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ASSOCIATED CYCLES OF HARISH-CHANDRA MODULES AND DIFFERENTIAL OPERATORS OF GRADIENT-TYPE

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ABSTRACT. We describe the associated cycles of irreducible Harish-Chandra modules with irreducible associated variety, in relation to the symbols of differential operators of gradient type. Also, the irreducibility is discussed for the isotropy representation which determines the multiplicity in the associated cycle.

CONTENTS

1. INTRODUCTION

Let G be a connected semisimple Lie group with finite center, and let K be a maximal compact subgroup of G. The corresponding Lie algebras are denoted by $\mathfrak{g}_0$ and $\mathfrak{k}_0$, and we write $\mathfrak{g}$ and $\mathfrak{k}$ for their complexifications, respectively. Then, $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ gives a complexified Cartan decomposition of $\mathfrak{g}$.

We consider an irreducible admissible representation π of G. There exist analytic or algebraic methods to associate with π certain nilpotent orbits in the Lie algebras. Let us give a quick review of such methods, and introduce some key notion which we use in this article.

On one hand, the analytic methods use the distribution character Θ_π of π, and also the Fourier transformation on the Lie algebras. In fact, by Barbash and Vogan [2], the leading term of the asymptotic expansion of Θ_π at the identity $e \in G$ gives rise to a tempered distribution on $\mathfrak{g}_0$. The support of its Fourier transform, say $\mathcal{AS}(\pi)$, turns to be a G-invariant closed cone in $\mathfrak{g}_0$. $\mathcal{AS}(\pi)$ is called the *asymptotic support* of π. Independently, Howe introduced in [9] the *wave front set* $\mathrm{WF}(\pi)$ of the representation π, which is also a union of some nilpotent G-orbits in $\mathfrak{g}_0$. Later, it was proved by Rossmann [15] that these two nilpotent invariants $\mathcal{AS}(\pi)$ and $\mathrm{WF}(\pi)$ of π coincide with each other.

On the other hand, the algebraic methods is based on the Harish-Chandra $(\mathfrak{g}, K)$-module $\boldsymbol{X}$ consisting of K-finite vectors for π. Let $U(\mathfrak{g})$ be the universal enveloping

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algebra of $\mathfrak{g}$. We write $S(\mathfrak{g})$ for the symmetric algebra of $\mathfrak{g}$. It is isomorphic to the graded algebra attached to $U(\mathfrak{g})$ through the natural increasing filtration. A good filtration of $\boldsymbol{X}$ provides us with a finitely generated, graded $(S(\mathfrak{g}), K)$-module $M = \operatorname{gr} \boldsymbol{X}$. Then, the support $\operatorname{Supp} M$ of the $S(\mathfrak{g})$-module M turns to be a $K_{\mathbb{C}}$-invariant affine algebraic variety contained in the set of nilpotent elements in $\mathfrak{p}$, which is in fact independent of the choice of a good filtration of $\boldsymbol{X}$. Here $K_{\mathbb{C}}$ denotes the complexification of K. The affine variety $\mathcal{V}(\boldsymbol{X}) := \operatorname{Supp} M$ is called the *associated variety* of $\boldsymbol{X}$ (see [1], [19], [20]). We note that the dimension of $\mathcal{V}(\boldsymbol{X})$ equals the Gelfand-Kirillov dimension of $\boldsymbol{X}$, and that the primitive ideal $\operatorname{Ann}_{U(\mathfrak{g})}\boldsymbol{X}$ defines the unique nilpotent $G_{\mathbb{C}}$-orbit in $\mathfrak{g}$ containing $\mathcal{V}(\boldsymbol{X})$, where $G_{\mathbb{C}}$ is a connected Lie group with Lie algebra $\mathfrak{g}$. (Cf. [18]) As investigated in [7], [13], [22] and [25] etc., the associated variety $\mathcal{V}(\boldsymbol{X})$ controls some key properties concerning the structure of original $(\mathfrak{g}, K)$-module $\boldsymbol{X}$.

By taking into account also the *multiplicity* of the $S(\mathfrak{g})$-module $M = \operatorname{gr}\boldsymbol{X}$ at each irreducible component of $\mathcal{V}(\boldsymbol{X})$, we are naturally led to a refinement of $\mathcal{V}(\boldsymbol{X})$, say $\mathcal{C}(\boldsymbol{X})$, which is called the *associated cycle* of $\boldsymbol{X}$. Also for the asymptotic support $\mathcal{AS}(\pi)$, one has a similar refinement of this nature, which is called the *asymptotic cycle* (or the *wave front cycle*) of π. Recently, it has been shown by Schmid and Vilonen [17] that the above two cycles are equal up to the Kostant-Sekiguchi correspondence of nilpotent orbits.

