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Hyperfunction Solutions of Invariant Linear Differential Equations on Prehomogeneous Vector Spaces.

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Abstract

How to determine invariant hyperfunction solutions of invariant linear differential equations with polynomial coefficients on the vector space of $n \times n$ real symmetric matrices is discussed in this work. The real special linear group of degree n naturally acts on the vector space of $n \times n$ real symmetric matrices. We can observe that every invariant hyperfunction solution is expressed as a linear combination of Laurent expansion coefficients of the complex power of the determinant function with respect to the parameter of the power. Then the problem can be reduced to the determination of Laurent expansion coefficients which is needed to express. We can give an algorithm to determine them by applying the author's result in [12]. Our method is applicable to other prehomogeneous vector spaces.

1 Introduction.

Let $V := \text{Sym}_n(\mathbb{R})$ be the space of $n \times n$ symmetric matrices over the real field \mathbb{R} and let $\mathbf{SL}_n(\mathbb{R})$ be the special linear group over \mathbb{R} of degree n . Then the group $G := \mathbf{SL}_n(\mathbb{R})$ acts on the vector space V by the representation

$$\rho(g) : x \longmapsto g \cdot x \cdot {}^t g, \quad (1)$$

with $x \in V$ and $g \in G$. Let $D(V)$ be the algebra of linear differential operators on V with polynomial coefficients and let $\mathcal{B}(V)$ be the space of hyperfunctions on V . We denote by $D(V)^G$ and $\mathcal{B}(V)^G$ the subspaces of G -invariant linear differential operators and of G -invariant hyperfunctions on V , respectively. For a given invariant differential operator $P(x, \partial) \in D(V)^G$ and an invariant hyperfunction $v(x) \in \mathcal{B}(V)^G$, we consider the linear differential equation

$$P(x, \partial)u(x) = v(x) \quad (2)$$

where the unknown function $u(x)$ is in $\mathcal{B}(V)^G$.

Our problem is the following. Let $P(x, \partial) \in D(V)^G$ be a given G -invariant homogeneous differential operator.

1. Construct a basis of G -invariant hyperfunction solutions $u(x) \in \mathcal{B}(V)^G$ to the differential equation

$$P(x, \partial)u(x) = 0.$$

2. Construct a G -invariant hyperfunction solution $u(x) \in \mathcal{B}(V)^G$ to the differential equation

$$P(x, \partial)u(x) = v(x),$$

for a given quasi-homogeneous hyperfunction $v(x) \in \mathcal{B}(V)^G$. In particular, when $v(x) = \delta(x)$, it is a problem to find a G -invariant fundamental solution.

2 Invariant Differential Operators.

We denote

$$x = (x_{ij})_{n \geq j \geq i \geq 1}, \quad \partial = (\partial_{ij}) = \left(\frac{\partial}{\partial x_{ij}} \right)_{n \geq j \geq i \geq 1}$$

$$x^\alpha = \prod_{n \geq j \geq i \geq 1} x_{ij}^{\alpha_{ij}}, \quad \partial^\beta = \prod_{n \geq j \geq i \geq 1} \partial_{ij}^{\beta_{ij}}$$

with

$$\begin{aligned} \alpha &= (\alpha_{ij}) \in \mathbb{Z}_{\geq 0}^m, & |\alpha| &= \sum_{n \geq j \geq i \geq 1} \alpha_{ij} \\ \beta &= (\beta_{ij}) \in \mathbb{Z}_{\geq 0}^m, & |\beta| &= \sum_{n \geq j \geq i \geq 1} \beta_{ij} \end{aligned}$$

and $m = n(n+1)/2$. Then $P(x, \partial) \in D(\mathbf{V})$ is expressed as

$$P(x, \partial) := \sum_{k \in \mathbb{Z}_{\geq 0}} \sum_{\substack{\alpha, \beta \in \mathbb{Z}_{\geq 0}^m \\ |\beta| = k}} a_{\alpha\beta} x^\alpha \partial^\beta. \quad (3)$$

We call the *order* of $P(x, \partial)$ the highest number k in the sum (3). On the other hand, for

$$P(x, \partial) := \sum_{k \in \mathbb{Z}} \sum_{\substack{\alpha, \beta \in \mathbb{Z}_{\geq 0}^m \\ |\alpha| - |\beta| = k}} a_{\alpha\beta} x^\alpha \partial^\beta \quad (4)$$

We call the *homogeneous part* of $P(x, \partial)$ of degree k $\sum_{\substack{\alpha, \beta \in \mathbb{Z}_{\geq 0}^m \\ |\alpha| - |\beta| = k}} a_{\alpha\beta} x^\alpha \partial^\beta$ in (4). Differential operators with only one homogeneous part is called *homogeneous differential operators*.

