}$, $C^{\natural}_{3,9}$, $B_{3,8}$, $C_{6,6}$, $B_{4,6}$ is proved as the above. Similarly the configuration $\{2A_2, D_{4,7}\}^{\star}$ exists and an example is given by $f_2 = (x+y)(-x+2y)$, $f_3 = y^2 - x^3$. In this case C has two irreducible components, a line Γ_1 and a quintic Γ_5. Note that $\Gamma_1 \cap \Gamma_5 = \{O\}$, Γ_5 has a singularity E_7 at O and 2 A_2 singularities.

Now we consider the configuration with i-vector (2,2,2).

The configuration $\{3A_5\}$ is given for instance by $f_2 = (x-1)^2 + y^2 - 1$ and $f_3 = x(y^2 - 1)$. Note that $C_2 \cap C_3$ consists of three simple tangent points. The configuration $\{3E_6\}$ is given by $f_2 = (x-1)^2 + y^2 - 1$ and $f_3 = (x-1)(x^2 - y^2)$. This case was studied by Oka [O4]. The existence of the configurations $\{2A_2, E_6\}$, $\{A_5, 2E_6\}$ is shown in Proposition 3.1.

Now we consider the configurations with i-vector (1,2,3). Examples of the configurations $\{A_2, A_5, \sigma\}$ and $\{A_2, E_6, \sigma\}$ with $\sigma = A_8, B_{3,6}, C_{3,7}$ are given in Proposition 3.1.

The configuration $\{A_2, A_5, C_{3,8}\}^{\star}$ exists and an example is given by $f_2 = y - x^2, f_3 = (-23x^3 - 15yx^2 + 27xy - 16y^3 + 27y^2)/27$. In this case, C has two irreducible components, a conic Γ_2 which is defined by $5x^2 - 16xy - 16y^2 - 9y = 0$ and a quartic Γ_4 which defined by $40x^4 - 10yx^3 - 117yx^2 + 51y^2x^2 + 18y^2x - 16xy^3 + 81y^2 + 16y^4 - 63y^3 = 0$. Note that $\Gamma_2 \cap \Gamma_4 = \{O, A\}$ where (C, O) is $C_{3,8}$, (C, A) is A_5 and Γ_4 has two singularities, A_3 at O and A_2 at $B = (1,1)$.

The configuration $\{A_2, A_5, C^{\sharp}_{3,8}\}^{\star}$ exists, by taking $f_2 = y - x^2$, $f_3 = 2y^3 + (3/2)yx - (7/2)xy^2 + yx^2 - x^3$. In this case, C is also has two irreducible components: they are a line $\Gamma_1 = \{y = 0\}$ and a quintic Γ_5. The line Γ_1 is a tangent direction of Γ_5 with multiplicity 5 at O, and they have no other intersection point except O. Moreover, Γ_5 has 3 singularities, an A_3 at O, an A_5 at $(1,1)$ and an A_2 at $(-1/4, 1/16)$.

The configuration $\{A_2, E_6, C^{\sharp}_{3,8}\}^{\star}$ exists, by taking $f_2 = y^2 + y(x - 1/4) - x^2$, $f_3 = y(-(3/8)x + (9/16)y) - x^3 + (19/8)x^2y - (63/32)y^3$. Again C is a union of a line and a quintic.

$\underline{n = 2:}$ The configurations with respect to i-vector (1,5) do exist by similar reason as the cases (1,1,4), (1,1,1,3) or (1,1,1,1,2).

Now we consider the configuration with i-vector (2,4). Examples of the configurations $\{A_5, \sigma\}$ and $\{E_6, \sigma\}$ with $\sigma = A_{11}, C_{3,9}^{\sharp}, B_{3,8}$ are given in Proposition 3.1.

There are 3 more configurations for i-vector is (2,4), in these cases the conic is a line with multiplicity 2. Hence at least f has two factors. Explicit examples are given as follows.

$\{A_5, B_{4,6}\}^\star$: $f_2(x,y) = x^2$ and $f_3(x,y) = y^2 - x^3$. C has two cubic components, defined by $y^2 - (1 \pm \sqrt{-1})x^3 = 0$. The singularities of C are a $B_{4,6}$ at O and an A_5 at $(0,1)$.

