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Algebraic structures of superconformal algebras

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1 Introduction

Let V be a vector space over $\mathbf{C}$ and let q be a nondegenerate quadratic form on V. Let $\mathrm{Cl}(V,q)$ denote the associated Clifford algebra. The exterior algebra $\bigwedge(V)$ is isomorphic to $\mathrm{Cl}(V,q)$ as vector spaces by the map $\bar{f}$ naturally determined by $f_k : V^{\otimes k} \to \mathrm{Cl}(V,q)$,

$$f_k(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^{\sigma} v_{\sigma(1)} v_{\sigma(2)} \cdots v_{\sigma(k)}. \tag{1.1}$$

Suppose $\dim V = 2$ and set $\Delta(x) = 2 - \frac{\deg x}{2}$ for $x \in \bigwedge(V)$. Let π_r denote the projection of $\bigwedge(V)$ to the subspace $\{x \in \bigwedge(V) | \Delta(x) = r\}$. Define

$$x_{\langle j\rangle'} y = \pi_{\Delta(x)+\Delta(y)-j-1}(\bar{f}^{-1}(\bar{f}(x)\bar{f}(y))), \tag{1.2}$$

for $x, y \in \bigwedge(V)$ and $j = 0, 1$. Then the space $\bigwedge(V) \otimes \mathbf{C}[t, t^{-1}]$ is given a simple Lie superalgebra structure by

$$[x \otimes t^m, y \otimes t^n] = (x_{\langle 0\rangle'} y) \otimes t^{m+n} - ((\Delta(x) - 1)n - (\Delta(y) - 1)m)(x_{\langle 1\rangle'} y) \otimes t^{m+n-1}, \tag{1.3}$$

which is isomorphic to the well-known $N = 2$ superconformal algebra, where the Virasoro subalgebra is given by $L_{m-1} = 1 \otimes t^m$. Thus, the triple $(\bigwedge(V), \langle 0\rangle', \langle 1\rangle')$ determines the $N = 2$ superconformal algebra.

In this article we formulate the "superalgebras" that determine superconformal algebras in the same way to the one described above. It is given as a new axiomatic description of Operator Product Expansion. As an application we classify infinite dimensional simple Lie superalgebras with physical OPE as conformal superalgebras in the sense of V.G.Kac. The detailed argument is described in [9].

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2 Conformal superalgebra

Let $\mathcal{G}$ be an infinite dimensional Lie superalgebra satisfying the following conditions.

(1) There exists a set of formal distributions $\mathcal{F} \subset \mathcal{G}[[z, z^{-1}]]$ such that $\mathcal{G}$ is spanned by the coefficients of the elements of $\mathcal{F}$.

(2) The Lie bracket of $\mathcal{G}$ is written by OPE, that is, for any $a, b \in \mathbf{C}[\partial]\mathcal{F}$, we have

$$[a(z), b(w)] = \sum_j \left(a_{(j)}b\right)(w)\frac{\partial_z{}^j}{j!}\delta(z-w), \tag{2.1}$$

$$\left(a_{(j)}b\right)(w) = \mathrm{Res}_z[a(z), b(w)](z-w)^j, \tag{2.2}$$

where the sum is finite.

(3) For some $L(z) \in \mathcal{F}$, the coefficients of $L(z)$ span a Virasoro subalgebra of $\mathcal{G}$.

The product defined by $a_{(j)}b$ for the pair (a, b) is called the *residue product.* The superconformal algebras, for example the Virasoro algebra, the Neveu-Schwarz algebra, and $N = 2, 3, 4$ superconformal algebras satisfy these conditions.

A conformal superalgebra (or a vertex Lie superalgebra in [7]) is an axiomatic description of Operator Product Expansion. Let us state the axioms for conformal superalgebras, following [4]. We denote $A^{(j)} = A^j/j!$, where A is an operator.

Definition 2.1 Let R be a $\mathbf{C}$-vector space $\mathbf{Z}/2\mathbf{Z}$-graded by a parity p equipped with countably many products

$$(n) : R \otimes R \to R, \quad (n \in \mathbf{N}),$$

and a linear map $\partial : R \to R$. The triple $(R, \{(n)\}_{n\in\mathbf{N}}, L)$ satisfying the following conditions for an even vector $L \in R$ is called a *conformal superalgebra:*