Now, we focus our attention on the associated cycles $\mathcal{C}(\boldsymbol{X})$ of irreducible $(\mathfrak{g}, K)$-modules $\boldsymbol{X}$. Throughout this article, we assume that the associated variety $\mathcal{V}(\boldsymbol{X})$ of $\boldsymbol{X}$ is the closure of a single nilpotent $K_{\mathbb{C}}$-orbit $\mathcal{O}$ in $\mathfrak{p}$. This assumption does not exclude important $(\mathfrak{g}, K)$-modules related to elliptic orbits. In reality, it is well-known that the $(\mathfrak{g}, K)$-modules of discrete series (more generally Zuckerman derived functor modules) and also the irreducible admissible highest weight modules of Hermitian Lie algebras satisfy this hypothesis. Let $I = I(\mathcal{V}(\boldsymbol{X}))$ denote the prime ideal of $S(\mathfrak{g})$ which defines the irreducible variety $\mathcal{V}(\boldsymbol{X})$. Then the multiplicity $\operatorname{mult}_I(\boldsymbol{X})$ of $\boldsymbol{X}$ at I is defined to be the length of the localization M_I of M at I as an $S(\mathfrak{g})_I$-module. The graded $(\mathfrak{g}, K)$-module M is not uniquely determined by $\boldsymbol{X}$, but the multiplicity $\operatorname{mult}_I(\boldsymbol{X})$ is actually an invariant of $\boldsymbol{X}$. In this setting, the associated cycle of $\boldsymbol{X}$ turns to be

$$\mathcal{C}(\boldsymbol{X}) = \operatorname{mult}_I(\boldsymbol{X}) \cdot [\overline{\mathcal{O}}]. \tag{1.1}$$

Take a point X of the orbit $\mathcal{O}$, and let $K_{\mathbb{C}}(X)$ denote the isotropy subgroup of $K_{\mathbb{C}}$ at X. Vogan [19] has shown that the multiplicity $\operatorname{mult}_I(\boldsymbol{X})$ in the associated cycle $\mathcal{C}(\boldsymbol{X})$ is not just a positive integer, but that it can be interpreted as the dimension of a certain finite-dimensional module $\mathcal{W}$ over $K_{\mathbb{C}}(X)$. We name $\mathcal{W}$ the *isotropy representation* of $K_{\mathbb{C}}(X)$ associated with $\boldsymbol{X}$ (see Definition 2.1 for the precise definition).

In this article, we first investigate the relationship between the associated cycles $\mathcal{C}(\boldsymbol{X})$ and the (polynomialized) *differential operators of gradient type* on $\mathfrak{p}$ whose kernel spaces realize the K-finite dual of $M = \operatorname{gr} \boldsymbol{X}$, by developing our argument in [25] for the case of highest weight representations in full generality. If the annihilator ideal of M coincides with the whole I by an appropriate choice of M, we get a quite natural understanding of the associated cycle $\mathcal{C}(\boldsymbol{X})$ and the isotropy representation $\mathcal{W}$ in terms of the symbol map $\boldsymbol{\sigma}$ of a differential operator of gradient type. Namely, the symbol $\boldsymbol{\sigma}$ yields a $K_{\mathbb{C}}$-vector bundle on each $K_{\mathbb{C}}$-orbit contained in the associated variety $\mathcal{V}(\boldsymbol{X})$, and the fibre space at $X \in \mathcal{O}$ realizes the dual of the isotropy representation $\mathcal{W}$ (see Theorem 4.1). This result applies well to the unitary highest weight modules, and also to the $(\mathfrak{g}, K)$-modules of discrete series with sufficiently regular Harish-Chandra parameters. In fact, the equality $I = \mathrm{Ann}_{S(\mathfrak{g})} M$ holds for these two types of $(\mathfrak{g}, K)$-modules $\boldsymbol{X}$, if we define M through the K-stable good filtration arising from the unique minimal K-type of $\boldsymbol{X}$ (see Theorems 2.4 and 2.5). Furthermore, the corresponding differential operators of gradient type are well understood for these $\boldsymbol{X}$ ([8], [16], [5], [6]; see also [21], [25], [26]).

Second, we are concerned with the irreducibility of the isotropy representation $\mathcal{W}$. This part is a work in progress. A criterion for the irreducibility of $\mathcal{W}$ is given in Theorem 5.1 under certain assumptions on $M = \mathrm{gr}\, \boldsymbol{X}$, where we bear the $(\mathfrak{g}, K)$-modules $\boldsymbol{X}$ with *small* nilpotent orbits $\mathcal{O}$ in mind. If $\boldsymbol{X}$ is a unitarizable highest weight module for Hermitian Lie algebras $\mathfrak{g}_0 = \mathfrak{sp}(n, \mathbb{R})$, $\mathfrak{su}(p, q)$ or $\mathfrak{o}^*(2n)$ of classical type, the isotropy representation $\mathcal{W}$ can be described by using the oscillator representations of reductive dual pairs (Proposition 5.3). In particular, we find that the assignment $\boldsymbol{X} \mapsto \mathcal{W}^*$ is essentially identical with the dual pair correspondence in the stable range case, and that the $K_{\mathbb{C}}(X)$-module $\mathcal{W}$ is irreducible in this case.

Some related works on the associated cycle $\mathcal{C}(\boldsymbol{X})$ from different approaches can be found in [3] and [4] for the discrete series (through D-modules on the flag variety), also in [11] and [14] for unitarizable highest weight modules (through the study of asymptotic K-types).

The detail of this article with complete proofs will appear elsewhere.