$\{A_5, D_{4,7}\}^\star$: $f_2 = -(x-y)^2$ and $f_3 = y^2 - x^3$. In this case, C has 3 irreducible components, a line Γ_1, a conic Γ_2 and a cuspidal cubic Γ_3. The singularities of C are a $D_{4,7}$ at O and an A_5 at $(1,1)$. Γ_2 intersects Γ_3 at O and $(1,1)$ with respective intersection multiplicity 3 and 1, while Γ_1 is tangent to both Γ_2 and Γ_3 at O.

$\{A_5, C_{6,6}\}^\star$: $f_2 = -x^2$ and $f_3 = (y-1)(x^2 - y^2)$. In this case, C has 2 irreducible components, both of them are nodal cubics. The singularities of C are a $C_{6,6}$ at O and an A_5 at (0,1).

i-vector is (3,3): Examples of the configurations $\{2A_8\}$, $\{A_8, B_{3,6}\}$ and $\{A_8, C_{3,7}\}$ are given in Proposition 3.1.

An example of $\{A_8, C_{3,8}^{\sharp}\}^\star$ is given by $f_2 = y - x^2, f_3 = -x^3 + x^2y + (9y + 16y^2)x/6 + y^2 + 32y^3/27$. In this case, C is a union of a line Γ_1 and a quintic Γ_5, where Γ_1 is a tangent direction of Γ_5 at O, and $\Sigma(\Gamma_5) = \{A_8, A_3\}$. The singularities of C are a $C_{3,8}^{\sharp}$ at O and an A_8 at (1,1).

An example of $\{2B_{3,6}\}^\star$ is given by $f_3 = x(x-2)y, f_2 = y^2 + x^2 - 2x$. In this case, C is a union of three smooth conics $\Gamma_2^{(i)}$, and they intersect each other other at two points O and $A = (2,0)$ with multiplicity 2.

<u>$n = 1$:</u> The existence in these cases is obvious from the local classification. The explicit equations for irreducible curves are given in Proposition 3.1. Explicit examples for the reducible cases are given as following.

$\{B_{3,12}\}^\star$: Let $f_2 = y - x^2, f_3 = y^3$. In this case, C consists of three smooth conics $\Gamma_2^{(i)}$, which are tangent to each other at O with multiplicity 4.

$\{C_{6,12}\}^\star$: Let $f_2 = -y^2, f_3 = x(y - x^2)$. In this case, $f = (x^3 - xy - y^3)(x^3 - xy + y^3)$, thus C is a union of two nodal cubics.

$\{Sp_2\}^\star$: Let $f_2 = -y^2, f_3 = y^2 - x^3$. In this case, $f = (x^3 - y^2 - y^3)(x^3 - y^2 + y^3)$, thus C is a union of two cuspidal cubics.

$\{B_{6,6}\}^\star$: Let $f_2 = y^2, f_3 = x^3$. Thus C consists of 6 concurrent lines.

To complete the proof of Theorem 2, it suffices to show the non-existence of the remaining configurations. □

Lemma 2.2. *There are no tame (2,3)-torus curves having a configuration of singularities in the following list.*

(1) $\{A_2, E_6, C_{3,8}\}^\star$, $\{A_2, E_6, C_{3,9}\}^\star$ *and* $\{A_2, A_5, C_{3,9}\}^\star$,

(2) $\{C_{3,k}, C_{3,l}\}^\star$ *for any* $6 \le k, l \le 9$, $(k,l) \ne (6,6)$. *Here we use the notation* $C_{3,6}$ *for* $B_{3,6}$ *for notation's consistency, and* $C_{3,8}^{\sharp}$ *is also included in these pairs.*

(3) $\{E_6, \sigma\}^\star$ *where* $\sigma = C_{6,6}, B_{4,6}, D_{4,7}$.

(4) $\{A_8, C_{3,8}\}^\star, \{A_8, C_{3,9}\}^\star$.