(C) For all $a, b, c \in R$,

(C0) there exists some $N \in \mathbf{N}$ such that for all $n \in \mathbf{N}$ satisfying $n \geq N$

$$a_{(n)}b = 0,$$

(C1) for all $n \in \mathbf{N}$,

$$(\partial a)_{(n)}b = -na_{(n-1)}b,$$

(C2) for all $n \in \mathbf{N}$,

$$a_{(n)}b = (-1)^{p(a)p(b)} \sum_{j=0}^{\infty} (-1)^{j+n+1} \partial^{(j)} b_{(n+j)}a,$$

(C3) for all $m, n \in \mathbf{N}$,

$$a_{(m)}(b_{(n)}c) = \sum_{j=0}^{\infty} \binom{m}{j} \left(a_{(j)}b\right)_{(n+m-j)} c + (-1)^{p(a)p(b)} b_{(n)}(a_{(m)}c).$$

(V) $L \in R$ satisfies $L_{(0)}L = \partial L$, $L_{(1)}L = 2L$, $L_{(2)}L = 0$, $L_{(0)} = \partial$ as operators on R, and $L_{(1)}$ is diagonalizable.

L is called the *conformal vector* of R. A homomorphism of conformal superalgebras from R to R' is a $K[\partial]$-module homomorphism $f : R \to R'$ compatible with the (n) products for all $n \in \mathbf{N}$ and maps L to the conformal vector of R'. An ideal of a conformal superalgebra is a $K[\partial]$-submodule that is closed under the left multiplication with respect to the (n) products for all $n \in \mathbf{N}$. A conformal superalgebra R with no ideals other than $\{0\}$ and R itself is called a *simple* conformal superalgebra. The ideal $\{c \in R | x_{(n)}c = 0, x \in R, n \in \mathbf{N}\}$ is called the *center* of R. When the center is $\{0\}$, the conformal superalgebra is said to be *centerless.*

The eigenvalue of $L_{(1)}$ is denoted by $\Delta(x)$ for an eigenvector x and is called the *conformal weight of* x. Define $R^k = \{x \in R | L_{(1)}x = kx\}$, $\Delta_R = \{k \in K | R^k \neq \{0\}\}$ and $\Delta'_R = \Delta_R \setminus \{0\}$.

A conformal superalgebra R is called a *superconformal algebra* if there exists a finite dimensional subspace $\mathcal{F}$ such that $R = \mathbf{C}[\partial]\mathcal{F}$, all conformal weights are non-negative half-integers, the even subspace $R_{\text{even}} = \bigoplus_{n\in\mathbf{N}} R^n$ and the odd subspace $R_{\text{odd}} = \bigoplus_{n\in\mathbf{N}+\frac{1}{2}} R^n$. We call a superconformal algebra R a *physical conformal superalgebra* if $\mathcal{F} \subset R^2 \oplus R^{\frac{3}{2}} \oplus R^1 \oplus R^{\frac{1}{2}}$ and $\mathcal{F} \cap R^2 = \mathbf{C}L$, following the terminology in [5].

3 New formulation

For a conformal superalgebra $(R, \{(n)\}_{n\in\mathbf{N}}, L)$, we shall call the subspace $\{x \in R | L_{(2)}x \in R^0\}$ the *reduced subspace* of R and denote it by $\check{R}$. Denote $\check{R}^k = \check{R} \cap R^k$, $\Delta_{\check{R}} = \{k \in K | \check{R}^k \neq \{0\}\}$ and $\Delta'_{\check{R}} = \Delta_{\check{R}} \setminus \{0\}$. Obviously we have $\check{R}^0 = R^0$.

We say a conformal superalgebra is *regular* if R^0 is the center and $R^{-\frac{n}{2}} = \{0\}$ for all $n = 1, 2, \cdots$ and if for each $k \in \Delta_R$ there exists some $M \in \mathbf{N}$ such that $k - m \notin \Delta_R$ for all $m \in \mathbf{N}$ satisfying $m \geq M$. For a regular conformal algebra R we have the following proposition by decomposing R into irreducible components as an sl_2-module by the actions of $L_{(0)}$, $L_{(1)}$, $L_{(2)}$.

Proposition 3.1 *Let $(R, \{(n)\}_{n\in\mathbf{N}}, L)$ be a regular conformal superalgebra and $\check{R}$ the reduced subspace of $(R, \{(n)\}_{n\in\mathbf{N}}, L)$. Then there exists a unique decomposition*

$$x = \sum_{j=0}^{m} \partial^{(j)} x^j \tag{3.1}$$

for any $x \in R$ for some $m \in \mathbf{N}$ where $x^0 \in \check{R}$ and $x^j \in \bigoplus_{k\in\Delta'_{\check{R}}} \check{R}^k$ for $j > 0$.