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2. Associated cycle and isotropy representation

As in Section 1, let $\boldsymbol{X}$ be an irreducible $(\mathfrak{g}, K)$-module with irreducible associated variety $\mathcal{V}(\boldsymbol{X}) = \overline{\mathcal{O}}$, where $\mathcal{O}$ is a nilpotent $K_{\mathbb{C}}$-orbit in $\mathfrak{p}$. We take an irreducible K-submodule V_τ of $\boldsymbol{X}$, and a K-stable good filtration of $\boldsymbol{X}$ as follows:

$$\begin{cases} \boldsymbol{X}_0 \subset \boldsymbol{X}_1 \subset \cdots \subset \boldsymbol{X}_n \subset \cdots, \\ \boldsymbol{X}_n := U_n(\mathfrak{g}) V_\tau \ (n = 0, 1, 2, \ldots), \end{cases} \tag{2.1}$$

where $U_0(\mathfrak{g}) := \mathbb{C}1$, and $U_n(\mathfrak{g})$ $(n \geq 1)$ denotes the subspace of $U(\mathfrak{g})$ generated by elements of the form $X_1 \cdots X_k$ $(X_1, \ldots, X_k \in \mathfrak{g},\ 0 \leq k \leq n)$. This filtration gives rise to a graded $(S(\mathfrak{g}), K)$-module

$$M = \mathrm{gr}\, \boldsymbol{X} = \bigoplus_{n=0}^{\infty} M_n \quad \text{with} \quad M_n := \boldsymbol{X}_n / \boldsymbol{X}_{n-1} \ (\boldsymbol{X}_{-1} := \{0\}). \tag{2.2}$$

We note that the action of $\mathfrak{k}$ on M vanishes since the filtration (2.1) is K-stable, and that

$$M_n = S^n(\mathfrak{g}) M_0 = S^n(\mathfrak{g}) V_\tau \simeq S^n(\mathfrak{p}) \otimes V_\tau \quad \text{as } K\text{-modules}, \tag{2.3}$$

where $S^n(\mathfrak{g}) \simeq U_n(\mathfrak{g}) / U_{n-1}(\mathfrak{g})$ is the subspace of $S(\mathfrak{g})$ consisting of all the homogeneous elements of degree n, and $S(\mathfrak{p}) = \oplus_{n \geq 0} S^n(\mathfrak{p})$ is the symmetric algebra of $\mathfrak{p}$ with homogeneous components $S^n(\mathfrak{p})$. By definition, the associated variety $\mathcal{V}(\boldsymbol{X})$ of $\boldsymbol{X}$ is expressed as

$$\mathcal{V}(\boldsymbol{X}) = \{Z \in \mathfrak{g} \mid f(Z) = 0 \text{ for all } f \in \mathrm{Ann}_{S(\mathfrak{g})}M\} \subset \mathfrak{p}, \tag{2.4}$$

where we identify $S(\mathfrak{g})$ with the ring of polynomial functions on $\mathfrak{g}$ through the Killing form B of $\mathfrak{g}$, and $\mathrm{Ann}_{S(\mathfrak{g})}M$ denotes the annihilator ideal in $S(\mathfrak{g})$ of M.

The Hilbert Nullstellensatz tells us that the radical of the ideal $\mathrm{Ann}_{S(\mathfrak{g})}M$ coincides with the prime ideal $I = I(\mathcal{V}(\boldsymbol{X}))$ that defines the irreducible variety $\mathcal{V}(\boldsymbol{X})$:

$$I = \sqrt{\mathrm{Ann}_{S(\mathfrak{g})}M}, \tag{2.5}$$

This implies that $I^{n_0}M = \{0\}$ for some positive integer n_0. Hence we get a finite decreasing filtration of $(S(\mathfrak{g}), K)$-module M as

$$M = I^0M \supset I^1M \supset \cdots \supset I^{n_0}M = \{0\}. \tag{2.6}$$

Now, we take an element X in the open orbit $\mathcal{O} \subset \mathcal{V}(\boldsymbol{X})$. Let $K_{\mathbb{C}}(X) := \{k \in K_{\mathbb{C}} | \ \mathrm{Ad}(k)X = X\}$ be the isotropy subgroup of $K_{\mathbb{C}}$ at X. We write $\mathfrak{m}(X)$ for the maximal ideal of $S(\mathfrak{g})$ which defines the one point variety $\{X\}$ in $\mathfrak{g}$:

$$\mathfrak{m}(X) := \sum_{Y \in \mathfrak{g}} (Y - B(Y, X))S(\mathfrak{g}) \quad \text{for } X \in \mathcal{O}. \tag{2.7}$$

Then we introduce a finite-dimensional $K_{\mathbb{C}}(X)$-module

$$\mathcal{W} := \bigoplus_{j=0}^{n_0-1} \mathcal{W}(j) \quad \text{with} \quad \mathcal{W}(j) := I^jM/\mathfrak{m}(X)I^jM. \tag{2.8}$$

Definition 2.1. We call the resulting representation ϖ of $K_{\mathbb{C}}(X)$ on $\mathcal{W} = \oplus_j \mathcal{W}(j)$ the *isotropy representation* of $K_{\mathbb{C}}(X)$ associated with the data $(\boldsymbol{X}, V_\tau, X)$, where V_τ defines the filtration of $\boldsymbol{X}$ that yields the graded module $M = \mathrm{gr}\,\boldsymbol{X}$.

