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Collision of two-dimensional elastic disks: Microscopic mechanism of emergence of restitution coefficient

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1 Introduction

The collision of particles with the internal degrees of freedom are inelastic in general. The inelastic collisions are abundant in nature[1]. Examples can be seen in collisions of atoms, molecules, elastic materials, balls in sports, and so on. In particular, recent extensive interest in granular materials[2] makes physicists to recognize fundamental roles of inelastic collisions.

The impact of macroscopic materials is characterized by the coefficient of restitution (COR) defined by

$$e = -v_r/v_i, \tag{1}$$

where v_i and v_r are the relative velocities of incoming and outgoing particles, respectively. COR e had been believed to be a material constant, since the classical experiment by Newton[3]. In general, however, experiments show that COR for three dimensional materials is not a constant even in approximate sense but depends strongly on the impact velocity[1, 4, 5].

The origin of the dissipation in inelastic collisions is the transfer of the kinetic energy of the center of mass into the internal degrees of freedom during the impacts. Systematic theoretical investigations of the impact have begun with the paper by Kuwabara and Kono[6]. Taking into account the viscous motion among the internal degrees of freedom, they derived the equation of the macroscopic deformation

$$\ddot{h} = -kh^{3/2} - \gamma\sqrt{h}\dot{h} \tag{2}$$

in a collision of two spheres, where the macroscopic deformation h is given by $h = R_1 + R_2 - |\mathbf{r}_1 - \mathbf{r}_2|$ with the radius $R_i (i = 1, 2)$ and the position of the center of the mass $\mathbf{r}_i$ of i th particle. $\dot{h}$ and $\ddot{h}$ are respectively dh/dt and d^2h/dt^2. k and γ are unimportant constants. The first term of the right hand side in eq.(2) represents the Hertzian contact force[7, 8, 9, 10] and the second term is the dissipation due to the internal motion. Later, Brilliantov *et al.*[11] and Morgado and Oppenheim[12] derived the identical equation to eq.(2) based on the different methods. Thus, eq.(2) is plausible in the quasi-static collisions of two spheres.

On the other hand, Gerl and Zippelius[13] performed the microscopic simulation of two-dimensional collision of an elastic disk with a wall. Their simulation is mainly based on the mode expansion of an elastic disk under the force free boundary condition. Then, they analyze Hamilton's equation ;

$$\dot{P}_{n,l} = -\frac{\partial H}{\partial Q_{n,l}}; \quad \dot{Q}_{n,l} = \frac{\partial H}{\partial P_{n,l}} \tag{3}$$

under the Hamiltonian

$$H = \frac{p_0^2}{2M} + \sum_{n,l}^{N} (\frac{P_{n,l}^2}{2M} + \frac{1}{2} M \omega_{n,l}^2 Q_{n,l}^2) + V_0 \int_{-\pi/2}^{\pi/2} d\phi e^{-ay(\phi,t)}. \tag{4}$$

Here $Q_{n,l}$ and $P_{n,l}$ are respectively the expansion coefficient of the elastic deformation and the canonical momentum, where n and l are the mode indices. $y(\phi, t)$ is the shape of the elastic disk in the polar coordinate[13, 14]. M is the mass of the disk, and p_0 satisfies $\dot{p}_0 = M\ddot{y}_0 = -(\partial H/\partial y_0)$ with the position of the center of mass y_0. V_0 and a are parameters to express the strength of the wall potential and $\omega_{n,l}$ is the angular frequency of the (n, l) mode[13]. Their results indicate that COR decreases with the impact velocity, which strongly depends on Poisson's ratio. Since the relation between quasi-static theory of impact[6, 11, 12] and their microscopic simulation[13] is not clear, we have to clarify the relation between two typical approaches.