Now we define the products $\langle n\rangle$ on $\check{R}$ by

$$\begin{array}{rcl}\langle n\rangle : \check{R}\times\check{R} & \to & \check{R}\\ (a,b) & \mapsto & a_{\langle n\rangle}b = (a_{(n)}b)^0,\end{array}$$

for each $n\in\mathbf{N}$, where we have identified $\check{R}$ with $R/\partial R$ by Proposition 3.1.

Consider the following properties of a triple $(P,\{\langle n\rangle\}_{n\in\mathbf{N}},L)$ where P is a vector space $\mathbf{Z}/2\mathbf{Z}$-graded by a parity p equipped with countably many products $\{\langle n\rangle\}_{n\in\mathbf{N}}$ on V where $L\in P$:

(P0) For $a,b\in P$ there exists some $N\in\mathbf{N}$ such that for all $n\in\mathbf{N}$ satisfying $n>N$,

$$a_{\langle n\rangle}b = 0.$$

(P2) For $a,b\in P$ and $n\in\mathbf{N}$,

$$a_{\langle n\rangle}b = -(-1)^{n+p(a)p(b)}b_{\langle n\rangle}a.$$

(P3) For $a,b,c\in P$ and $n,m\in\mathbf{N}$,

$$\begin{aligned}&\sum_{j=0}^{m}\binom{m}{j}G(\Delta(b),\Delta(c),n,j)a_{\langle m-j\rangle}b_{\langle n+j\rangle}c\\ &\qquad -(-1)^{p(a)p(b)}\sum_{j=0}^{n}\binom{n}{j}G(\Delta(a),\Delta(c),m,j)b_{\langle n-j\rangle}a_{\langle m+j\rangle}c\\ &= \sum_{j=0}^{m+n}F(\Delta(a),\Delta(b),m,n,j)\left(a_{\langle j\rangle}b\right)_{\langle m+n-j\rangle}c,\end{aligned}$$

where

$$G(\Delta(a),\Delta(b),n,j) = \begin{cases}\frac{(2\Delta(a)-n-j-1;j)}{(2(\Delta(a)+\Delta(b)-n-j-1);j)} = \prod_{k=0}^{j-1}\frac{(2\Delta(a)-n-j-1+k)}{(2(\Delta(a)+\Delta(b)-n-j-1)+k)} & \text{for } \Delta(a)+\Delta(b)-n-j-1\notin -\frac{1}{2}\mathbf{N},\\ 1, & \text{for } \Delta(a)+\Delta(b)-n-1=0, j=0,\\ 0, & \text{otherwise,}\end{cases}$$

$$\begin{aligned}&F(\Delta(a),\Delta(b),m,n,t)\\ &= \sum_{k=0}^{t}\binom{m}{t-k}\binom{m+n+k-t}{k}(-1)^k G(\Delta(a),\Delta(b),t-k,k),\end{aligned}$$

and $(r;j) = r(r+1)(r+2)\cdots(r+j-1)$.

(PV) L is even and satisfies $L_{\langle 0\rangle}a=0$, $L_{\langle 1\rangle}L=2L$, $L_{\langle 2\rangle}a\in P^0$ for all $a\in P$. The operator $L_{\langle 1\rangle}$ is diagonalizable. P^0 is central, $\Delta_P\cap(-\frac{1}{2}\mathbf{N})\subset\{0\}$, and for all $k\in\Delta_P$ there exists some $M\in\mathbf{N}$ such that $k-m\notin\Delta_P$ for all $m\in\mathbf{N}$ satisfying $m\geq M$, where $P^k=\{a\in P|\ L_{\langle 1\rangle}a=ka\}$ and $\Delta_P=\{k\in K|\ P^k\neq\{0\}\}$.

The triple $(\check{R}, \{\langle n\rangle\}_{n\in\mathbf{N}}, L)$ satisfies (P0), (P2), (P3), and (PV). Conversely we have the following theorem.

Theorem 3.2 *For a triple $(P, \{\langle n\rangle\}_{n\in\mathbf{N}}, L)$ satisfying* (P0), (P2), (P3) *and* (PV), *there exists a regular conformal superalgebra $(R_P, \{(n)\}_{n\in\mathbf{N}}, L)$ whose reduced subspace is P and the products satisfies $(a_{(n)}b)^0 = a_{\langle n\rangle}b$ for all $a, b \in P$, $n \in \mathbf{N}$. Furthermore the conformal superalgebra is unique up to isomorphisms.*