Remark 2.2. Let $g \in K_{\mathbb{C}}$. Then one has $K_{\mathbb{C}}(\mathrm{Ad}(g)X) = gK_{\mathbb{C}}(X)g^{-1}$, and the representation operator on I^jM defined by the element g gives a linear isomorphism

$$I^jM/\mathfrak{m}(X)I^jM \xrightarrow{\sim} I^jM/\mathfrak{m}(\mathrm{Ad}(g)X)I^jM,$$

which intertwines the action of $k \in K_{\mathbb{C}}(X)$ on $I^jM/\mathfrak{m}(X)I^jM$ with that of $gkg^{-1} \in K_{\mathbb{C}}(\mathrm{Ad}(g)X)$ on $I^jM/\mathfrak{m}(\mathrm{Ad}(g)X)I^jM$. In particular, the dimension of the isotropy representation ϖ is independent of the choice of a point $X \in \mathcal{O}$.

The argument of Vogan in [19, Section 2] allows us to deduce the following proposition.

Proposition 2.3. *The multiplicity* $\mathrm{mult}_I(\boldsymbol{X})$ *in the associated cycle* $\mathcal{C}(\boldsymbol{X})$ *of* $\boldsymbol{X}$ *(cf.* (1.1)) *is equal to the dimension of the isotropy representation* ϖ *associated with* $(\boldsymbol{X}, V_\tau, X)$ $(X \in \mathcal{O})$: $\mathrm{mult}_I(\boldsymbol{X}) = \dim \mathcal{W}$. *Especially, we have*

$$\mathrm{mult}_I(\boldsymbol{X}) \geq \dim \mathcal{W}(0) = \dim M/\mathfrak{m}(X)M > 0. \tag{2.9}$$

The equality

$$\mathrm{mult}_I(\boldsymbol{X}) = \dim M/\mathfrak{m}(X)M \tag{2.10}$$

holds if $\mathrm{Ann}_{S(\mathfrak{g})}M$ *coincides with the whole* I.

The following two theorems say that the above simple formula (2.10) is applicable to some important $(\mathfrak{g}, K)$-modules $\boldsymbol{X}$ related to elliptic orbits.

First, we assume that $\operatorname{rank} G = \operatorname{rank} K$, which is a necessary and sufficient condition for G to admit irreducible unitary representations of discrete series. Let $\mathfrak{t}$ be the complexification of a compact Cartan subalgebra of $\mathfrak{g}_0$. We write Δ for the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$.

Theorem 2.4. *Let $\boldsymbol{X}$ be an irreducible $(\mathfrak{g}, K)$-module of discrete series with Harish-Chandra parameter $\Lambda \in \mathfrak{t}^*$. Take the positive system Δ^+ for which Λ is dominant.*

(1) *Then the associated variety $\mathcal{V}(\boldsymbol{X})$ is described as*

$$\mathcal{V}(\boldsymbol{X}) = \operatorname{Ad}(K_{\mathbb{C}})\mathfrak{p}_-,$$

where $\mathfrak{p}_-$ is the subspace of $\mathfrak{p}$ generated by root vectors corresponding to noncompact negative roots. In particular, $\mathcal{V}(\boldsymbol{X})$ is the closure of the unique nilpotent $K_{\mathbb{C}}$-orbit $\mathcal{O}$ such that $\mathcal{O} \cap \mathfrak{p}_-$ is open in $\mathfrak{p}_-$.

(2) *Let V_τ be the lowest K-type of $\boldsymbol{X}$. Then, the annihilator of the graded $S(\mathfrak{g})$-module $M = \operatorname{gr} \boldsymbol{X}$ defined through V_τ is a radical ideal: $\operatorname{Ann}_{S(\mathfrak{g})} M = I$.*

The assertion (1) of this theorem is well-known. (2) follows from our earlier works [23] and [24] on the associated variety of discrete series representations, where an elementary proof of (1), and a combinatorial description of the open orbit $\mathcal{O} \subset \mathcal{V}(\boldsymbol{X})$ for the case $SU(p,q)$, are also provided. More generally, the same claims appear to be true for the derived functor modules $\mathcal{A}_{\mathfrak{q}}(\lambda)$ with sufficiently regular parameters λ (the irreducibility of the associated variety is well-known).

Second, suppose that G is a simple Lie group of Hermitian type. We denote by $\mathfrak{g} = \mathfrak{p}_+ + \mathfrak{k} + \mathfrak{p}_-$ the $K_{\mathbb{C}}$-stable triangular decomposition of $\mathfrak{g}$, where $\mathfrak{p}_+$ (resp. $\mathfrak{p}_-$) is a nilpotent abelian Lie subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$, which is isomorphic to the holomorphic (resp. anti-holomorphic) tangent space of G/K at the origin. Then, the assertion (2) of the following theorem is due to Joseph [12, Lem.2.4 and Th.5.6].