In this letter, we will perform the microscopic simulation of the impact of a two-dimensional elastic disk with a wall. We introduce two methods of simulation; One is based on the lattice model (model A) and another is continuum model (model B) which is identical to that by Gerl and Zippelius[13]. Through our simulation, we will demonstrate that (i) the effect of temperature (the initial internal motion) is important, (ii) COR is suddenly dropped by the plastic deformation which is enhanced by the initial temperature, and (iii) the continuum model (model B) does not recover the quasi-static theories[6, 11, 12]. Part of this letter can been seen as a conference report.[14]

2 Our models

Let us explain our model. In both models, the wall exists at $y = 0$, and the center of mass keeps the position at $x = 0$. The disk approaches from $y > 0$ region and is rebounded by the wall. The disk in model A consists of some mass points (with the mass m) on the triangular lattice. All the mass points are combined with linear springs with the spring constant κ[14]. In the limit of large number of the mass points, this disk corresponds to the continuum circular disk with the Young's modulus $Y = 2\kappa/\sqrt{3}$ and Poisson's ratio 1/3[15]. The position of each mass point of model A is governed by the following equation:

$$m\frac{d^2\mathbf{r}_p}{dt^2} = -\kappa\sum_{i=1}^{6}(d_0 - |\mathbf{r}_p - \mathbf{r}_i|)\frac{\mathbf{r}_p - \mathbf{r}_i}{|\mathbf{r}_p - \mathbf{r}_i|} + \hat{y}aV_0e^{-ay_p} \tag{5}$$

where d_0 is the lattice constant, $\mathbf{r}_i$ is the position of the nearest neighbor mass points of $\mathbf{r}_p$, m is the mass of the mass points, y_p is the y coordinate of $\mathbf{r}_p$, and $\hat{y}$ is the unit vector of y direction. As in the simulation by Gerl and Zippelius[13] we introduce the wall whose potential is given by V_0e^{-ay} in both models, where $V_0 = a/2$ and $a = 100/d_0$ for model A and $a = 500/R$ with the radius of the disk R for model B. These choices of a are aimed to simulate the collision between two identical disks, though we have not extrapolated our results as $a \to \infty$. The number of the mass points is fixed to 1459 in model A, since the rough evaluation of convergence of the results has been checked in this model. For the sake of simplicity and comparison between two different models, we only simulate the case of Poisson's ratio $\sigma = 1/3$. The numerical scheme of the integration of model A is the classical fourth order Runge-Kutta method with $\Delta t = 1.6 \times 10^{-3}\sqrt{m/\kappa}$. For model B, we adopt the fourth order symplectic integral method with $\Delta t = 5.0 \times 10^{-3} R/c$ with $c = \sqrt{Y/\rho}$ for model B where Y is Young's modulus and ρ is the density. In both models, we have checked the conservation of the total energy.

The summary of differences between model A and B is as follows: (i) All of the mass points in model A interact with the wall but, in model B, only exterior boundary has the influence of the potential as in (4). (ii) Model A can have nonlinear deformations because of the directional projection, but model B is based on the theory of linear elasticity. (iii) Model A can express some plastic deformations, but model B cannot. (iv) Model A has the six folds symmetry but model B has the rotational symmetry.

3 Simulation

At first, we carry out the simulation of model A and model B with the initial condition at $T = 0$ (*i.e.* no internal motion). Figure 3 is the plot of the COR against the impact velocity for both model A and model B. When the impact velocity v_i is larger than $0.1c$ with $c = \sqrt{Y/\rho}$, the value of COR of model A is almost identical to that of model B. Each line decreases smoothly as impact velocity increases. At present, we do not know the reason why the significant difference between two models exists at low impact velocity.