Example 2.1. 1. We define ∂^* by

$$\partial^* = (\partial_{ij}^*) = \left(\epsilon_{ij} \frac{\partial}{\partial x_{ij}} \right), \quad \text{and } \epsilon_{ij} := \begin{cases} 1 & i = j \\ 1/2 & i \neq j \end{cases} \quad (5)$$

2. Let h and n be positive integers with $1 \leq h \leq n$. A sequence of increasing integers $p = (p_1, \dots, p_h) \in \mathbb{Z}^h$ is called an *increasing sequence in $[1, n]$ of length h* if it satisfies $1 \leq p_1 < \dots < p_h \leq n$. We denote by $IncSeq(h, n)$ the set of increasing sequences in $[1, n]$ of length h .
3. For two sequences $p = (p_1, \dots, p_h)$ and $q = (q_1, \dots, q_h) \in IncSeq(h, n)$ and for an $n \times n$ symmetric matrix $x = (x_{ij}) \in \text{Sym}_n(\mathbb{R})$, we define an $h \times h$ matrix $x_{(p,q)}$ by

$$x_{(p,q)} := (x_{p_i, q_j})_{1 \leq i \leq j \leq h}.$$

In the same way, for an $n \times n$ symmetric matrix $\partial = (\partial_{ij})$ of differential operators, we define an $h \times h$ matrix $\partial_{(p,q)}$ of differential operators by

$$\partial_{(p,q)}^* := (\partial_{p_i, q_j}^*)_{1 \leq i \leq j \leq h}.$$

4. For an integer h with $1 \leq h \leq n$, we define

$$P_h(x, \partial) := \sum_{p, q \in IncSeq(h, n)} \det(x_{(p,q)}) \det(\partial_{(p,q)}^*). \quad (6)$$

5. In particular, $P_n(x, \partial) = \det(x) \det(\partial^*)$ and Euler's differential operator is given by

$$P_1(x, \partial) = \sum_{n \geq j \geq i \geq 1} x_{ij} \frac{\partial}{\partial x_{ij}} = \text{tr}(x \cdot \partial^*). \quad (7)$$

These are all homogeneous differential operators of degree 0 and invariant under the action of $\mathbf{GL}(V)$, and hence it is also invariant under the action of $G_1 := \mathbf{SL}_n(\mathbb{R}) \subset \mathbf{GL}(V)$.

6. $\det(x)$ and $\det(\partial^*)$ are homogeneous differential operators of degree n and $-n$, respectively. They are invariant under the action of $G := \mathbf{SL}_n(\mathbb{R})$, and relatively invariant differential operators under the action of $\mathbf{GL}_n(\mathbb{R})$, with characters $\chi(g) := \det(g)^2$ and $\chi^{-1}(g) := \det(g)^{-2}$, respectively.

Proposition 2.1.

1. Every $\mathbf{GL}_n(\mathbb{R})$ -invariant differential operator on V can be expressed as a polynomial in $P_i(x, \partial)$ ($i = 1, \dots, n$) defined in (6).
2. Every $\mathbf{SL}_n(\mathbb{R})$ -invariant differential operator on V can be expressed as a polynomial in $P_i(x, \partial)$ ($i = 1, \dots, n-1$), $\det(x)$ and $\det(\partial^*)$.

For the proof see H. Maass [5] “*Siegel's Modular Forms and Dirichlet Series*, Lecture Notes in Mathematics, vol. 216, Springer-Verlag, 1971”.

3 Some definitions and Propositions.

We denote $P(x) := \det(x)$ and we set $S := \{x \in V \mid \det(x) = 0\}$. The subset $V - S$ decomposes into $n + 1$ connected components,

$$V_i := \{x \in \text{Sym}_n(\mathbb{R}) \mid \text{sgn}(x) = (i, n - i)\} \quad (8)$$

with $i = 0, 1, \dots, n$. Here, $\text{sgn}(x)$ for $x \in \text{Sym}_n(\mathbb{R})$ is the signature of the quadratic form $q_x(\vec{v}) := {}^t \vec{v} \cdot x \cdot \vec{v}$ on $\vec{v} \in \mathbb{R}^n$. We define the complex power function of $P(x)$ by

$$|P(x)|_i^s := \begin{cases} |P(x)|^s & \text{if } x \in V_i, \\ 0 & \text{if } x \notin V_i. \end{cases} \quad (9)$$

for a complex number $s \in \mathbb{C}$.