Theorem 2.5. *Let $\boldsymbol{X}$ be an irreducible $(\mathfrak{g}, K)$-module with highest weight.*

(1) *The associated variety $\mathcal{V}(\boldsymbol{X})$ is the closure of a single $K_{\mathbb{C}}$-orbit $\mathcal{O}$ in $\mathfrak{p}_+$.*

(2) *Let V_τ be the irreducible K-submodule of $\boldsymbol{X}$ generated by its highest weight vector. Define an $(S(\mathfrak{g}), K)$-module $M = \operatorname{gr} \boldsymbol{X}$ through V_τ. If $\boldsymbol{X}$ is unitarizable, the annihilator in $S(\mathfrak{g})$ of any nonzero element of M is equal to the prime ideal $I = I(\mathcal{V}(\boldsymbol{X}))$.*

3. Differential operator of gradient type

Let V be a finite-dimensional K-module. We can define a graded $(S(\mathfrak{g}), K)$-module structure on the tensor product $S(\mathfrak{p}) \otimes V = \oplus_{n \geq 0} S^n(\mathfrak{p}) \otimes V$ by

$$\begin{cases} D' \cdot (D \otimes v) := D'D \otimes v & (D' \in S(\mathfrak{p})), \\ Z \cdot (D \otimes v) := 0 & (Z \in S(\mathfrak{k})), \\ k \cdot (D \otimes v) := \operatorname{Ad}(k)D \otimes kv & (k \in K), \end{cases} \tag{3.1}$$

where $D \otimes v \in S(\mathfrak{p}) \otimes V$ with $D \in S(\mathfrak{p})$ and $v \in V$. We write $\mathcal{P}(\mathfrak{p}, V^*) = \oplus_{n \geq 0} \mathcal{P}^n(\mathfrak{p}, V^*)$ for the algebra of polynomial functions on $\mathfrak{p}$ with values in $V^* = \operatorname{Hom}_{\mathbb{C}}(V_\tau, \mathbb{C})$, where the subspace $\mathcal{P}^n(\mathfrak{p}, V^*)$ consists of homogeneous polynomials of degree n. Then, $\mathcal{P}(\mathfrak{p}, V^*)$ turns to be an $(S(\mathfrak{g}), K)$-module by the actions:

$$\tag{3.2} \begin{cases} (D' \cdot f)(Y) := (\partial(D')f)(Y) & (D' \in S(\mathfrak{p})), \\ (Z \cdot f)(Y) := 0 & (Z \in S(\mathfrak{k})), \\ (k \cdot f)(Y) := k \cdot f(\mathrm{Ad}(k)^{-1}Y) & (k \in K), \end{cases}$$

for $f \in \mathcal{P}(\mathfrak{p}, V^*)$ and $Y \in \mathfrak{p}$. Here $D' \mapsto \partial(D')$ denotes the algebra isomorphism from $S(\mathfrak{p})$ onto the algebra of constant coefficient differential operators on $\mathfrak{p}$, defined by

$$\tag{3.3} \partial(Y')f(Y) := \frac{d}{dt} f(Y + tY')_{|t=0} \quad \text{for} \quad Y' \in \mathfrak{p}.$$

It is standard to verify that the bilinear form

$$\tag{3.4} (S(\mathfrak{p}) \otimes V) \times \mathcal{P}(\mathfrak{p}, V^*) \ni (D \otimes v, f) \longmapsto \langle D \otimes v, f \rangle := ((\partial({}^T D)f)(0), v)_{V^* \times V} \in \mathbb{C}$$

sets up a nondegenerate $(S(\mathfrak{g}), K)$-invariant pairing on $S(\mathfrak{p}) \otimes V \times \mathcal{P}(\mathfrak{p}, V^*)$, where T denotes the principal automorphism of $S(\mathfrak{p})$ such that ${}^T Y = -Y$ for $Y \in \mathfrak{p}$, and $(\,\cdot\,,\,\cdot\,)_{V^* \times V}$ is the dual pairing on $V^* \times V$.

Now, let N be any graded $(S(\mathfrak{g}), K)$-submodule of $S(\mathfrak{p}) \otimes V$. We consider the quotient $(S(\mathfrak{g}), K)$-module

$$M := (S(p) \otimes V)/N.$$

The bilinear form (3.4) naturally gives rise to a nondegenerate invariant pairing of $(S(\mathfrak{g}), K)$-modules

$$\tag{3.5} M \times N^{\perp} \longrightarrow \mathbb{C},$$

where $N^{\perp}$ is the orthogonal of N in $\mathcal{P}(\mathfrak{p}, V^*)$. This implies that the K-finite dual M^* of the quotient $(S(\mathfrak{g}), K)$-module M is isomorphic to the submodule $N^{\perp}$ of $\mathcal{P}(\mathfrak{p}, V^*)$.

We wish to characterize $N^{\perp} \simeq M^*$ as the kernel of a certain differential operator on $\mathfrak{p}$ of gradient type. For this, we first take two bases $(X_1, \dots, X_s)$ and $(X_1^*, \dots, X_s^*)$ of the vector space $\mathfrak{p}$ such that

$$\tag{3.6} B(X_i, X_j^*) = \delta_{ij} \quad \text{with Kronecker's } \delta_{ij}.$$

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_s)$ of nonnegative integers α_i $(1 \leq i \leq s)$, we set

$$X^{\alpha} := X_1^{\alpha_1} \cdots X_s^{\alpha_s}, \quad (X^*)^{\alpha} := (X_1^*)^{\alpha_1} \cdots (X_s^*)^{\alpha_s}.$$