Hence the properties (P0), (P2), (P3) and (PV) give another formulation of regular conformal superalgebras. For such a triple $(P, \{\langle n\rangle\}_{n\in\mathbf{N}}, L)$ the Lie superalgebra associated to the conformal superalgebra R_P is nothing but the space $P \otimes \mathbf{C}[t, t^{-1}]$ with the Lie bracket

$$[a\otimes t^m, b\otimes t^n] = \sum_{j=0}^{\infty} F(\Delta(a), \Delta(b), m, n, j)\left(a_{\langle j\rangle}b\right)\otimes t^{m+n-j}. \tag{3.2}$$

We denote it by $\mathrm{Lie}(P, \{\langle n\rangle\}_{n\in\mathbf{N}}, L)$. A Lie superalgebra has regular OPE only when it is isomorphic to some $\mathrm{Lie}(P, \{\langle n\rangle\}_{n\in\mathbf{N}}, L)$ or to the quotient by an ideal where $(P, \{\langle n\rangle\}_{n\in\mathbf{N}}, L)$ is a triple satisfying (P0), (P2), (P3) and (PV).

4 Physical conformal superalgebra

As an application we classify the simple physical conformal superalgebras. Let R be a physical conformal superalgebra. A regular conformal superalgebra R is physical if and only if the reduced subspace $\check{R}$ satisfies the following.

- Eigenvalues of $L_{(1)}$ on $\check{R}$ are 2, $\frac{3}{2}$, 1 and $\frac{1}{2}$.
- $\check{R}^2 = \mathbf{C}L$.
- $\check{R}^{3/2}$ and $\check{R}^{1/2}$ are odd subspaces.
- $\check{R}^1$ and $\check{R}^2$ are even subspaces.

We denote the homogeneous subspaces $\check{R}^{\frac{3}{2}}$, $\check{R}^1$ and $\check{R}^{\frac{1}{2}}$ by V, A and F respectively, following the notations in [5].

Consider the products on $\check{R}$ defined by

$$a\circ b = \begin{cases} \frac{a_{\langle 1\rangle}b}{\Delta(a)+\Delta(b)-2}, & \text{for } \Delta(a)+\Delta(b)-2\neq 0,\\ 0, & \text{otherwise,}\end{cases}$$

and

$$a\bullet b = a_{\langle 0\rangle}b.$$

Then, the formulation in Section 3 is equivalent to the conditions below for a system $(L, V, A, F, \bullet, \circ)$:

(H0) $L\circ = id$, $L\bullet = 0$ as operators on $\check{R}$,

(H1) $(\check{R}, \bullet)$ is a Lie superalgebra,

(H2) $(\check{R}, \circ)$ is an associative commutative superalgebra,

(H3) $A\bullet$ gives derivations with respect to $\circ$,

(H4) $(u\circ + u\bullet)^2 = (u\bullet u)\circ$ as operators on $\check{R}$ for $u \in V$,

where $\check{R} = \mathbf{C}L \oplus V \oplus A \oplus F$, under the assumption that $\mathbf{C}L \oplus A$ is the even subspace of $\check{R}$ and $V \oplus F$ is the odd subspace, and that $\Delta(L) = 2$, $\Delta(V) = \frac{3}{2}$, $\Delta(A) = 1$, $\Delta(F) = \frac{1}{2}$, $\Delta(a \bullet b) = \Delta(a) + \Delta(b) - 1$, and $\Delta(a \circ b) = \Delta(a) + \Delta(b) - 2$.

Let q be the quadratic form on V determined by $u \bullet u = q(u)L$.

Proposition 4.1 *Let R be a physical conformal superalgebra with $V \neq \{0\}$. Then, R is simple if and only if q is nondegenerate and $F^3 = 0$, where $F^3 = \{f \in F | v^1{}_{(0)} v^2{}_{(0)} v^3{}_{(0)} f = 0$ for all $v^k \in V\}$.*

Proposition 4.2 *Let R be a simple physical conformal superalgebra. Then the map*

$$\iota : \mathrm{Cl}(V, q) \to \check{R},$$

$$v_1 v_2 \cdots v_r \mapsto (v_1\circ + v_1\bullet)(v_2\circ + v_2\bullet) \cdots (v_r\circ + v_r\bullet)L,$$

is surjective unless $V \circ V \circ V = 0$ with $V \neq \{0\}$.

Let R_ι denote the conformal sub-superalgebra generated by $\mathrm{Im}\iota$. To classify simple physical conformal superalgebras, we will first list up all possible physical conformal superalgebra structures that is simple or with $V \circ V \circ V = \{0\}$ on a quotient space of $\mathrm{Cl}(V, q)$ by an left ideal, where V is an arbitrary finite dimensional vector space and q is a nondegenerate quadratic form on V.

