数理解析研究所講究録
1182 巻 2001 年 80-82

Problem of Fenchel on the complex projective plane and representations of the 3rd braid group

Takanori MATSUNO
Department of Liberal Arts and Sciences
Osaka Prefectural College of Technology

1 Abstract

We denote by $\mathbf{P}^2$ the complex projective plane. Let $C = \{(X_0 : X_1 : X_2) \in \mathbf{P}^2 | X_2X_0^2 - X_1^3 = 0\}$ be a curve of $\mathbf{P}^2$. Let $L_\infty = \{(X_0 : X_1 : X_2) \in P^2 | X_2 = 0\}$ be the line of $\mathbf{P}^2$, which we call line at infinity. C is a rational curve of degree 3 with a cusp at $(0 : 0 : 1)$. C and L_∞ are tangent at $(1 : 0 : 0)$. Let e_1, e_2 be positive integers greater than 1. Put $D = e_1C + e_2L_\infty$. We consider the following problem and give here a partial answer by constructing representations of the 3rd braid group.

Fenchel's Problem Give a condition on the pair (e_1, e_2) for the existence of a finite Galois covering $\pi : X \to \mathbf{P}^2$ which branches at D.

2 Elementary facts

We choose a point $p_0 \in \mathbf{P}^2 - \{C \cup L_\infty\}$ and fix it. The fundamental group $\pi_1(\mathbf{P}^2 - \{C \cup L_\infty\}, p_0)$ is isomoriphic to $< \alpha, \beta, \delta | \alpha\beta\alpha = \beta\alpha\beta = \delta^{-1} >$ the 3rd braid group. This group is isomorphic to $< \gamma, \delta | \gamma^3 = \delta^2 >$. This isomorphism is given by $\gamma \mapsto (\alpha\beta)^{-1}, \delta \mapsto (\alpha\beta\alpha)^{-1}$. We identify α (resp. β, resp. δ) with a closed path in $\mathbf{P}^2 - \{C \cup L_\infty\}$ which rounds counterclockwise direction once around non-singular points P_α of C (resp. P_β of C, resp. P_δ of L_∞). Let J be the smallest normal subgroup of $\pi_1(\mathbf{P}^2 - \{C \cup L_\infty\}, p_0)$ which contains α^{e_1} and δ^{e_2}. There is a finite Galois covering which branches at D if and only if there is a normal subgroup K of $\pi_1(\mathbf{P}^2 - \{C \cup L_\infty\}, p_0)$ of finite index with $J \subset K$, which satisfies the following conditions: (1)If $\alpha^k \in K$ then $k \equiv 0 \pmod{e_1}$ and (2)If $\delta^l \in K$ then $l \equiv 0 \pmod{e_2}$. However it is difficult to look for such a K.

Let G be a finite group generated by two elements A, B, which satisfy the relation $ABA = BAB, A^{e_1} = B^{e_1} = 1, (ABA)^{e_2} = 1$. Obviously A and B are conjugate to each other. If there is a finite group G as above, we have a surjective homomorphism $\Phi : \pi_1(\mathbf{P}^2 - \{C \cup L_\infty\}, p_0) \to G$. Then the kernel of Φ corresponds to a finite Galois covering $\pi : X \to \mathbf{P}^2$ which branches at D.

Put $Q = ABA$. It is easy to see:

Lemma 2.1 *If G is abelian, then G is a cyclic group.*

Since Q^2 is an element of the center of G,

Lemma 2.2 *If the order of Q is odd, then G is abelian (G is a cyclic group).*

Hence we have:

Theorem 2.1 *If e_2 is odd, then any covering $\pi : X \to \mathbf{P}^2$ which branches at D is cyclic .*

Trivially we have:

Propsition 2.1 *For given odd number e_2, if $e_2 \equiv 0$ (mod 3) put $e_1 = e_2/3$, otherwise put $e_1 = e_2$. Then there exists $\pi : X \to \mathbf{P}^2$ which branches at D.*

It is well-known (see for example [1]):

Lemma 2.3 *For given positive integer n there is a finte group G generated by two elements $\hat{Q}$ of order 2 and $\hat{R}$ of order 3 with $\hat{Q}\hat{R}$ of order n.*

By putting $\hat{Q} = \hat{A}\hat{B}\hat{A}$ and $\hat{R} = \hat{A}\hat{B}$, we have:

Theorem 2.2 *If e_2 is 2, then for any positive integer e_1 greater than 1 there is a covering $\pi : X \to \mathbf{P}^2$ which branches at D.*

Let D be as before and let $D' = e_1'C + e_2'L_\infty$. Let e_j'' be the LCM $< e_j, e_j' >$ $(j = 1, 2)$ and put $D'' = e_1''C + e_2''L_\infty$.