Then, the elements X^{α} (resp. $(X^*)^{\alpha}$) with $|\alpha| = n$ form a basis of $S^n(\mathfrak{p})$ for every integer $n \geq 1$, where $|\alpha| := \alpha_1 + \cdots + \alpha_s$ is the length of α. We now introduce a *gradient map* ∇^n of order n by

$$\tag{3.7} (\nabla^n f)(Y) := \sum_{|\alpha|=n} \frac{1}{\alpha!} (X^*)^{\alpha} \otimes \partial(X^{\alpha}) f(Y) \quad \text{for } f \in \mathcal{P}(\mathfrak{p}, V^*),$$

where $\alpha! = \alpha_1! \cdots \alpha_n!$. It is easy to observe that $\nabla^n f$ is independent of the choice of two bases $(X_i)_{1 \leq i \leq s}$ and $(X_j^*)_{1 \leq j \leq s}$ of $\mathfrak{p}$ with the property (3.6). Furthermore, ∇^n gives an $(S(\mathfrak{g}), K)$-homomorphism

$$\mathcal{P}(\mathfrak{p}, V^*) \ni f \stackrel{\nabla^n}{\longmapsto} \nabla^n f \in \mathcal{P}(\mathfrak{p}, S^n(\mathfrak{p}) \otimes V^*).$$

Second, we note that our submodule N is finitely generated over $S(\mathfrak{g})$, since the ring $S(\mathfrak{g})$ is Noetherian and since $S(\mathfrak{p}) \otimes V = S(\mathfrak{p}) \cdot V$. Hence, there exist a finite number of homogeneous K-submodules $W_u \subset N$ $(u = 1, \ldots, q)$ which generate N over $S(\mathfrak{g})$:

$$N = S(\mathfrak{g}) \cdot W_1 + \cdots + S(\mathfrak{g}) \cdot W_q \quad \text{with} \quad W_u \subset S^{i_u}(\mathfrak{p}) \otimes V \tag{3.8}$$

for some integers $i_u \geq 0$ arranged as $i_1 < \cdots < i_q$. For each $u = 1, \ldots, q$, let P_u denote the K-homomorphism from $\mathcal{P}^{i_u}(\mathfrak{p}, V^*)$ to W_u^* defined by

$$P_u(h)(w) := \langle w, h \rangle \quad (w \in W_u) \tag{3.9}$$

for $h \in \mathcal{P}^{i_u}(\mathfrak{p}, V^*)$.

We now introduce an $(S(\mathfrak{g}), K)$-homomorphism

$$\mathcal{D} : \mathcal{P}(\mathfrak{p}, V^*) \to \mathcal{P}(\mathfrak{p}, W^*) \quad \text{with} \quad W^* := \oplus_{u=1}^q W_u^*, \tag{3.10}$$

by putting

$$(\mathcal{D}f)(Y) := \sum_{u=1}^{q} P_u(\nabla^{i_u} f(Y)) \quad (Y \in \mathfrak{p}; f \in \mathcal{P}(\mathfrak{p}, V^*)), \tag{3.11}$$

where $\nabla^{i_u} f(Y) \in S^{i_u}(\mathfrak{p}) \otimes V^*$ is identified with a polynomial in $\mathcal{P}^{i_u}(\mathfrak{p}, V^*)$ through the Killing form of $\mathfrak{g}$.

Definition 3.1. We call $\mathcal{D}$ the *differential operator of gradient type* associated with (V^*, W^*).

The space of solutions of the differential equation $\mathcal{D}f = 0$ is characterized as follows.

Proposition 3.2. *One gets $N^\perp = \operatorname{Ker} \mathcal{D}$. Hence, the kernel of the differential operator $\mathcal{D}$ is isomorphic to the K-finite dual M^* of $M = (S(\mathfrak{p}) \otimes V)/N$, as $(S(\mathfrak{g}), K)$-modules.*

Let us define a map σ from $\mathfrak{p} \times V^*$ to W^* by

$$\sigma(X, v^*) := \sum_{u=1}^{q} P_u(X^{i_u} \otimes v^*) \quad \text{for} \quad (X, v^*) \in \mathfrak{p} \times V^*, \tag{3.12}$$

which we call the *symbol map* of $\mathcal{D}$.

For any fixed $X \in \mathfrak{p}$, it is not hard to prove

Proposition 3.3. *The natural map*

$$V \hookrightarrow S(\mathfrak{p}) \otimes V \to M = (S(\mathfrak{p}) \otimes V)/N \to M/\mathfrak{m}(X)M$$

from V to $M/\mathfrak{m}(X)M$ induces a $K_{\mathbb{C}}(X)$-isomorphism

$$(M/\mathfrak{m}(X)M)^* \simeq \operatorname{Ker} \sigma(X, \cdot\,) \tag{3.13}$$

by passing to the dual. Here $\mathfrak{m}(X)$ is the maximal ideal of $S(\mathfrak{g})$ as in (2.7), and $K_{\mathbb{C}}(X)$ is the isotropy subgroup of $K_{\mathbb{C}}$ at X.