Fix a finite dimensional vector space V with a nondegenerate quadratic form q and consider an orthonormal basis $\{e_1, e_2, \cdots, e_N\}$ of (V, q). Set $D_k^0 = D_k = \frac{1}{\sqrt{2}}(e_{2k-1} + \sqrt{-1}e_{2k})$, $D_k^1 = \bar{D}_k = D_{\bar{k}} = \frac{1}{\sqrt{2}}(e_{2k-1} - \sqrt{-1}e_{2k})$, and $D^w = D_1^{w_1} D_2^{w_2} \cdots D_n^{w_n}$ where $n = \lfloor N/2 \rfloor$, $w \in (\mathbf{Z}/2\mathbf{Z})^n$ and w_i denotes the ith binary digit of w.

Theorem 4.3 ([2]) *The left $\mathrm{Cl}(V, q)$-module $\mathrm{Cl}(V, q)$ is completely reducible. The irreducible decomposition is given as follows. If $N = 2n$ then*

$$\mathrm{Cl}(V, q) = \bigoplus_{w \in (\mathbf{Z}/2\mathbf{Z})^n} M(w), \tag{4.1}$$

where $M(w) = \mathrm{Cl}(V, q)D^w$. If $N = 2n + 1$ then

$$\mathrm{Cl}(V, q) = \bigoplus_{w \in (\mathbf{Z}/2\mathbf{Z})^n} (M^+(w) \oplus M^-(w)), \tag{4.2}$$

where $M^\pm(w) = \mathrm{Cl}(V, q)D^w(1 \pm e_N)$.

It is obvious that $\check{R}_\iota = \iota(\mathrm{Cl}^4(V,q))$ where $\mathrm{Cl}^n(V,(\cdot,\cdot)) = \mathrm{Span}\{v_1v_2\cdots v_k|v_i \in V,\ k \leq n\}$, hence by Theorem 4.3 we have $\dim V \leq 8$. Hence at most finitely many vector spaces exist as the candidates for R_ι, and in fact the list of such conformal superalgebras is

$$Vir, K_1, K_2, K_3, S_2, N_4^\alpha\ (\alpha \in \mathbf{C}/\pm 1,\ \alpha^2 \neq 1), CK_6. \tag{4.3}$$

(See below.) All conformal superalgebras in the list are simple, and only S_2 is with $V \circ V \circ V = \{0\}$. By Proposition 4.1 we have $\dim F \leq \binom{\dim V}{3}$, hence at most finitely many vector spaces exist as the candidates for a simple physical conformal superalgebra R that have the subalgebra R_ι isomorphic to S_2. In fact the list of such conformal superalgebras is

$$S_2, W_2, N_4. \tag{4.4}$$

Hence the complete list of simple physical conformal superalgebras is

$$Vir, K_1, K_2, K_3, S_2, W_2, N_4, N_4^\alpha\ (\alpha \in \mathbf{C}/\pm 1,\ \alpha^2 \neq 1), CK_6. \tag{4.5}$$

However, $\mathrm{Lie}(N_4)$ and $\mathrm{Lie}(N_4^\alpha)$ are isomorphic for each $\alpha \in \mathbf{C}/\pm 1$, $\alpha^2 \neq 1$, and for any other pair of the simple physical conformal superalgebras the associated Lie superalgebras are not isomorphic to each other. All the Lie superalgebras associated to the list are simple except N_4; the Lie superalgebra $\mathrm{Lie}(N_4)$ has one dimensional center, which is a cocycle of the simple Lie superalgebra $\mathrm{Lie}(K_4)' = [\mathrm{Lie}(K_4), \mathrm{Lie}(K_4)]$.

Thus the complete list of the simple Lie superalgebras with physical OPE is

$$Vir, K_1, K_2, K_3, S_2, W_2, K_4', CK_6, \tag{4.6}$$

where we omitted $\mathrm{Lie}(\cdot)$. Here, Vir is the Virasoro algebra, K_j is the $N = j$ superconformal algebra for each $j = 1, 2, 3$, S_2 is the (small) $N = 4$ superconformal algebra, W_2 is a superconformal algebra with 4 supercharges, and CK_6 is the $N = 6$ superconformal algebra discovered by Cheng-Kac in [1]. N_4 and N_4^αs are simple physical conformal superalgebras with 4 supercharges described in [9]. $\mathrm{Lie}(N_4^0)$, a central extension of $\mathrm{Lie}(K_4)'$, is also known as the large $N = 4$ superconformal algebra in [6] and [8].

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