By constructing the fiber product, we have:

Propsition 2.2 *If there is a covering $\pi : X \to \mathbf{P}^2$ which branches at D and there is a covering $\pi' : X' \to \mathbf{P}^2$ which branches at D', then there is a covering $\pi'' : X'' \to \mathbf{P}^2$ which branches at D''*

3 Cyclic extension

We denote by S_n the symmetric group of n letters. Let $\hat{G} \subset S_r$ be a finite group generated by two permutations $\hat{Q}$, $\hat{R}$, which satisfy the relation $\hat{Q}^2 = \hat{R}^3 = 1$. Then $\hat{Q}$ is a product of cycles of length 2 with no common letters and $\hat{R}$ is a product of cycles of length 3 with no common letters.

We may assume $\hat{G}$ has the following properties. (1)transitivity: For each letters x, y there is a permutaion of $\hat{G}$ which maps x to y. (2)simplicity: If a permutation of $\hat{G}$ fixes a letter, then it is the unit element of $\hat{G}$.

Now by showing examples, we give a method to construct a cyclic extension $G \subset S_{rq}$ of $\hat{G} \subset S_r$ by an element of its center.

The case $r = 3$. Put $\hat{Q} = (a\ b)$ and $\hat{R} = (a\ b\ c)$. In this case $\hat{G} = S_3$ and non-abelian. We need to assume q is odd. Put

$$Q = \begin{pmatrix} a_1 & a_2 & \dots & a_q & b_1 & b_2 & \dots & b_{q-1} & b_q \\ b_1 & b_2 & \dots & b_q & a_2 & a_3 & \dots & a_q & a_1 \end{pmatrix} \begin{pmatrix} c_1 & c_2 & \dots & c_p & c_{p+1} & c_{p+2} & \dots & c_q \\ c_{p+2} & c_{p+3} & \dots & c_q & c_1 & c_2 & \dots & c_{p+1} \end{pmatrix}$$

$$R = \begin{pmatrix} a_1 & a_2 & \dots & a_q & b_1 & b_2 & \dots & b_q & c_1 & c_2 & \dots & c_{q-1} & c_q \\ b_1 & b_2 & \dots & b_q & c_1 & c_2 & \dots & c_q & a_2 & a_3 & \dots & a_q & a_1 \end{pmatrix}.$$

Then

$$F = Q^2 = R^3 = (a_1\ \dots\ a_q)(b_1\ \dots\ b_q)(c_1\ \dots\ c_q)$$

and

$$A = R^{-1}Q = (a_1\ c_{p+1}\ a_{p+2}\ c_1\ \dots)$$

where $q = 2p + 1$. The order of A is $2q$.

Let G be a finite group generated by two permutations Q, R. F is a center of G. In a natural way we have the following exact sequence:

$$1 \to < F >^G \to G \to \hat{G} \to 1$$

where $< F >^G$ is a subgroup of G generated by F. $< F >^G$ is a cyclic group of order q. Then we can have a surjective homomorphism $\Phi : \pi_1(\mathbf{P}^2 - \{C \cup L_\infty\}, p_0) \to G$. Hence we have:

Theorem 3.1 *If q is odd, then there is a finite Galois covering $\pi : X \to \mathbf{P}^2$ which branches at* $2qC + 2qL_\infty$

The case $r = 4$. Put $\hat{Q} = (a\ b)$ and $\hat{R} = (b\ c\ d)$. In this case $\hat{G} \subset S_4$ and non-abelian. For the extension we need to assume the LCM$< 6, q >= 1$. In a similar way, we have:

Theorem 3.2 *If q is as above, then there is a finite Galois covering $\pi : X \to \mathbf{P}^2$ which branches at* $4qC + 2qL_\infty$

The case $r = 12$. Put $\hat{Q} = (a\ j)(b\ d)(c\ h)(e\ l)(f\ i)(g\ k)$ and $\hat{R} = (a\ b\ c)(d\ e\ f)(g\ h\ i)(j\ k\ l)$. In this case $\hat{G} \subset S_{12}$ and non-abelian. In a similar way, we have:

Theorem 3.3 *There is a finite Galois covering $\pi : X \to \mathbf{P}^2$ which branches at* $3q(q-1)C + 2qL_\infty$

References

[1] R.H.Fox, *On Fenchel's conjecture about F-groups*, Mat. Tidsskrift, vol B (1952) 61-65

[2] M.Namba, *Branched coverings and algebraic functions*, Research Notes in Math. (1987) vol 161 Pitman-Longman