4. Application to Harish-Chandra modules

As in Section 2, let $\boldsymbol{X}$ be an irreducible $(\mathfrak{g}, K)$-module with associated variety $\mathcal{V}(\boldsymbol{X}) = \overline{\mathcal{O}}$, and let $M = \operatorname{gr} \boldsymbol{X}$ be the graded $(S(\mathfrak{g}), K)$-module defined by the filtration of $\boldsymbol{X}$ arising from an irreducible K-submodule $V_\tau \subset \boldsymbol{X}$. Since $M = S(\mathfrak{p})V_\tau$, there exists a unique surjective $(S(\mathfrak{g}), K)$-homomorphism

$$\pi : S(\mathfrak{p}) \otimes V_\tau \to M$$

such that π restricted to V_τ is the identity operator. Setting $N := \operatorname{Ker} \pi$, we have

$$M \simeq (S(\mathfrak{p}) \otimes V_\tau)/N \quad \text{as } (S(\mathfrak{g}), K)\text{-modules.}$$

By virtue of Proposition 2.3, we can now apply our observation in Section 3 to get the following characterization of the associated cycle $\mathcal{C}(\boldsymbol{X}) = \operatorname{mult}_I(\boldsymbol{X}) \cdot [\overline{\mathcal{O}}]$ of $\boldsymbol{X}$.

Theorem 4.1. *Under the above notation, let $\boldsymbol{\sigma}$ be the symbol map of the differential operator $\mathcal{D}$ on $\mathfrak{p}$ of gradient type whose kernel equals $N^\perp \simeq M^*$ (cf. Proposition 3.2). Then one has,*

(1) $\mathcal{V}(\boldsymbol{X}) = \{X \in \mathfrak{p} |\ \operatorname{Ker} \boldsymbol{\sigma}(X, \cdot) \neq \{0\}\}$.

(2) *If X lies in the open orbit $\mathcal{O}$ in $\mathcal{V}(\boldsymbol{X})$, the $K_\mathbb{C}(X)$-module* $\operatorname{Ker} \boldsymbol{\sigma}(X, \cdot)$ *is isomorphic to the submodule $\mathcal{W}(0)^*$ of the dual $\mathcal{W}^* = \oplus_j \mathcal{W}(j)^*$ of the isotropy representation $\mathcal{W}$ associated with $(\boldsymbol{X}, V_\tau, X)$. Moreover, we have the isomorphisms*

$$\operatorname{Ker} \boldsymbol{\sigma}(X, \cdot) \simeq \mathcal{W}(0) \simeq \mathcal{W} \tag{4.1}$$

of $K_\mathbb{C}(X)$-modules if $\operatorname{Ann}_{S(\mathfrak{g})} M$ *coincides with the prime ideal $I = I(\mathcal{V}(\boldsymbol{X}))$.*

(3) *One gets* $\operatorname{mult}_I(\boldsymbol{X}) \geq \dim \operatorname{Ker} \boldsymbol{\sigma}(X, \cdot) > 0$ *for any $X \in \mathcal{O}$. The equality*

$$\operatorname{mult}_I(\boldsymbol{X}) = \dim \operatorname{Ker} \boldsymbol{\sigma}(X, \cdot) \tag{4.2}$$

holds if $I = \operatorname{Ann}_{S(\mathfrak{g})} M$.

This together with Theorems 2.4 and 2.5 immediately implies the following

Theorem 4.2. *The isomorphisms* (4.1) *and the equality* (4.2) *hold if $\boldsymbol{X}$ is either a discrete series $(\mathfrak{g}, K)$-module with sufficiently regular Harish-Chandra parameter, or a unitarizable highest weight module, where $M = \operatorname{gr} \boldsymbol{X}$ is defined through the unique minimal K-type V_τ of $\boldsymbol{X}$.*

For two types of $(\mathfrak{g}, K)$-modules $\boldsymbol{X}$ in Theorem 4.2, the K-submodules W_u of $S(\mathfrak{p}) \otimes V_\tau$ which generate N over $S(\mathfrak{g})$, are intrinsically understood. Moreover, the corresponding differential operator $\mathcal{D}$ on the Lie algebra $\mathfrak{p}$ naturally extend to a G-invariant differential operator $\tilde{\mathcal{D}}$ defined on the vector bundle $G \times_K V_\tau^*$ over G/K, and the full kernel space of $\tilde{\mathcal{D}}$ gives the maximal globalization of the dual Harish-Chandra module $\boldsymbol{X}^*$.

5. Irreducibility of isotropy representation

We now discuss the irreducibility of the isotropy $K_\mathbb{C}(X)$-representation $\mathcal{W}$, bearing the Harish-Chandra modules $\boldsymbol{X}$ with small nilpotent orbits $\mathcal{O}$ in mind. At the moment, our result is the following criterion.

Theorem 5.1. *Let $\boldsymbol{X}$ be an irreducible $(\mathfrak{g}, K)$-module with associated variety $\mathcal{V}(\boldsymbol{X}) = \overline{\mathcal{O}}$, and let $M = \operatorname{gr} \boldsymbol{X}$ be the $(S(\mathfrak{g}), K)$-module defined through an irreducible K-submodule $V_\tau \subset \boldsymbol{X}$. Suppose that (i) the codimension of the boundary $\partial\mathcal{O} := \overline{\mathcal{O}} \backslash \mathcal{O}$ in $\overline{\mathcal{O}}$ is at least two, and that (ii) the annihilator $\operatorname{Ann}_{S(\mathfrak{g})}(m)$ in $S(\mathfrak{g})$ of any nonzero vector $m \in M$ equals $I = I(\mathcal{V}(\boldsymbol{X}))$. Then, the following two conditions (a) and (b) are equivalent with each other.*

(a) The isotropy representation $\mathcal{W}$ of $K_{\mathbb{C}}(X)$ associated with $(\boldsymbol{X}, V_\tau, X)$, $X \in \mathcal{O}$, is irreducible.