Proof. (1) For a singular point $P \in C$, recall that $\mu'(P) := \mu(P) + r(C,P) - 1$. We first consider the case $\{A_2, E_6, C_{3,8}\}^\star$, $\{A_2, E_6, C_{3,9}\}^\star$ and $\{A_2, A_5, C_{3,9}\}^\star$. Note that $\chi(C') = 4$ for each of them. Thus $r(C) \ge 2$. In fact, we have observed before any generic member of the moduli of $\{A_2, A_5, C_{3,8}\}^\star$ consists of a conic and a quartic. Each of the above three configurations should be a degeneration of a family of $\{A_2, A_5, C_{3,8}\}^\star$. In the case of $\{A_2, E_6, C_{3,8}\}^\star$

and $\{A_2, E_6, C_{3,9}\}^\star$ (respectively in the case of $\{A_2, A_5, C_{3,9}\}^\star$), a quartic can not contain singularities A_2 and E_6 simultaneously (resp. A_2 and A_6) by the inequality (2.2), which proves the non-existence of the case (1).

(2) Now we consider the configurations $\{C_{3,k}, C_{3,l}\}^\star$ for any $6 \le k, l \le 9$. Note that $\mu'(C_{3,\ell}) = 12, 12, 14, 14$ respectively and the tangent cones are irreducible. Thus if C has any two of them simultaneously, we need to have $r(C) \ge 3$. Furthermore if $C_{3,8}$ or $C_{3,9}$ is included, $r(C) \ge 4$, in which case, C has at least two line components. If C has two line components, we have an obvious contradiction by the number of the tangential cone argument. Thus the remaining cases are $\{C_{3,k}, C_{3,l}\}, k, l = 6, 7$ and C has either 3 conics or one line and two other components of degree 2 and 3 respectively. Observe that $C_{3,7}$ has irreducible singularity A_4, which can not exist on a cubic or a conic. Thus the non-existence is proved except $\{C_{3,6}, C_{3,6}\}^\star = \{B_{3,6}, B_{3,6}\}^\star$. This exceptional case exists as we have seen before.

(3) Assume that the singularity at O is $C_{6,6}$, $B_{4,6}$ or $D_{4,7}$. Then the conic C_2 consists of two different lines ℓ_1, ℓ_2 or a line with multiplicity two by the local classification argument in §2. Note that in any case, the intersection multiplicity of a line (or the reduced line) in C_2 and C_3 at O is 2. Thus there exists a simple intersection outside of O. In the first case, the other partner singularities are two A_2, which is not the case (3). If C_2 is a double line, the other partner singularity is A_5 as we have seen in Corollary 1.3, which is also not in case (3).

(4) Assume that C has a configuration $\{A_8, C_{3,8}\}^\star$ or $\{A_8, C_{3,9}\}^\star$. Then they must be a degeneration of the configuration of type $\{A_2, A_5, C_{3,8}\}^\star$ whose generic member consists of a conic and a quartic. Thus by the same argument as above, this is impossible. □

I am grateful to Professor Oka for pointing out some missing cases in Table B and showing the above non-computational proof of Lemma 2.2.

3. The global degeneration

3.1. **A certain degenerations.** In this section we consider the degenerations of irreducible tame torus curves of type (2,3), our aim is finding the list of all maximal curves. Here an irreducible sextic C of a torus type is called *maximal* if C does not have any degeneration in the space of irreducible sextics of torus type.

The degenerations between the curves in the table B may come from the configurations in the same level of i-vectors in the table B, and also from different levels of i-vectors. We first show the following.

Proposition 3.1. *In the same i-vectors level, the degeneration families are given in* Table B'.

Proof. See in [P]. □

A configuration at the end of each degeneration family in the above proposition is called *semi-maximal.* Next we consider degenerations between the semi-maximal curves. For later purpose, we first give the following degenerations, which will be used to prove Theorem 3.4.