We consider a linear combination of the hyperfunctions $|P(x)|_i^s$

$$P^{[\vec{a}, s]}(x) := \sum_{i=0}^n a_i \cdot |P(x)|_i^s \quad (10)$$

with $s \in \mathbb{C}$ and $\vec{a} := (a_0, a_1, \dots, a_n) \in \mathbb{C}^{n+1}$. Then $P^{[\vec{a}, s]}(x)$ is a hyperfunction with a meromorphic parameter $s \in \mathbb{C}$, and depends on $\vec{a} \in \mathbb{C}^{n+1}$ linearly.

Proposition 3.1. $P^{[\vec{a}, s]}(x)$ is holomorphic with respect to $s \in \mathbb{C}$ except for the poles at $s = -(k + 1)/2$ with $k = 1, 2, \dots$. The possible highest order of the pole of $P^{[\vec{a}, s]}(x)$ at $s = -(k + 1)/2$ is

$$\begin{cases} \lfloor \frac{k+1}{2} \rfloor & (k = 1, 2, \dots, n-1), \\ \lfloor \frac{n}{2} \rfloor & (k = n, n+1, \dots, \text{ and } k+n \text{ is odd}), \\ \lfloor \frac{n+1}{2} \rfloor & (k = n, n+1, \dots, \text{ and } k+n \text{ is even}). \end{cases} \quad (11)$$

Definition 3.1. Let $\lambda \in \mathbb{C}$ be a fixed complex number.

1. We denote by $PHO(\lambda)$ the possible highest order of $P^{[\vec{a},s]}(x)$ at $s = \lambda$. Namely we define

$$PHO(\lambda) := \begin{cases} \lfloor \frac{k+1}{2} \rfloor & \lambda = -\frac{k+1}{2} \quad (k = 1, 2, \dots, n-1), \\ \lfloor \frac{n}{2} \rfloor & \lambda = -\frac{k+1}{2} \quad (k = n, n+1, \dots, \text{ and } k+n \text{ is odd}), \\ \lfloor \frac{n+1}{2} \rfloor & \lambda = -\frac{k+1}{2} \quad (k = n, n+1, \dots, \text{ and } k+n \text{ is even}), \\ 0 & \text{otherwise.} \end{cases} \quad (12)$$

2. Let $q \in \mathbb{Z}$. We define a vector subspace $A(\lambda, q)$ of \mathbb{C}^{n+1} by

$$A(\lambda, q) := \{\vec{a} \in \mathbb{C}^{n+1} \mid P^{[\vec{a},s]}(x) \text{ has a pole of order } \leq q \text{ at } s = \lambda\}. \quad (13)$$

Then we have $A(\lambda, q-1) \subset A(\lambda, q)$ by definition. We define $\overline{A(\lambda, q)}$ by

$$\overline{A(\lambda, q)} := A(\lambda, q)/A(\lambda, q-1) \quad (14)$$

3. We define $o(\vec{a}, \lambda) \in \mathbb{Z}$ by

$$o(\vec{a}, \lambda) := \text{the order of pole of } P^{[\vec{a},s]}(x) \text{ at } s = \lambda. \quad (15)$$

We have $p = o(\vec{a}, \lambda)$ if and only if $\vec{a} \in A(\lambda, p)$ and $[\vec{a}] \in \overline{A(\lambda, p)}$ is not zero.

4. Let $\vec{a} \in \mathbb{C}^{n+1}$ and let $p = o(\vec{a}, \lambda) \in \mathbb{Z}_{\geq 0}$. This means that $P^{[\vec{a},s]}(x)$ has a pole of order p at $s = \lambda$. Then we have the Laurent expansion of $P^{[\vec{a},s]}(x)$ at $s = \lambda$,

$$P^{[\vec{a},s]}(x) = \sum_{w=-p}^{\infty} P_w^{[\vec{a},\lambda]}(x) (s-\lambda)^w. \quad (16)$$

We often denote by

$$Laurent_{s=\lambda}^{(w)}(P^{[\vec{a},s]}(x)) := P_w^{[\vec{a},\lambda]}(x) \quad (17)$$

the w -th Laurent expansion coefficient of $P^{[\vec{a},s]}(x)$ at $s = \lambda$ in (16).

Proposition 3.2. *Let $\vec{a}, \vec{b} \in \mathbb{C}^{n+1}$ and let $p = PHO(\lambda)$.*

1. *Let q be an integer in $q \leq p$. We have*

$$\vec{a} - \vec{b} \in A(\lambda, q)$$

if and only if

$$Laurent_{s=\lambda}^{(w)}(P^{[\vec{a},s]}(x)) = Laurent_{s=\lambda}^{(w)}(P^{[\vec{b},s]}(x))$$

for $w = -p, -p+1, \dots, -q-1$. In particular,

$$P^{[\vec{a},s]}(x) = P^{[\vec{b},s]}(x)$$

if $\vec{a} - \vec{b} \in A(\lambda, q)$ for some $q < 0$.