(b) For any nonzero $(S(\mathfrak{g}), K)$-submodule L of M, the Gelfand-Kirillov dimension of M/L is smaller than $\dim \mathcal{O}$, or equivalently, the annihilator of M/L in $S(\mathfrak{g})$ is strictly bigger than I.

Example 5.2. Let G be a connected simple Lie group of Hermitian type, and let $\mathfrak{g} = \mathfrak{p}_+ + \mathfrak{k} + \mathfrak{p}_-$ be the triangular decomposition of $\mathfrak{g}$ as in Theorem 2.5. Every unitarizable highest weight $(\mathfrak{g}, K)$-module satisfies the assumption (ii) of the above theorem by virtue of Theorem 2.5. The condition (i) is also fulfilled except for the case that G/K is of tube type and $\mathcal{V}(\boldsymbol{X}) = \mathfrak{p}_+$ (see [25, Section 3.1]).

We end this article by illustrating a simple but interesting example of the description of the isotropy representation $\mathcal{W}$. Let G be one of the classical groups $SU(p,q)$, $Sp(n,\mathbb{R})$, or $SO^*(2n)$ of Hermitian type. For each positive integer k, let us consider a reductive dual pair (G, G'_k) in $Sp(N,\mathbb{R})$ for some N, with $G'_k = U(k)$, $O(k)$, or $Sp(k)$ respectively. Then, the oscillator representation ω_k of the pair (G, G'_k) decomposes into irreducibles as

$$\omega_k \simeq \bigoplus_{\sigma \in \Xi_k} \boldsymbol{X}(\sigma) \hat{\otimes} U_\sigma \quad \text{as } (\mathfrak{g}, K) \times G'_k\text{-modules}, \tag{5.1}$$

where Ξ_k denotes a set of equivalence classes of irreducible (finite-dimensional) unitary representations (σ, U_σ) of the compact group G'_k. Note that we must go up to the metaplectic double cover for the case $G = Sp(n,\mathbb{R})$ with odd k. It is well-known that each $\boldsymbol{X}(\sigma)$ is a unitarizable highest weight $(\mathfrak{g}, K)$-module, and that $\sigma \mapsto \boldsymbol{X}(\sigma)$ sets up a one-to-one correspondence from Ξ_k onto a set of equivalence classes of irreducible unitary representations of G with highest weights. Moreover, $\boldsymbol{X}(\sigma)$'s exhaust all the unitarizable highest weight modules for $G = Sp(n,\mathbb{R})$ and $SU(p,q)$. (See for example [6], [10].)

In this setting, we can describe the isotropy representation $\mathcal{W}_\sigma$ associated with $(\boldsymbol{X}(\sigma), V_\tau, X)$, where V_τ is the extreme K-type of $\boldsymbol{X}(\sigma)$. The idea is as follows. By using a realization of ω_k on the space $\mathbb{C}[M_k]$ of polynomial functions on a vector space M_k, we first specify the quotient $\mathbb{C}[M_k]/\mathfrak{m}(X)\mathbb{C}[M_k]$ by some algebraic and geometric methods, and then decompose it into irreducibles as G'_k-modules in order to identify the isotropy module $\mathcal{W}_\sigma$ over $K_{\mathbb{C}}(X)$ for every $\sigma \in \Xi_k$. To state the result, let r be the real rank of G, and we set $m_k := \min(k, r)$ for each integer $k > 0$. The direct product group $G'_{m_k} \times G'_{k-m_k}$ embeds into G'_k diagonally, where G'_{k-m_k} should be understood as $\{e\}$ (the identity group) if $k \leq r$.

Proposition 5.3. *Assume that G/K is of tube type, or the pair (G, G'_k) is in the stable range with smaller G'_k : $k \leq r$. Then, by an appropriate choice of $X \in \mathcal{O}$, there exists a* *surjective group homomorphism β from $K_{\mathbb{C}}(X)$ to $G'_{m_k} \times \{e\}$ such that*

$$\mathcal{W}_\sigma \simeq (\delta_k \cdot (\sigma^* \circ \beta),\ (U_\sigma^*)^{G'_{k-m_k}}) \quad \textit{as } K_{\mathbb{C}}(X)\textit{-modules,} \tag{5.2}$$

for every $\sigma \in \Xi_k$, where δ_k is a one-dimensional character of $K_{\mathbb{C}}(X)$, and $(U_\sigma^*)^{G'_{k-m_k}}$ is the subspace of all G'_{k-m_k}-fixed vectors in U_σ^* viewed as a $K_{\mathbb{C}}(X)$-module through $\delta_k \cdot (\sigma^* \circ \beta)$. In particular, if $r \leq k$, the isotropy representation $\mathcal{W}_\sigma$ ($\sigma \in \Xi_k$), equivalent to $(\delta_k \cdot (\sigma^* \circ \beta),\ U_\sigma^*)$, is irreducible, and so the $(S(\mathfrak{g}), K)$-module $M = \operatorname{gr} \boldsymbol{X}(\sigma)$ satisfies the property (b) in Theorem 5.1.

We refer to [25, Section 5] for more detailed account of this proposition, where the remaining case that G/K is not tube type and $k > r$ is also studied.

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