Proposition 3.2. *Among the semi-maximal curves and together* $\{6A_2\}$, *there are following degeneration families:*

(1) $\{6A_2\} \to \{4A_2, E_6\} \to \{2A_2, 2E_6\} \to \{3E_6\} \to \{E_6, B_{3,8}\}$
(2) $\{A_2, E_6, C_{3,7}\} \to \{E_6, B_{3,8}\}$
(3) $\{2A_2, B_{4,6}\} \to \{A_2, Sp_1\}$

Proof. See in [P]. □

Table B': Global degenerations (with same i-vector)

$$\{A_{17}\} \to \{C_{3,15}\} \qquad \{A_{17}\} \to \{C_{9,9}\}$$

$$\{A_2, A_{14}\} \to \{A_2, C_{3,12}\} \to \{A_2, B_{3,10}\}$$
$$\{A_2, A_{14}\} \to \{A_2, C_{6,9}\} \to \{A_2, Sp_1\}$$

$$\begin{array}{ccccc} \{A_5, A_{11}\} & \to & \{A_5, C^\natural_{3,9}\} & \to & \{A_5, B_{3,8}\} \\ \downarrow & & \downarrow & & \downarrow \\ \{E_6, A_{11}\} & \to & \{E_6, C^\natural_{3,9}\} & \to & \{E_6, B_{3,8}\} \end{array} \qquad (*)$$

$$\{2A_8\} \to \{A_8, B_{3,6}\} \to \{A_8, C_{3,7}\}$$

$$\begin{array}{ccccc} \{2A_2, A_{11}\} & \to & \{2A_2, C^\natural_{3,9}\} & \to & \{2A_2, B_{3,8}\} \\ & \searrow & & & \downarrow \\ & & \{2A_2, C_{6,6}\} & \to & \{2A_2, B_{4,6}\} \end{array}$$

$$\begin{array}{ccccc} \{A_2, A_5, A_8\} & \to & \{A_2, A_5, B_{3,6}\} & \to & \{A_2, A_5, C_{3,7}\} \\ \downarrow & & \downarrow & & \downarrow \\ \{A_2, E_6, A_8\} & \to & \{A_2, E_6, B_{3,6}\} & \to & \{A_2, E_6, C_{3,7}\} \end{array} \qquad (**)$$

$$\{3A_5\} \to \{2A_5, E_6\} \to \{A_5, 2E_6\} \to \{3E_6\}$$

$$\{3A_2, A_8\} \to \{3A_2, B_{3,6}\} \to \{3A_2, C_{3,7}\} \to \{3A_2, C_{3,8}\} \to \{3A_2, C_{3,9}\}$$

$$\{2A_2, 2A_5\} \to \{2A_2, A_5, E_6\} \to \{2A_2, 2E_6\}$$

$$\{4A_2, A_5\} \to \{4A_2, E_6\}$$

3.2. **Maximal sextics.** First we remark the following.

Lemma 3.3. *Assume that we have an analytic family of plane curves ($|t| \le \varepsilon$) such that C_t has only isolated singularities in a fixed open neighborhood U of the origin for any t and O is the unique singularity of C_0. We assume that $P_{t,1}, \dots, P_{t,\nu}$ are the singular points of $C_t \cap U$ which converges to O for $t = 0$. Then we have*

(1) *If $\nu \ge 2$, $\sum_{i=1}^{\nu} \mu(C_t, P_{t,i}) > \mu(C_0, O)$.*

(2) *If $\nu = 1$, $\mu(C_0, O) \ge \mu(C_t, P_{t,1})$ for $t \ne 0$ and the equality holds if and only if (C_0, O) is equivalent to $(C_t, P_{t,1})$ for $t \ne 0$.*

Proof. The first assertions follows from the vanishing theorem of Lefschetz number of the monodromy ([A1], [Lê2]) and the second assertion is due to ([Lê2],[LR]). □

Using Propositions 3.1 and 3.2 we obtain the following theorem.

Theorem 3.4. *The maximal sextics of torus type has the following configurations.*

(1) $n = 1 : \{C_{3,15}\}, \{C_{9,9}\}$.

(2) $n = 2 : \{A_2, B_{3,10}\}, \{A_2, Sp_1\}, \{E_6, B_{3,8}\}, \{A_8, C_{3,7}\}$.

(3) $n = 4 : \{3A_2, C_{3,9}\}$.