2. *Let $r = o(\vec{a}, \lambda) \in \mathbb{Z}_{\geq 0}$, i.e., the order of pole of $P^{[\vec{a},s]}(x)$ at $s = \lambda$. Then the Laurent expansion coefficients at $s = \lambda$*

$$\{Laurent_{s=\lambda}^{(-r+i)}(P^{[\vec{a},s]}(x))\}_{i=0,1,2,\dots}$$

are linearly independent.

3. Let $\vec{a}_1, \dots, \vec{a}_k \in \mathbb{C}^{n+1}$ be the vectors satisfying that they are linearly independent in the quotient space $\mathbb{C}^{n+1}/A(\lambda, q-1)$ with a positive integer q . Then, for an integer w with $w \geq -q$, the hyperfunctions

$$\{Laurent_{s=\lambda}^{(w)}(P^{[\vec{a}, s]}(x))\}_{i=1,2,\dots,k}$$

are linearly independent.

Definition 3.2. We say that $v(x) \in \mathcal{B}(\mathbf{V})$ is *quasi-homogeneous* of degree $\lambda \in \mathbb{C}$ if and only if there exists an positive integer q such that

$$\underbrace{F_{k,\lambda} \circ F_{k,\lambda} \circ \dots \circ F_{k,\lambda}}_q(v) = 0$$

for all $k \in \mathbb{R}_{>0}$ where

$$F_{k,\lambda}(v) := v(k \cdot x) - k^\lambda v(x).$$

Definition 3.3. We use the following notations.

1. $QH(\lambda) := \{u(x) \in \mathcal{B}(\mathbf{V}) \mid u(x) \text{ is quasi-homogeneous of degree } \lambda \in \mathbb{C}\}$
2. $QH(\lambda)^\mathcal{G} := QH(\lambda) \cap \mathcal{B}(\mathbf{V})^\mathcal{G}$
3. $QH := \bigoplus_{\lambda \in \mathbb{C}} QH(\lambda)$
4. $QH^\mathcal{G} := \bigoplus_{\lambda \in \mathbb{C}} QH(\lambda)^\mathcal{G}$

Proposition 3.3. Let $p \in \mathbb{Z}$ be the order of the pole of $P^{[\vec{a}, s]}(x)$ at $s = \lambda$.

1. Then the Laurent expansion coefficient of $P^{[\vec{a}, s]}(x)$ at $s = \lambda$ defined by (17)

$$Laurent_{s=\lambda}^{(w)}(P^{[\vec{a}, s]}(x)) = P_w^{[\vec{a}, \lambda]}(x)$$

is a quasi-homogeneous hyperfunction of degree $n \cdot \lambda$ of quasi-degree $p + w$. Conversely, let $v(x) \in QH(n \cdot \lambda)^\mathcal{G}$. Then $v(x)$ is written as a linear combination of Laurent expansion coefficients of $|P(x)|_i^s$ at $s = \lambda$.

2. Let

$$LC(\lambda, w) = \{Laurent_{s=\lambda}^{(w)}(P^{[\vec{a}, s]}(x)) \mid \vec{a} \in \mathbb{C}^{n+1}\},$$

the vector space of w -th Laurent expansion coefficients of $P^{[\vec{a}, s]}(x)$. Then we have the direct sum decomposition

$$QH(n \cdot \lambda)^\mathcal{G} = \bigoplus_{\substack{w \in \mathbb{Z}, \\ w \geq PHO(\lambda)}} LC(\lambda, w) \quad (18)$$

Namely let $v(x) \in QH^\mathcal{G}(n \cdot \lambda)$. Then $v(x)$ is written as a linear combination of Laurent expansion coefficients of $|P(x)|_i^s$ at $s = \lambda$.

Proposition 3.4. Let $P(x, \partial) \in D(\mathbf{V})^\mathcal{G}$ be a homogeneous differential operator.

1. The homogeneous degree of $P(x, \partial)$ is in $(n \cdot \mathbb{Z})$. Namely the homogeneous degree is divisible by n .

2. If the homogeneous degree of $P(x, \partial)$ is nk with $k \in \mathbb{Z}$, then we have

$$P(x, \partial)(\det x)^s = b_P(s)(\det x)^{s+k} \quad (19)$$

where $b_P(s)$ is a polynomial in $s \in \mathbb{C}$ and $x \in \text{Sym}_n(\mathbb{R})$ is positive definite. We have also

$$\begin{aligned} P(x, \partial)P^{[\bar{a}, s]}(x) &= b_P(s) \det(x)^k P^{[\bar{a}, s]}(x) \\ &= b_P(s) \text{sgn}(\det(x)) P^{[\bar{a}, s+k]}(x) \end{aligned} \quad (20)$$

for all $x \in V - S$.