Second, we investigate the force acting on the center of mass of the disk caused by the interaction with the wall in model B. In the limit of $v_i \to 0$ we expect that the Hertzian contact theory can be used[9, 10, 13]. The small amount of transfer from the translational motion to the internal motion is the macroscopic dissipation. Thus, we can check the validity of quasi-static approaches[6, 11, 12] from our simulation by the difference between the observed force acting on the center of mass and

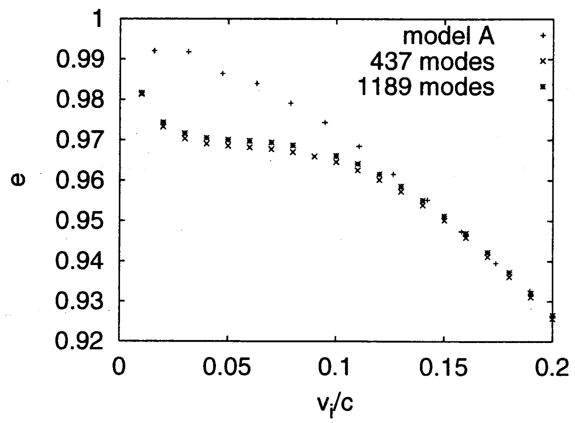


Figure 1: Coefficient of restitution for normal collision of the Model A and Model B as a function of impact velocity, where $c = \sqrt{Y/\rho}$ with the Young's modulus Y and the density ρ.

the Hertzian contact force. The two dimensional Hertzian contact law[10, 13] is given by the relation between the macroscopic deformation of the center of mass h and the elastic force F_{el} as

$$h \simeq -\frac{F_{el}}{\pi Y}\{\ln\left(\frac{4\pi Y R}{F_{el}\,(1-\sigma^2)}\right) - 1\}, \tag{6}$$

where Y, σ and R are the Young modulus, Poisson's ratio and the radius of the disk without deformation, respectively. If h is given, we can calculate the elastic force by solving eq.(6) numerically. Since in the limit of $v_i \to 0$ we may replace eq.(6) by $F_{el} \simeq -\pi Y h/\ln(4R/h)$[13]. Thus, the dissipative force F_{dis} in the two-dimensional quasi-static theory is expected to $F_{dis} \propto -\pi Y\dot{h}/\ln(4R/h)$.

Figure 2 is the comparison of our simulation in model B (1189 modes) with the Hertzian contact theory (6). The result of our simulation at the impact velocity $v_i = 0.01c$ with $c = \sqrt{Y/\rho}$ shows the beautiful hysteresis as suggested in the simulation at $v_i = 0.1c$ in ref.[13]. This means the compression and rebound are not symmetric. The hysteresis curve is still self-similar even at $v_i = 0.04c$ but the loop becomes noisy at $v_i = 0.1c$.

For the low impact velocity $v_i = 0.001c$, the hysteresis loop disappears but the total force

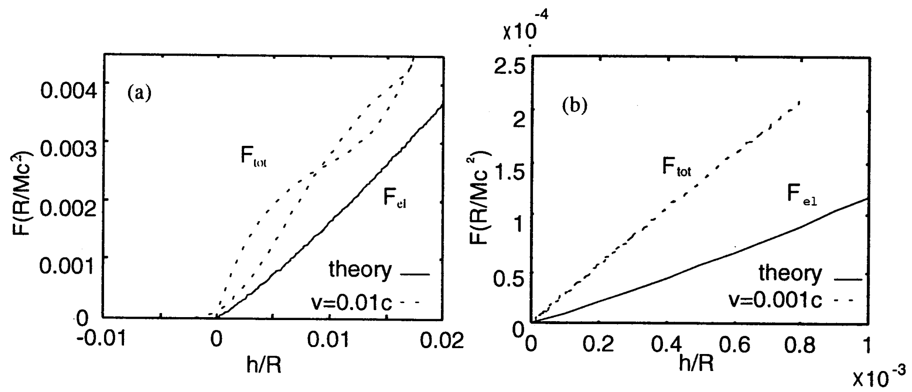


Figure 2: The comparison of the Hertzian force in eq.(6) with our simulation at $v_i = 0.01c$ (a) and $v_i = 0.001c$(b) at $T = 0$ in model B. at $v_i = 0.001c$ and $T = 0$ in model B.