Proof. Assume that there is a family of degeneration $C_t \to C_0$ for $t \to 0$. By Lemma 3.3, the sum of Milnor numbers is strictly increasing for $t \to 0$. For $\{C_{3,15}\}, \{C_{9,9}\}$, the assertion is obvious as they have $\mu = 19$ and no other place to degenerate. Now we consider the configurations $\{A_2, B_{3,10}\}, \{A_2, Sp_1\}, \{E_6, B_{3,8}\}$. The total Milnor numbers are 20 for each of them and they are obviously maximal. For $\{A_8, C_{3,7}\}$, the total Milnor number is 19 and the possibility of the degeneration is to one of $\{A_2, B_{3,10}\}, \{A_2, Sp_1\}, \{E_6, B_{3,8}\}$. If this is the case, there must be degenerations of each of the two singularities to the corresponding singularities in the above configurations. Then $C_{3,7}$ has to degenerate into either Sp_1 or $B_{3,10}$ or $B_{3,8}$, and then A_8 has no partner to degenerate.

It remains to show that the configuration $\{3A_2, C_{3,9}\}$ can not degenerate to any configuration in (1) or (2). By the total Milnor number argument, the possibility is to $\{A_2, B_{3,10}\}, \{A_2, Sp_1\}$ or $\{E_6, B_{3,8}\}$. Assume that $\{3A_2, C_{3,9}\} \to \{E_6, B_{3,8}\}$. As $2A_2 \to E_6$ but $3A_2 \not\to E_6$, we need to have a degeneration $A_2 + C_{3,9} \to B_{3,8}$ which is ridiculous as the Milnor number is decreasing. The impossibly of the degeneration $\{3A_2, C_{3,9}\} \to \{A_2, B_{3,10}\}$ or $\{A_2, Sp_1\}$ can be proved by computation, see in [P]. □

Remark 3.5. The impossibility of the above degeneration can be shown by looking the dual curves. Suppose that we have a degenerating family C_t with C_t has $\{3A_2, C_{3,9}\}$ for $t \neq 0$ and C_0 has the configuration $\{A_2, Sp_1\}$. We assume also that the singularities $C_{3,9}$ and Sp_1 are at the origin, with $y = 0$ as the tangent cone. Then we first notice that the dual curve of C_t and C_0 has degree 6 ([O4]). This implies that the corresponding dual curve can be considered as an analytic family of sextics. Secondly we can see that the dual singularity of $C_{3,9}$ is again $C_{3,9}$ at $(0,1,0)$ and C_t^* has further three A_2 singularities also. On the other hand, the dual singularity of Sp_1 is A_8 by Theorem 14, [O4]. Let L be the line supporting the tangent cone of C_0 at the singularity Sp_1. Then $L \cap C_0 = \{O\}$. This implies that in the dual curve C_0^*, it has A_8 singularity at $(0,1,0)$. Then in the family $C_t^* \to C_0^*$, we have the degeneration $C_{3,9} \to A_8$. This is impossible as the Milnor number is decreasing. The detail will be studied in our next paper [OP].

4. New Zariski pairs

We first modify the definition of a Zariski pair given in [Ar]. Our definition is weaker than the original one.

Definition 4.1. A pair of plane curves (C, C') is called a *weak Zariski pair* if they have same degree and $\Sigma(C) \sim \Sigma(C')$, but $\mathbb{P}^2 - C$ is not homeomorphic to $\mathbb{P}^2 - C'$.

We do not ask the respective tubular neighborhood of C and C' are homeomorphic. The notation $\Sigma(C) \sim \Sigma(C')$ means that there is a bijection $\phi : \Sigma(C) \to \Sigma(C')$ such that (C, ξ) is topological equivalent to $(C', \phi(\xi))$ for any $\xi \in \Sigma(C)$.

As a corollary of Theorem 2 we have two new Zariski pairs with respect to the following configurations

a) $\{A_2, A_5, C_{3,8}\}^\star$ and $\{A_2, A_5, C_{3,8}^\sharp\}^\star$.

b) $\{3A_2, C_{3,8}\}$ and $\{3A_2, C_{3,8}^\sharp\}^\star$.

In the first pair, $\{A_2, A_5, C_{3,8}\}^\star$ is a union of a conic and a quartic, while $\{A_2, A_5, C_{3,8}^\sharp\}^\star$ is a union of a line and a quintic.

In the second pair, $\{3A_2, C_{3,8}\}$ is irreducible, while $\{3A_2, C_{3,8}^\sharp\}^\star$ is reducible, a union of a line and a quintic.

The explicit equations of the above pairs and the discussions about geometry of these curves are given in the previous section.

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