3. If $k < 0$, then $b^{-k}(s-1) | b_P(s)$ where $b^{-k}(s-1) := b(s-1)b(s-2) \cdots b(s-(-k))$ with $b(s) := \prod_{i=1}^n (s + \frac{i+1}{2})$.

Definition 3.4 (b_P -function). Let $P(x, \partial) \in D(V)^{\mathbf{G}}$ be a homogeneous differential operator. We call $b_P(s)$ in (19) the b_P -function of $P(x, \partial)$. Namely let $P(x, \partial)$ be a \mathbf{G} -invariant homogeneous differential operator of homogeneous degree nk ($k \in \mathbb{Z}$). (Homogeneous degree of \mathbf{G} -invariant differential operator is divisible by n .) Then we have

$$P(x, \partial)(\det(x))^s = {}^{\exists} b_P(s)(\det(x))^{s+k}$$

with $s \in \mathbb{C}$ and $x > 0$, i.e., positive definite. Here $b_P(s)$ is a polynomial in \mathbb{C} . We call $b_P(s)$ the b_P -function of $P(x, \partial)$.

Example 3.1. 1. For $P_h(x, \partial) := \sum_{p, q \in \text{IncSeq}(h, n)} \det(x_{(p, q)}) \det(\partial_{(p, q)}^*)$ (homogeneous degree $kn = 0$) defined by (6)

$$b_P(s) = \text{const.}(s)(s + \frac{1}{2}) \cdots (s + \frac{h-1}{2}).$$

2. For $P(x, \partial) = \det(\partial^*)$ (homogeneous degree $kn = -n$),

$$b_P(s) = \text{const.}(s)(s + \frac{1}{2}) \cdots (s + \frac{n-1}{2}).$$

3. For $P(x, \partial) = \det(x)$ (homogeneous degree $kn = n$),

$$b_P(s) = 1.$$

4 Main theorems.

We have the following theorems.

Theorem 4.1. Let $P(x, \partial) \in D(V)^{\mathbf{G}}$ be a non-zero homogeneous differential operator with homogeneous degree kn .

1. We suppose that

$$\text{the degree of } b_P(s) = \text{the order of } P(x, \partial). \quad (21)$$

The space of \mathbf{G} -invariant hyperfunction solutions of the differential equation $P(x, \partial)u(x) = 0$ is finite dimensional. The solutions $u(x)$ are given as finite linear combinations of quasi-homogeneous \mathbf{G} -invariant hyperfunctions.

2. We suppose that

$$b_P(s) \neq 0. \quad (22)$$

Let $v(x)$ be a quasi-homogeneous \mathbf{G} -invariant hyperfunction. Then there is a solution $u(x) \in \mathcal{B}(\mathbf{V})^{\mathbf{G}}$ of the differential equation $P(x, \partial)u(x) = v(x)$. The solutions $u(x)$ are given as finite linear combinations of quasi-homogeneous \mathbf{G} -invariant hyperfunctions.

3. Let $P(x, \partial) \in D(\mathbf{V})^{\mathbf{G}}$ be a non-zero homogeneous differential operator satisfying the condition (21) and let $Q(x, \partial) \in D(\mathbf{V})^{\mathbf{G}}$ be a homogeneous differential operator satisfying the condition (21) with the same homogeneous degree kn as $P(x, \partial)$ and suppose that $b_P(s) = b_Q(s)$. Then the \mathbf{G} -invariant solution space of the differential equation $P(x, \partial)u(x) = v(x)$ coincides with that of the differential equation $Q(x, \partial)u(x) = v(x)$.
4. We can give an algorithm to compute all the \mathbf{G} -invariant hyperfunction solutions of the differential equation $P(x, \partial)u(x) = v(x)$ provided that we can calculate the total homogeneous degree of $P(x, \partial)$ and the explicit form of $b_P(s)$ in some way.

5 Algorithms for constructing solutions.

We define a standard basis of \mathbb{C}^{n+1} .

Definition 5.1 (Standard basis). Let

$$SB := \{\bar{a}_0, \bar{a}_1, \dots, \bar{a}_n\} \quad (23)$$

be a basis of \mathbb{C}^{n+1} . We say that SB is a *standard basis* of \mathbb{C}^{n+1} at $s = \lambda$ if the following property holds: there exists an increasing integer sequence

$$0 \leq k(0) < k(1) < \dots < k(\text{PHO}(\lambda)) = n \quad (24)$$

such that

$$SB_q := \{\bar{a}_0, \bar{a}_1, \dots, \bar{a}_{k(q)}\}$$

is a basis of $A(\lambda, q)$ for each q in $0 \leq q \leq \text{PHO}(\lambda)$.