observed in our simulation is almost a linear function of h which is deviated from Hertzian contact theory and quasi-static theory. In particular, the turning point at $\dot{F} = 0$ is deviated from the Hertzian curve. This deviation is clearly contrast to the quasi-static theory, because the dissipative force in the quasi-static theory must be zero at the turning point at which $\dot{h} = 0$ should satisfy. This tendency is invariant even for the simulation of model A, though the data becomes noisy. The linearity of the total repulsion force is not surprising, because $e^{-ay(\phi,t)}$ in the potential term in eq.(4) can be expanded by series of $Q_{n,l}$ for very slow impact[13, 14]. Although we cannot judge whether the model itself is not appropriate for slow impact or the quasi-static theory is wrong, the validity of the quasi-static theory cannot be supported by our microscopic simulation. However, the contact time τ in the impact evaluated by the quasi-static theory [13] can be evaluated as $\tau \simeq (\pi R/c)\sqrt{\ln(4c/v_i)}$ is close to the results of our simulation of model A[14]. Thus, the relation between dynamical impact theory and quasi-static elastic theory is not trivial.

We also investigate the impact with the finite temperature. The temperature is introduced as follows: In model A, we prepare the Maxwellian for the initial velocity distribution of mass points, while each position of the points is located at equilibrium. From the variance of the Maxwellian

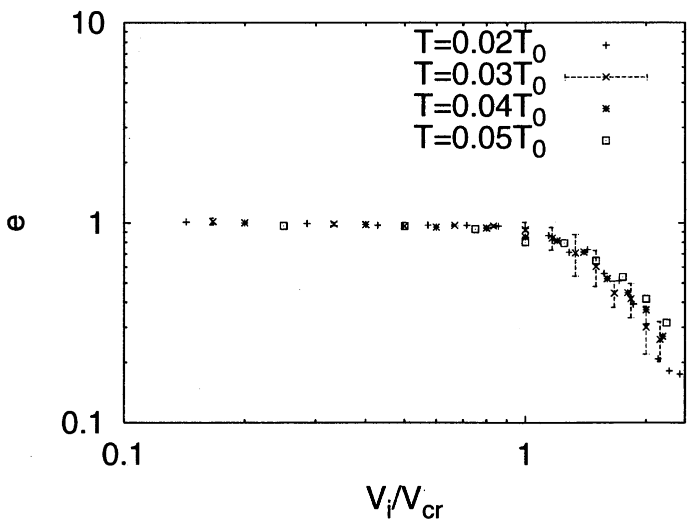


Figure 3: The relation between the coefficient of restitution and the impact velocity rescaled by the critical velocity for each temperature. Curves are plotted in the log-log scale. The temperature is scaled by $T_0 = mc^2/k_B$ with the mass of the mass points m and Boltzmann constant k_B. Note that the error bars are plotted only in the case $T/T_0 = 0.03$ but is the same order even at other T.

we can introduce the temperature as a parameter. To perform the simulation, we prepare 10 independent samples obeying the Maxwellian with the aid of normal random number. In model B, we prepare samples in which the absolute value of each mode satisfies equipartition law correctly. The sign of each mode is assumed to be at random. From the equipartition law we can introduce the temperature as a parameter of simulation.

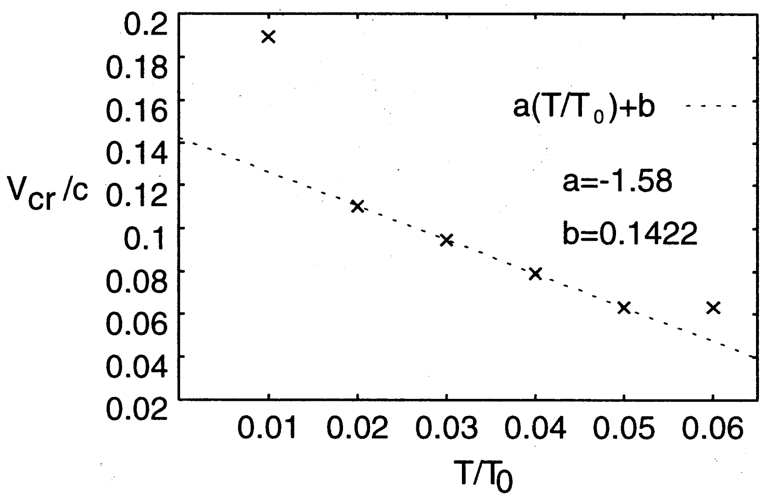
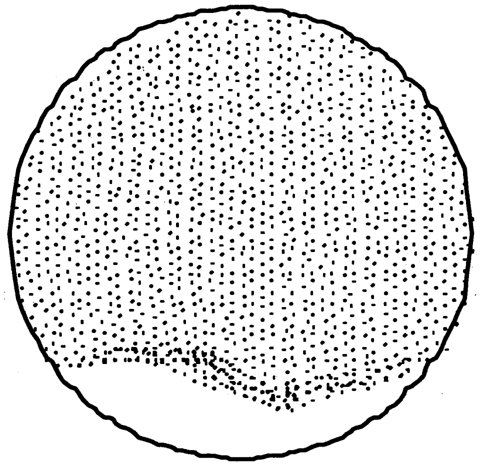


Figure 4: The plot of the initial temperature and the critical velocity causing the plastic deformation. $v_{cr}/c = a(T/T_0) + b$ is the fitting curve line from the data between $T/T_0 =$ 0.02 and 0.05.

In this letter, we focus on the case of large impact velocity, though the results of the low impact cases are suggestive[14]. For large impact velocity, we do not observe any definite temperature effect in model B but we find drastic decreases of COR in model A. It seems that COR can be on a universal curve when the impact velocity is scaled by the critical velocity above which COR drops abruptly (Fig.3). The relation between the critical velocity and the initial temperature at the intermediate impact velocities is shown in the Fig.3. The critical velocity seems to obey a linear function of T at the intermediate T.

(a)



(b)

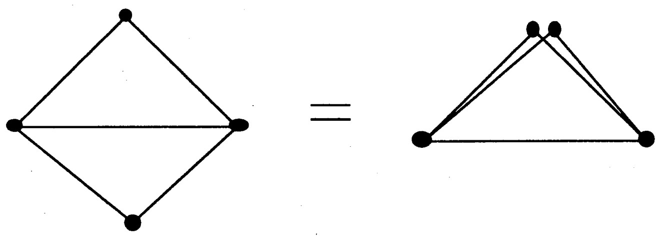


Figure 5: (a) Plastic deformation of model A with $v_i = 0.22$ at $T = 0.03$. The solid circle represents the initial circle. The points are positions of the mass points after the collision. (b) Two configurations are energetically equivalent.

We investigate what happens in the disk above the critical velocity and find the existence of plastic deformation of the disk (Fig.3(a)). Actually, there is no energy differences between two configurations in Fig.3(b) in model A which can occur after the strong compression during the impact but cannot be released after the impact is over. It is known that the plastic deformation causes the drop of COR[10], but our finding is something new, because (i) this process is excited by the temperature and (ii) COR decreases more rapidly like $e \sim v_i^{-1.2}$ than that for the conventional plastic deformation $e \sim v_i^{-1/3}$ [14]. The mechanism how to occur the plastic deformation is not clear at present including the linear law in Fig.3.

4 Conclusion

We have numerically studied the impact of a two dimensional elastic disk with the wall with the aid of model A and model B. The result can be summarized as (i) The coefficient of restitution (COR) decreases with the impact velocity. (ii) The result of our simulation is not consistent with the result of the two-dimensional quasi-static theory. For large impact velocity, there is hysteresis in the deformation of the center of mass. For small velocity, there remains the inelastic force even at $\dot{h} = 0$. (iii) There are drastic effects of temperature in both small and large impact velocity. (iv) In particular, for large impact velocity of model A, we have found the abrupt drop of COR above the critical impact velocity by the plastic deformation. The critical velocity of the plastic deformation seems to obey a simple linear function of temperature.

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