When $\lambda \notin \frac{1}{2}\mathbb{Z}$, any basis is a standard basis since all $P^{[\bar{a}, s]}(x)$ is holomorphic at $s = \lambda$. When λ is in $\frac{1}{2}\mathbb{Z}$, we can easily choose one standard basis for a given λ . However, it is sufficient only to consider the three kinds of standard basis, SB^{half} , SB^{even} and SB^{odd} .

Algorithm 5.1 (The case of homogeneous degree zero). For a given non-zero $\mathbf{SL}_n(\mathbb{R})$ -invariant differential operator $P(x, \partial) \in D(\mathbf{V})_0^{\mathbf{G}}$ of homogeneous degree 0 satisfying the condition

$$\text{the degree of } b_P(s) = \text{the order of } P(x, \partial), \quad (25)$$

one algorithm to compute a basis of the $\mathbf{SL}_n(\mathbb{R})$ -invariant differential equation $P(x, \partial)u(x) = 0$ is given in the following.

Input A non-zero $\mathbf{SL}_n(\mathbb{R})$ -invariant differential operator $P(x, \partial) \in D(\mathbf{V})_0^{\mathbf{G}}$ satisfying the condition (25).

Output A basis of the $\mathbf{SL}_n(\mathbb{R})$ -invariant hyperfunctions to the differential equation $P(x, \partial)u(x) = 0$.

Procedure

1. Compute the b_P -function for $P(x, \partial)$. It is denoted by

$$b_P(s) = (s - \lambda_1)^{k_1} \cdots (s - \lambda_p)^{k_p}.$$

2. For each λ_i ($i = 1, \dots, p$), take one standard basis at $s = \lambda_i$

$$SB^{\lambda_i} = \{\bar{a}_0(\lambda_i), \dots, \bar{a}_n(\lambda_i)\},$$

which is defined in Definition 5.1.

3. Compute the Laurent expansion coefficients

$$Laurent_{s=\lambda_i}^{(k)}(P^{[\bar{a}_j(\lambda_i), s]}(x))$$

for each $\bar{a}_j(\lambda_i)$ ($i = 1, \dots, p, j = 0, \dots, n$) and $k = -o_{ij}, -o_{ij} + 1, \dots$ with $o_{ij} := o(\bar{a}_j(\lambda_i), \lambda_i)$ until all the generators of (26) will be obtained.

$$L_{ij} := \{Laurent_{s=\lambda_i}^{(k)}(P^{[\bar{a}_j(\lambda_i), s]}(x))\}_{k=-o_{ij}, \dots, -o_{ij}+k_i-1} \quad (26)$$

4. Then

$$\bigoplus_{\substack{i=1, \dots, p \\ j=0, \dots, n}} L_{ij} \quad (27)$$

forms a basis of the G -invariant hyperfunction solution space to $P(x, \partial)u(x) = 0$.

Algorithm 5.2 (The case of positive homogeneous degree). For a given non-zero $\mathbf{SL}_n(\mathbb{R})$ -invariant differential operator $P(x, \partial) \in D(\mathbf{V})_q^G$ of positive homogeneous degree $q > 0$ satisfying the condition

$$\text{the degree of } b_P(s) = \text{the order of } P(x, \partial), \quad (28)$$

one algorithm to compute a basis of the $\mathbf{SL}_n(\mathbb{R})$ -invariant differential equation $P(x, \partial)u(x) = 0$ is given in the following.

Input A non-zero $\mathbf{SL}_n(\mathbb{R})$ -invariant differential operator $P(x, \partial) \in D(\mathbf{V})_q^G$ with $q > 0$ satisfying the condition (28).

Output A basis of the $\mathbf{SL}_n(\mathbb{R})$ -invariant hyperfunctions to the differential equation $P(x, \partial)u(x) = 0$.

Procedure

1. Consider the set $R := R_1 \cup R_2$ with

$$R_1 := \{\lambda_i := -\frac{i+1}{2} \mid i = 1, 2, \dots, n+2q-2\},$$

$$R_2 := \{\lambda \in \mathbb{C} \mid b_P(\lambda) = 0\}.$$

Let p be the number of elements of the set $R_2 - R_1$. We denote by

$$\lambda_{n+2q-1}, \lambda_{n+2q}, \dots, \lambda_{n+2q+p-2}$$

the elements of $R_2 - R_1$. Then we can write the elements of R by

$$R = \{\lambda_1, \lambda_2, \dots, \lambda_{n+2q+p-2}\}.$$

2. We define the multiplicity k_i of λ_i by

$$k_i := \begin{cases} \text{the multiplicity of } s - \lambda_i \text{ in } b_P(s) & \text{if } b_P(\lambda_i) = 0 \\ 0 & \text{if } b_P(\lambda_i) \neq 0 \end{cases} \quad (29)$$

3. For each λ_i ($i = 1, \dots, n + 2q + p - 2$), take one standard basis

$$SB^{\lambda_i} = \{\bar{a}_0(\lambda_i), \dots, \bar{a}_n(\lambda_i)\}$$

at $s = \lambda_i$, which is the standard basis SB^{half} , SB^{even} and SB^{odd} when $\lambda_i \in \frac{1}{2}\mathbb{Z}$ and the one defined in Definition 5.1 otherwise.

4. For each λ_i , we associate an finite increasing integer sequence $\{l(u)\}_{u=0,1,2,\dots}$ with the last term n . If $\lambda_i \in \frac{1}{2}\mathbb{Z}$, then we define $\{l(u)\}_{u=0,1,2,\dots}$. If $\lambda_i \notin \frac{1}{2}\mathbb{Z}$, then we define it by $\{l(0) = n\}$.

5. Compute the Laurent expansion coefficients

$$Laurent_{s=\lambda_i}^{(k)}(P^{[\bar{a}_j(\lambda_i), s]}(x))$$

for each $\bar{a}_j(\lambda_i)$ ($i = 1, \dots, n + 2q + p - 2, j = 0, \dots, n$) and $k = -o_{ij}, -o_{ij} + 1, \dots$ with $o_{ij} := o(\bar{a}_j(\lambda_i), \lambda_i)$ until all the generators of (30) and (31) are obtained. For λ_i in $1 \leq i \leq n + 2q + p - 2$, we put

$$\begin{aligned} p_1 &:= PHO(\lambda_i), \\ p_2 &:= PHO(\lambda_i + q). \end{aligned}$$

If $\bar{a}_j(\lambda_i) \notin SB_{l(p_2)}^{\lambda_i}$, then we set

$$L_{ij} := \{Laurent_{s=\lambda_i}^{(w)}(P^{[\bar{a}_j(\lambda_i), s]}(x))\}_{-o_{ij} \leq w \leq -p_2 + k_i - 1}. \quad (30)$$

If $\bar{a}_j(\lambda_i) \in SB_{l(p_2)}^{\lambda_i}$, then we set

$$L_{ij} := \{Laurent_{s=\lambda_i}^{(w)}(P^{[\bar{a}_j(\lambda_i), s]}(x))\}_{-o_{ij} \leq w \leq -o_{ij} + k_i - 1}. \quad (31)$$

6. Then

$$\bigoplus_{\substack{i=1, \dots, n+2q+p-2 \\ j=0, \dots, n}} L_{ij} \quad (32)$$

forms a basis of the solution space.

Algorithm 5.3 (The case of negative homogeneous degree). For a given non-zero $\mathbf{SL}_n(\mathbb{R})$ -invariant differential operator $P(x, \partial) \in D(\mathbf{V})_{-q}^{\mathbf{G}}$ of negative homogeneous degree $-q < 0$ satisfying the condition

$$\text{the degree of } b_P(s) = \text{the order of } P(x, \partial), \quad (33)$$

one algorithm to compute a basis of the $\mathbf{SL}_n(\mathbb{R})$ -invariant differential equation $P(x, \partial)u(x) = 0$ is given in the following.

Input A non-zero $\mathbf{SL}_n(\mathbb{R})$ -invariant differential operator $P(x, \partial) \in D(\mathbf{V})_{-q}^{\mathbf{G}}$ with $-q < 0$ satisfying the condition (33).

Output A basis of the $\mathbf{SL}_n(\mathbb{R})$ -invariant hyperfunctions to the differential equation $P(x, \partial)u(x) = 0$.

Procedure

1. Let $b_P(s)$ be the b_P -function of $P(x, \partial)$. Then it is decomposed to

$$b_P(s) = b_1(s) \cdot b_2(s)$$

with

$$\begin{aligned} b_1(s) &:= \prod_{k=0}^{q-1} b(s-k) \\ b_2(s) &:= b_P(s)/b_1(s) \end{aligned}$$

where $b(s) = s(s + \frac{1}{2}) \cdots (s + \frac{n-1}{2})$.

2. Consider the set $R := R_1 \cup R_2$ with

$$\begin{aligned} R_1 &:= \{\lambda_i := -\frac{n-i}{2} \mid i = 1, 2, \dots, n+2q-2\}, \\ R_2 &:= \{\lambda \in \mathbb{C} \mid b_2(\lambda) = 0\}. \end{aligned}$$

Let p be the number of elements of the set $R_2 - R_1$. We denote by

$$\lambda_{n+2q-1}, \lambda_{n+2q}, \dots, \lambda_{n+2q+p-2}$$

the elements of $R_2 - R_1$. Then we can write the elements of R by

$$R = \{\lambda_1, \lambda_2, \dots, \lambda_{n+2q+p-2}\}.$$

3. We define the multiplicity k_i of λ_i by

$$k_i := \begin{cases} \text{the multiplicity of } s - \lambda_i \text{ in } b_2(s) & \text{if } b_2(\lambda_i) = 0 \\ 0 & \text{if } b_2(\lambda_i) \neq 0 \end{cases} \quad (34)$$

4. For each λ_i ($i = 1, \dots, n+2q+p-2$), take one standard basis

$$SB^{\lambda_i} = \{\vec{a}_0(\lambda_i), \dots, \vec{a}_n(\lambda_i)\}$$

at $s = \lambda_i$, which is the standard basis defined SB^{half} , SB^{even} and SB^{odd} when $\lambda_i \in \frac{1}{2}\mathbb{Z}$ and the one defined in Definition 5.1 otherwise.

5. For each λ_i , we associate an finite increasing integer sequence $\{l(u)\}_{u=0,1,2,\dots}$ with the last term n . If $\lambda_i \in \frac{1}{2}\mathbb{Z}$, then we define $\{l(u)\}_{u=0,1,2,\dots}$. If $\lambda_i \notin \frac{1}{2}\mathbb{Z}$, then we define it by $\{l(0) = n\}$.

6. Compute the Laurent expansion coefficients

$$Laurent_{s=\lambda_i}^{(k)}(P^{[\vec{a}_j(\lambda_i), s]}(x))$$

for each $\vec{a}_j(\lambda_i)$ ($i = 1, \dots, n+2q+p-2, j = 0, \dots, n$) and $k = -o_{ij}, -o_{ij} + 1, \dots$ with $o_{ij} := o(\vec{a}_j(\lambda_i), \lambda_i)$ until all the generators of (35), (36) and (37) are obtained. For λ_i in $1 \leq i \leq n+2q+p-2$, we put

$$\begin{aligned} p_1 &:= PHO(\lambda_i) \\ p_2 &:= PHO(\lambda_i - q) \end{aligned}$$

and let

$$a_{ij} := p_2 - o(\vec{a}_j(\lambda_i), \lambda_i - q).$$

If $\vec{a}_j(\lambda_i) \notin SB_{l(p_2)}^{\lambda_i}$, then we set

$$L_{ij} := \{0\}. \quad (35)$$

If $\vec{a}_j(\lambda_i) \in SB_{l(p_2)}^{\lambda_i} - SB_{l(p_1)}^{\lambda_i}$, then we set

$$L_{ij} := \{Laurent_{s=\lambda_i}^{(w)}(P^{[\vec{a}_j(\lambda_i), s]}(x))\}_{-o_{ij} \leq w \leq -o_{ij} + a_{ij} + k_i - 1}. \quad (36)$$

If $\vec{a}_j(\lambda_i) \in SB_{l(p_1)}^{\lambda_i}$, then we set

$$L_{ij} := \{Laurent_{s=\lambda_i}^{(w)}(P^{[\vec{a}_j(\lambda_i), s]}(x))\}_{-o_{ij} \leq w \leq -o_{ij} + (p_2 - p_1) + k_i - 1}. \quad (37)$$

7. Then

$$\bigoplus_{\substack{i=1, \dots, n+2q+p-2 \\ j=0, \dots, n}} L_{ij} \quad (38)$$

forms a basis of the solution space.

6 Examples.

Let us consider the case of $P(x, \partial) = \det(x)$. Then the total homogeneous degree of $P(x, \partial)$ is n and $b_P(s) = 1$. We can prove by our algorithm that the \mathbf{G} -invariant solution space of the differential equation $\det(x)u(x) = 0$ is generated by the \mathbf{G} -invariant measures on all the singular orbits (i.e., \mathbf{G} -orbits contained in $\det(x) = 0$), and hence, it is $\frac{n(n+1)}{2}$ -dimensional (= the number of singular orbits). Here the \mathbf{G} -invariant measure on each singular orbit is a relatively invariant hyperfunction.

Similar argument is possible for the case of $P(x, \partial) = \det(\partial)$ operators. In this case, the total homogeneous degree of $P(x, \partial)$ is $(-n)$ and we see that $b_P(s) = \prod_{i=1}^n (s + \frac{i-1}{2})$. The solution space of $\det(\partial)u(x) = 0$ is just the Fourier transform of that of $\det(x)u(x) = 0$, and hence it is $\frac{n(n+1)}{2}$ -dimensional and generated by relatively invariant hyperfunctions. We can construct them from the complex power of $\det(x)$

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